Proving existence of densities for solutions of SDE's via the lent particle method

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Abstract

The aim of this paper is to apply the lent particle method in order to prove existence of density for solutions of sde's driven by a Poisson measure. We claim that this method, based on the theory of Dirichlet forms and the representation of a well-chosen gradient, simplifies the proofs and permits an explicit expression of the carré du champ matrix i.e. the Malliavin matrix.

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1 Introduction

This article is based on 2 recent works ([3, 4]) in which we introduce a new method called *the lent particle method* and apply it to prove existence of density for Poisson functionals including solutions of Poisson driven SDE's. There is a huge literature on this subject and particularly on Malliavin calculus on the Poisson space, we refer to [4] and the bibliography at the end of this article for a non-exhaustive list of works on this subject. Roughly speaking, to prove existence of density using Malliavin calculus on the Poisson space, either one has to deal with finite difference operators either one has to derivate w.r.t. the times of the jumps either one has to derivate w.r.t. the size of the jumps. Our method corresponds to the last case and is based on the theory of Dirichlet forms and the fundamental property (EID) satisfied by local Dirichlet forms. This approach simplifies the method and permits to obtain explicit formulae for the gradient and the carré du champ. Here, we have chosen to present the results without recalling all the technical hypotheses neither the proofs, we just want to insist on the simplicity of the method, all the details may be found in [3, 4], see also our first article in this volume.

2 The framework

We consider $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$: a local symmetric Dirichlet structure which admits a carré du champ operator i.e. $(X, \mathcal{X}, \nu)$ is a measured space, ν is σ-finite and the bilinear form

$$e[f,g] = \frac{1}{2}\int \gamma[f,g]\, d\nu,$$

is a local Dirichlet form with domain $\mathbf{d} \subset L^2(\nu)$ and carré du champ operator γ.
We assume also technical conditions that we do not recall here (see [3]) and that $(\mathbf{d}, e)$ satisfies (EID).
Let N be a Poisson random measure on $[0, +\infty[\times X$ with intensity $dt \times \nu(du)$ defined on the probability space $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ where Ω_1 is the **configuration space**, $\mathcal{A}_1$ the σ-field generated by N and $\mathbb{P}_1$ the law of N.
Following [3], we construct a Dirichlet structure on the upper space, that we denote by $(\mathbb{D}, \mathcal{E})$. It is a Dirichlet form on $L^2(\Omega_1, \mathbb{P}_1)$. Before recalling its main properties, we give an example of Dirichlet structure on the *bottom space* that we shall consider in the case of Lévy processes.

2.1 Main example in $\mathbb{R}^d$

Let $(Y_t)_{t\geqslant 0}$ be a d-dimensional Lévy process, with Lévy measure $\nu = kdx$. Under standard hypotheses, we have the following representation:

$$Y_t = \int_0^t \int_{\mathbb{R}^d} u\tilde{N}(ds, du),$$

where $\tilde{N}$ is a compensated Poisson measure with intensity $dt \times kdx$. In this case, the idea is to introduce an ad-hoc Dirichlet structure on $\mathbb{R}^d$
The following example gives a case of such a structure $(\mathbf{d}, e)$ which satisfies all the required hypotheses and which is flexible enough to encompass many cases:

Lemma 1. *Let $r \in \mathbb{N}^*$, $(X, \mathcal{X}) = (\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ and $\nu = kdx$ where k is non-negative and Borelian. We are given $\xi = (\xi_{ij})_{1\leqslant i,j\leqslant r}$ an $\mathbb{R}^{r\times r}$-valued and symmetric Borel function. We assume that there exist an open set $O \subset \mathbb{R}^r$ and a function ψ continuous on O and null on $\mathbb{R}^r \setminus O$ such that*

1. *$k > 0$ on O ν-a.e. and is locally bounded on O*
2. *ξ is locally bounded and locally elliptic on O.*
3. *$k \geqslant \psi > 0$ ν-a.e. on O.*
4. *for all $i, j \in \{1, \cdots, r\}$, $\xi_{i,j}\psi$ belongs to $H^1_{loc}(O)$.*

We denote by H the subspace of functions $f \in L^2(\nu) \cap L^1(\nu)$ such that the restriction of f to O belongs to $C_c^\infty(O)$. Then, the bilinear form defined by

$$\forall f, g \in H,\ e(f, g) = \sum_{i,j=1}^{r} \int_O \xi_{i,j}(x)\partial_i f(x)\partial_j g(x)\psi(x)\, dx$$

is closable in $L^2(\nu)$. Its closure, $(\mathbf{d}, e)$, is a local Dirichlet form on $L^2(\nu)$ which admits a carré du champ γ.

$$\forall f \in \mathbf{d},\ \gamma(f)(x) = \sum_{i,j=1}^{r} \xi_{i,j}(x)\partial_i f(x)\partial_j f(x)\frac{\psi(x)}{k(x)}.$$

Moreover, it satisfies property (EID) *i.e. for any d and for any $\mathