An Alternative to Stochastic Volatility Models

Andrea Macrina

Department of Mathematics, King's College London

Institute of Economic Research, Kyoto University

Osaka University
Graduate School of Engineering Sciences

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Implied density

We consider the problem of how to model the price dynamics of an asset when we are given option prices for a range of strikes and maturities as initial data.

Our approach will be to model the probability density process for the asset price.

We have a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a Brownian filtration $\{\mathcal{F}_t\}$.

A non-dividend-paying asset A^i of limited liability $\left(A^i_t > 0, \forall\, t > 0\right)$ is chosen as numeraire, and henceforth all prices A^j_t are expressed in units of the numeraire asset A^i:

$$A^j_s = A^i_s \, \mathbb{E}^{\mathbb{Q}}\left[\frac{A^j_t}{A^i_t} \,\middle|\, \mathcal{F}_t \right]. \qquad (1)$$

The measure $\mathbb{Q}$ has the property that the price processes of all non-dividend-paying assets, when expressed in units of the selected numeraire, are martingales.

We refer to $\mathbb{Q}$ as the "martingale measure" associated with the numeraire.

Let $\{A_t\}$ denote the price process of a generic non-dividend-paying asset, expressed in units of the selected numeraire.

Then $\{A_t\}$ is a $\mathbb{Q}$-martingale, and the $\mathbb{Q}$-dynamical equation of $\{A_t\}$ is of the form

$$\mathrm{d}A_t = \sigma_t^A \, \mathrm{d}W_t. \tag{2}$$

Here $\{\sigma_t^A\}$ is the $\{\mathcal{F}_t\}$-adapted absolute volatility process, and $\{W_t\}$ is a (multi-dimensional) Brownian motion.

We fix a time $T > 0$, and introduce a continuous random variable A_T.

We assume for $0 \leq t < T$ the existence of an $\mathcal{F}_t$-conditional $\mathbb{Q}$-probability density $f_{tT}(x)$ for the random value A_T of the chosen asset at time T.

We assume, for all $x \in \mathbb{R}$ with the property that there exists an $\omega \in \Omega$ such that $A_T(\omega) = x$, that $f_{tT}(x)$ is defined and takes positive values.

Let us write $\mathcal{D}$ for the domain of the conditional density.

We require that $f_{tT}(x)$, $x \in \mathcal{D}$, should satisfy the following conditions:

1. For all bounded, measurable functions $g(x)$, $x \in \mathcal{D}$, it holds that
$$\int_{\mathcal{D}} g(x) f_{tT}(x) \mathrm{d}x = \mathbb{E}^{\mathbb{Q}} \left[g(X_T) \,|\, \mathcal{F}_t \right]. \tag{3}$$
2. For $0 \leq s \leq t < T$ and for each $x \in \mathcal{D}$ the process $\{f_{tT}(x)\}$ is a martingale:
$$\mathbb{E}^{\mathbb{Q}} \left[f_{tT}(x) \,|\, \mathcal{F}_s \right] = f_{sT}(x). \tag{4}$$

The asset price A_t at time t can now be expressed in terms of
$$A_t = \int_{\mathcal{D}} x \, f_{tT}(x) \mathrm{d}x, \tag{5}$$
from which the martingale condition follows:
$$A_t = \mathbb{E}_t^{\mathbb{Q}} \left[A_T \right]. \tag{6}$$

Since the density process $\{f_{tT}(x)\}$ is a positive $\mathbb{Q}$-martingale, for each $x \in \mathcal{D}$ the associated dynamical equation takes the form
$$\mathrm{d} f_{tT}(x) = f_{tT}(x) \, \sigma_{tT}^{f}(x) \, \mathrm{d}W_t, \tag{7}$$

Here $\sigma^f_{tT}(x)$ is the "relative" volatility of the density at x.

We require that the following normalization condition be satisfied:

$$\int_{\mathcal{D}} f_{tT}(x)\mathrm{d}x = 1. \tag{8}$$

The normalization condition is satisfied for all $t \in [0, T)$ if it holds in particular at $t = 0$, and if

$$\sigma^f_{tT}(x) = \sigma_{tT}(x) - \int_{\mathcal{D}} \sigma_{tT}(y) f_{tT}(y)\mathrm{d}y, \tag{9}$$

for some process $\{\sigma_{tT}(x)\}$. This can be shown by considering

$$f_{tT}(x) = \frac{g_{tT}(x)}{\int_{\mathcal{D}} g_{tT}(y)\mathrm{d}y}, \tag{10}$$

where it is assumed that

$$\frac{\mathrm{d}g_{tT}(x)}{g_{tT}(x)} = \mu_{tT}(x)\mathrm{d}t + \sigma_{tT}(x)\mathrm{d}W_t. \tag{11}$$

By use of (9), the dynamical equation of the density process $\{f_{tT}(x)\}$ takes the form

$$\mathrm{d}f_{tT}(x) = f_{tT}(x)\left[\sigma_{tT}(x) - \int_{\mathcal{D}} \sigma_{tT}(y) f_{tT}(y)\mathrm{d}y\right]\mathrm{d}W_t. \tag{12}$$

We call this infinite-dimensional SDE, the master equation.

By a "model" for the density process $\{f_{tT}(x)\}$ we understand solutions of the "master equation" (12) satisfying the normalization condition (8), along with:

1. The specification of the *initial density* $f_{0T}(x)$
2. The specification of the *volatility structure* $\{\sigma_{tT}(x)\}$ in the form of a functional
$$\sigma_{tT}(x) = \Phi[f_{tT}(\cdot), t, x]. \tag{13}$$

The initial density $f_{0T}(x)$ can be determined by the specification of initial option price data for the maturity T and for all strikes $K \in \mathcal{D}$.

Let C_{tT} denote the price at time t of a T-maturity, K-strike call option, then:

$$C_{tT}(K) = \mathbb{E}^{\mathbb{Q}}[(A_T - K)^+ \,|\, \mathcal{F}_t] = \int_{\mathcal{D}} (x - K)^+ f_{tT}(x)\mathrm{d}x. \tag{14}$$

In particular, we have:

$$C_{0T}(K) = \mathbb{E}^{\mathbb{Q}}[(A_T - K)^+] = \int_{\mathcal{D}} (x - K)^+ f_{0T}(x)\mathrm{d}x. \tag{15}$$

By the Breeden & Litzenberger (1978) device we see, in the present context, that

$$f_{0T}(x) = \frac{\partial^2 C_{0T}(K)}{\partial K^2}. \tag{16}$$

In practice, one would like to specify Φ modulo enough parametric freedom to allow the input of additional option price data.

What form this additional data might take depends on the nature of the market under consideration and the class of valuation problems being undertaken.

Integral form of the master equation

To proceed further, it is useful to derive an integral form of the master equation that incorporates the initial condition explicitly.

The first step is to integrate the master equation (12) to obtain

$$f_{tT}(x) = f_{0T}(x) \exp\left[\int_0^t \left[\sigma_{sT}(x) - \langle\sigma_{sT}\rangle\right] \mathrm{d}W_s - \tfrac{1}{2}\int_0^t \left[\sigma_{sT}(x) - \langle\sigma_{sT}\rangle\right]^2 \mathrm{d}s\right]. \tag{17}$$

where the bracket notation is defined by

$$\langle\sigma_{tT}\rangle = \int_{\mathcal{D}} \sigma_{tT}(x) f_{tT}(x)\mathrm{d}x, \tag{18}$$

By expanding the exponent we obtain the following expression:

$$f_{tT}(x) = f_{0T}(x) \frac{\exp\left[\int_0^t \sigma_{sT}(x)\left(\mathrm{d}W_s + \langle\sigma_{sT}\rangle \mathrm{d}s\right) - \frac{1}{2}\int_0^t {\sigma_{sT}}^2(x)\mathrm{d}s\right]}{\exp\left[\int_0^t \langle\sigma_{sT}\rangle\left(\mathrm{d}W_s + \langle\sigma_{sT}\rangle \mathrm{d}s\right) - \frac{1}{2}\int_0^t \langle\sigma_{sT}\rangle^2 \mathrm{d}s\right]} \tag{19}$$

Next we introduce a process $\{Z_t\}$, defined by

$$Z_t = W_t + \int_0^t \langle \sigma_{sT} \rangle \mathrm{d}s, \tag{20}$$

so that we can write

$$f_{tT}(x) = f_{0T}(x) \frac{\exp\left[\int_0^t \sigma_{sT}(x)\mathrm{d}Z_s - \frac{1}{2}\int_0^t {\sigma_{sT}}^2(x)\mathrm{d}s\right]}{\exp\left[\int_0^t \langle \sigma_{sT} \rangle \mathrm{d}Z_s - \frac{1}{2}\int_0^t \langle \sigma_{sT} \rangle^2 \mathrm{d}s\right]}. \tag{21}$$

We apply the normalization condition for the conditional density process to obtain

$$\begin{aligned} &\exp\left(\int_0^t \langle \sigma_{sT} \rangle \mathrm{d}Z_s - \frac{1}{2}\int_0^t \langle \sigma_{sT} \rangle^2 \mathrm{d}s\right) \\ &\qquad = \int_{\mathcal{D}} f_{0T}(x) \exp\left(\int_0^t \sigma_{sT}(x)\mathrm{d}Z_s - \frac{1}{2}\int_0^t {\sigma_{sT}}^2(x)\mathrm{d}s\right)\mathrm{d}x. \end{aligned} \tag{22}$$

As a consequence we have the following integral form for the master equation of the conditional density process:

$$f_{tT}(x) = \frac{f_{0T}(x)\exp\left(\int_0^t \sigma_{sT}(x)\mathrm{d}Z_s - \frac{1}{2}\int_0^t {\sigma_{sT}}^2(x)\mathrm{d}s\right)}{\int_{\mathcal{D}} f_{0T}(y)\exp\left(\int_0^t \sigma_{sT}(y)\mathrm{d}Z_s - \frac{1}{2}\int_0^t {\sigma_{sT}}^2(y)\mathrm{d}s\right)\mathrm{d}y}. \tag{23}$$

Of course, one has not "solved" the master equation yet, since $\{Z_t\}$ implicitly involves the density process, via (18) and (20).

Nevertheless, we can use (23) as a starting point for obtaining solutions, as we shall see next.

Finite-time horizon density models

We consider the case in which the random variable A_T has *a priori* probability density $\bar{f}_{0T}(x)$ given by

$$\bar{f}_{0T}(x)\mathrm{d}x = \mathbb{Q}\left[A_T \in \mathrm{d}x\right]. \tag{24}$$

We define the conditional density $f_{tT}(x)$ by

$$f_{tT}(x)\mathrm{d}x = \mathbb{Q}\left[A_T \in \mathrm{d}x \,|\, \mathcal{F}_t\right]. \tag{25}$$

In order for this definition to make sense we need to specify the filtration $\{\mathcal{F}_t\}$.

We introduce a Brownian motion $\{B_t\}$ that is taken to be independent of A_T and construct a process $\{\xi_{tT}\}_{0\le t\le T}$ given by

$$\xi_{tT} = \sigma\, A_T\, t + \beta_{tT}. \tag{26}$$

Here σ is a constant and $\{\beta_{tT}\}_{0\le t\le T}$ is a Brownian bridge constructed by

$$\beta_{tT} = B_t - \frac{t}{T}B_T. \tag{27}$$

Next we assume that $\{\mathcal{F}_t\}$ is generated by $\{\xi_{tT}\}$:

$$\mathcal{F}_t = \sigma\left(\{\xi_{sT}\}_{0\leq s\leq t}\right). \tag{28}$$

Clearly $\{\xi_{tT}\}$ is $\{\mathcal{F}_t\}$-adapted, and A_T is $\mathcal{F}_T$-measurable since $A_T = \xi_{TT}/\sigma\, T$. It is shown in Brody *et al.* (2007), (2008) that $\{\xi_{tT}\}$ is an $\{\mathcal{F}_t\}$-Markov process.

Proposition. Let the initial density $\bar{f}_{0T}(x)$ be specified, and let the volatility function $\sigma_{tT}(x)$ be given by

$$\sigma_{tT}(x) = \sigma\,\frac{T}{T-t}\,x, \tag{29}$$

for $0 \leq t < T$. Let the filtration $\{\mathcal{F}_t\}$ be defined by (28). Then the process $\{W_t\}_{0\leq t<T}$ defined by

$$W_t = \xi_{tT} - \int_0^t \frac{1}{T-s}\left(\sigma\, T\, \mathbb{E}\left[A_T \,|\, \xi_{st}\right] - \xi_{sT}\right) \mathrm{d}s, \tag{30}$$

is an $\{\mathcal{F}_t\}$-Brownian motion and the process $\{f_{tT}(x)\}$ given by

$$f_{tT}(x) = \frac{\bar{f}_{0T}(x)\exp\left[\frac{T}{T-t}\left(\sigma\xi_{tT}x - \frac{1}{2}\,\sigma^2x^2t\right)\right]}{\int_{\mathcal{D}} \bar{f}_{0T}(y)\exp\left[\frac{T}{T-t}\left(\sigma\xi_{tT}y - \frac{1}{2}\,\sigma^2y^2t\right)\right]\mathrm{d}y}, \tag{31}$$

satisfies the master equation (12) with the given initial condition.

Proof. The fact that $\{W_t\}_{0\leq t\leq T}$ is an $\{\mathcal{F}_t\}$-Brownian motion, is proven in Brody *et al.* (2007), (2008) by making use of Lévy's characterization theorem.

Next we calculate the conditional expectation $\mathbb{E}[A_T \,|\, \xi_{tT}]$ by use of the Bayes formula. We have:

$$\mathbb{E}\left[A_T \,|\, \xi_{tT}\right] = \int_{\mathcal{D}} x\, f_{tT}(x)\mathrm{d}x, \tag{32}$$

where the conditional density $\{f_{tT}(x)\}$ is given by

$$f_{tT}(x) = \frac{\bar{f}_{0T}\,\rho\left(\xi_{tT}\,|\,A_T = x\right)}{\int_{\mathcal{D}} \bar{f}_{0T}(y)\,\rho\left(\xi_{tT}\,|\,A_T = y\right)\mathrm{d}y}. \tag{33}$$

Conditional on the value of A_T, the random variable ξ_{tT} has a Gaussian density:

$$\rho\left(\xi_{tT}\,|\,A_T = x\right) = \sqrt{\frac{T}{2\pi\, t(T-t)}}\,\exp\left[-\frac{1}{2}\,\frac{T}{t(T-t)}\left(\xi_{tT} - \sigma t x\right)^2\right]. \tag{34}$$

Inserting the conditional density $\rho\left(\xi_{tT}\,|\,A_T = x\right)$ in (33), one obtains the expression (31) for the density process $\{f_{tT}(x)\}$ after some simplifications.

With this intermediate result at hand, we can write the process $\{W_t\}_{0\leq t\leq T}$ in terms of the density $\{f_{tT}(x)\}$:

$$W_t = \xi_{tT} - \int_0^t \frac{1}{T-s}\left(\sigma T\int_{\mathcal{D}} x\, f_{sT}(x)\mathrm{d}x - \xi_{sT}\right)\mathrm{d}s. \tag{35}$$

Next we show that the density process given by (31) satisfies the master equation (12).

We recall that the master equation (12) can be written in the form

$$f_{tT}(x) = \frac{f_{0T}(x)\exp\left(\int_0^t \sigma_{sT}(x)\mathrm{d}Z_s - \frac{1}{2}\int_0^t \sigma_{sT}(x)^2\mathrm{d}s\right)}{\int_{\mathcal{D}} f_{0T}(x)\exp\left(\int_0^t \sigma_{sT}(x)\mathrm{d}Z_s - \frac{1}{2}\int_0^t \sigma_{sT}(x)^2\mathrm{d}s\right)\mathrm{d}x}, \tag{36}$$

where

$$Z_t = W_t + \int_0^t \int_{\mathcal{D}} \sigma_{sT}(x) f_{sT}(x)\mathrm{d}x\,\mathrm{d}s. \tag{37}$$

We show now that (36) reduces to (31) if we insert (35) into (37), and use

$$\sigma_{tT}(x) = \sigma\,\frac{T}{T-t}\,x. \tag{38}$$

For the process $\{Z_t\}$ given in (37) we obtain,

$$Z_t = \xi_{tT} + \int_0^t \frac{\xi_{sT}}{T-s}\,\mathrm{d}s. \tag{39}$$

By inserting this expression for $\{Z_t\}$ in the integral form of the master equation (36), and thereafter integrating the exponent

$$\int_0^t \sigma_{sT}(x)\mathrm{d}Z_s - \tfrac{1}{2}\int_0^t \sigma_{sT}(x)^2\mathrm{d}s, \tag{40}$$

we obtain the expression (31) for the density process $\{f_{tT}(x)\}$. That is:

$$f_{tT}(x) = \frac{\bar{f}_{0T}(x)\exp\left[\frac{T}{T-t}\left(\sigma\xi_{tT}x - \frac{1}{2}\,\sigma^2x^2t\right)\right]}{\int_{\mathcal{D}}\bar{f}_{0T}(y)\exp\left[\frac{T}{T-t}\left(\sigma\xi_{tT}y - \frac{1}{2}\,\sigma^2y^2t\right)\right]\mathrm{d}y}. \tag{41}$$

□

Bachelier model on a finite-time horizon

We consider the Bachelier asset price model, characterised by

$$A_t = \gamma W_t, \tag{42}$$

over all times up to T. The process $\{W_t\}$ is an $\{\mathcal{F}_t\}$-adapted Brownian motion and γ is a constant.

The conditional density process $\{f^B_{tT}(x)\}$ of the Bachelier asset price process is given by

$$f^B_{tT}(x) = \frac{\exp\left[-\frac{1}{2}\frac{1}{\gamma^2(T-t)}(x-\gamma W_t)^2\right]}{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\frac{1}{\gamma^2(T-t)}(y-\gamma W_t)^2\right]\mathrm{d}y}. \tag{43}$$

It turns out that the Bachelier asset price model in finite time is a special case of the family of models presented above.

Let the initial density $\bar{f}_{0T}(x)$ be given by

$$\bar{f}_{0T}(x) = \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\,\exp\left(-\tfrac{1}{2}\,\sigma^2\,Tx^2\right), \tag{44}$$

and let the volatility function $\sigma_{tT}(x)$ be given by

$$\sigma_{tT}(x) = \sigma\,\frac{T}{T-t}\,x, \tag{45}$$

where $\sigma = 1/(\gamma T)$.

Then the density process $\{f_{tT}(x)\}$ defined by

$$f_{tT}(x) = \frac{\bar{f}_{0T}(x)\exp\left[\frac{T}{T-t}\left(\sigma\xi_{tT}x - \frac{1}{2}\,\sigma^2x^2t\right)\right]}{\int_{\mathcal{D}}\bar{f}_{0T}(y)\exp\left[\frac{T}{T-t}\left(\sigma\xi_{tT}y - \frac{1}{2}\,\sigma^2y^2t\right)\right]\mathrm{d}y}, \tag{46}$$

reduces to the expression for the process $\{f^B_{tT}(x)\}$ given in (43).

Infinite-time horizon density models

In this part of the presentation we generalise the results shown so far in two ways.

(1) We extend the time horizon to infinity.

(2) The volatility function for the SDE of the conditional density process shall be defined in terms of a general deterministic function $v(t, x)$ of two variables.

Conjecture. Let $\bar{f}_0(x) : \mathbb{R} \to \mathbb{R}^+$ be a density function. A filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ can be constructed along with

(i) an $\mathcal{F}_\infty$-measurable random variable X_∞ with density $\bar{f}_0(x)$,
(ii) an $\{\mathcal{F}_t\}$-adapted density process $\{f_t(x)\}$,
(iii) an $\{\mathcal{F}_t\}$-adapted Brownian motion $\{W_t\}$

such that (a) for some function $\gamma(t)$ on $[0,\infty)$, (b) for some function $g(x)$ that is invertible onto $\mathbb{R}$, and (c) for some suitably integrable function $v(t,x)$ on $[0,\infty) \times \mathbb{R}$ with the property that

$$\lim_{t\to\infty} \gamma(t) \int_0^t v(s,x)\mathrm{d}s = g(x), \tag{47}$$

the following hold for all $t \in [0,\infty)$:

$$\mathbb{Q}\left[X_\infty \in \mathrm{d}x \,|\, \mathcal{F}_t\right] = f_t(x)\mathrm{d}x, \tag{48}$$

and

$$f_t(x) = \bar{f}_0(x) + \int_0^t f_s(x)\left[v(s,x) - \int_{-\infty}^{\infty} v(s,y)f_s(y)\mathrm{d}y\right]\mathrm{d}W_s. \tag{49}$$

Proposition. Let $\bar{f}_0(x) : \mathbb{R} \to \mathbb{R}^+$ be a density function. Let $\gamma(t)$ be a function on $[0, \infty)$ such that $\lim_{t\to\infty} \sqrt{t}\,\gamma(t) = 0$. Let the function $g(x)$ be invertible onto $\mathbb{R}$. Let the function $v(t, x)$ on $[0, \infty) \times \mathbb{R}$ satisfy (47). Let $\{B_t\}$ be a Brownian motion and let X_∞ be an independent random variable with density $\bar{f}_0(x)$. Let $\{\mathcal{F}_t\}$ denote the filtration generated by the process $\{I_t\}$ defined by

$$I_t = B_t + \int_0^t v(s, X_\infty)\mathrm{d}s. \tag{50}$$

Let the $\{\mathcal{F}_t\}$-adapted density process $\{f_t(x)\}$ be defined by

$$f_t(x) = \frac{\bar{f}_0(x) \exp\left[\int_0^t v(s, x)\mathrm{d}I_s - \frac{1}{2}\int_0^t v^2(s, x)\mathrm{d}s\right]}{\int_{-\infty}^{\infty} \bar{f}_0(y) \exp\left[\int_0^t v(s, y)\mathrm{d}I_s - \frac{1}{2}\int_0^t v^2(s, y)\mathrm{d}s\right] \mathrm{d}y}. \tag{51}$$

Then (a) the random variable X_∞ is $\mathcal{F}_\infty$-measurable, (b) the process $\{W_t\}$ defined by

$$W_t = I_t - \int_0^t \mathbb{E}^{\mathbb{Q}}\left[v(s, X_\infty) \,|\, \mathcal{F}_s\right] \mathrm{d}s \tag{52}$$

is an $\{\mathcal{F}_t\}$-adapted Brownian motion, and (c) equations (56) and (49) are satisfied.

Proof: We consider the filtration $\{\mathcal{G}_t\}$ defined by $\mathcal{G}_t = \sigma\left(\{B_s\}_{0\leq s\leq t}, X_\infty\right)$ and notice that $\mathcal{G}_t \supset \mathcal{F}_t$.

We introduce a $(\{\mathcal{G}_t\}, \mathbb{Q})$-martingale $\{M_t\}$ defined by

$$M_t = \exp\left[-\int_0^t v(s, X_\infty)\mathrm{d}B_s - \tfrac{1}{2}\int_0^t v^2(s, X_\infty)\mathrm{d}s\right]. \tag{53}$$

The martingale $\{M_t\}$ can be used to go from $\mathbb{Q}$ to a new measure $\mathbb{B}$.

We observe that $\mathrm{d}I_t = \mathrm{d}B_t + v(t, X_\infty)\mathrm{d}t$.

It follows that $\{I_t\}$ is a $(\{\mathcal{G}_t\}, \mathbb{B})$-Brownian motion. It can be shown that I_t is $\mathbb{B}$-independent of X_∞ for any t.

We also note that the process $\{M_t^{-1}\}$ is a $(\{\mathcal{G}_t\}, \mathbb{B})$-martingale.

Let $H \in \mathcal{B}$ where $\mathcal{B}$ is the space of bounded functions on $\mathbb{R}$.

Then the generalized Bayes formula states that

$$\mathbb{E}^{\mathbb{Q}}\left[H(X_\infty)|\,\mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{B}}\left[M_t^{-1}H(X_\infty)\,|\,\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{B}}\left[M_t^{-1}\,|\,\mathcal{F}_t\right]}. \tag{54}$$

Making use of the fact that X_∞ is $\mathbb{B}$-independent of I_t for any $t \in [0, \infty)$, we obtain

$$\mathbb{E}^{\mathbb{Q}}\left[H(X_\infty) \,|\, \mathcal{F}_t\right] = \frac{\int_{-\infty}^{\infty} \bar{f}_0(x) H(x) \exp\left(\int_0^t v(s,x)\mathrm{d}I_s - \frac{1}{2}\int_0^t v^2(s,x)\mathrm{d}s\right)\mathrm{d}x}{\int_{-\infty}^{\infty} \bar{f}_0(y) \exp\left(\int_0^t v(s,y)\mathrm{d}I_s - \frac{1}{2}\int_0^t v^2(s,y)\mathrm{d}s\right)\mathrm{d}y}. \tag{55}$$

In particular, by setting $H(X_\infty) = \mathbf{1}(X_\infty \leq x)$, we deduce that

$$\mathbb{Q}\left[X_\infty \in \mathrm{d}x \,|\, \mathcal{F}_t\right] = f_t(x)\mathrm{d}x \tag{56}$$

is satisfied, where

$$f_t(x) = \frac{\bar{f}_0(x) \exp\left[\int_0^t v(s,x)\mathrm{d}I_s - \frac{1}{2}\int_0^t v^2(s,x)\mathrm{d}s\right]}{\int_{-\infty}^{\infty} \bar{f}_0(y) \exp\left[\int_0^t v(s,y)\mathrm{d}I_s - \frac{1}{2}\int_0^t v^2(s,y)\mathrm{d}s\right]\mathrm{d}y}. \tag{57}$$

The proof of the proposition continuous with showing (i) that (57) satisfies the master equation (49), (ii) that the process $\{W_t\}$ is an $(\{\mathcal{F}_t\}, \mathbb{Q})$-Brownian motion, and (iii) that X_∞ is $\mathcal{F}_\infty$-measurable.

Relation to finite-time density models

We recall that under $\mathbb{B}$ the information process $\{I_t\}$ is a Brownian motion over the interval $t \in [0, T)$, and that the filtration $\{\mathcal{F}_t\}$ is generated by $\{I_t\}$.

We can construct a $\mathbb{B}$-Brownian bridge by use of the $\mathbb{B}$-Brownian motion $\{I_t\}$ as follows: On $[0, T)$ we set

$$\xi_{tT} = (T - t) \int_0^t \frac{1}{T - s} \, \mathrm{d}I_s. \tag{58}$$

Next we recall that

$$I_t = B_t + \int_0^t v(s, X_\infty) \mathrm{d}s, \tag{59}$$

where here the volatility function $v(t, x)$ need only be defined on $[0, T) \times \mathbb{R}$. Then we have:

$$\xi_{tT} = (T - t) \int_0^t \frac{\mathrm{d}B_s}{T - s} + (T - t) \int_0^t \frac{1}{T - s} \, v(s, X_\infty) \, \mathrm{d}s. \tag{60}$$

The first integral defines a $(\{\mathcal{G}_t\}, \mathbb{Q})$-Brownian bridge process over the interval $[0, T)$ which we denote $\{\beta_{tT}\}$.

For the volatility function $v(t,x)$ we set

$$v(t,x) = \sigma \frac{T}{T-t}\, x. \tag{61}$$

This leads to

$$\xi_{tT} = \sigma\, X_\infty\, (T-t) T \int_0^t \frac{1}{(T-s)^2}\, \mathrm{d}s + \beta_{tT}, \tag{62}$$

and therefore further to

$$\xi_{tT} = \sigma\, X_\infty\, t + \beta_{tT}. \tag{63}$$

Since $\{\mathcal{F}_t\}$ is generated by $\{I_t\}$, so it is equivalently by $\{\xi_{tT}\}$.

Due to the special form of the volatility function (61), the random variable X_∞ becomes $\mathcal{F}_T$-measurable.

Hence we see that the role of A_T appearing in the definition (26) of the process $\{\xi_{tT}\}$,

$$\xi_{tT} = \sigma\, A_T\, t + \beta_{tT}, \tag{64}$$

is taken over by X_∞ in (63). Thus the relation between infinite-time horizon and finite-time horizon density models is established.

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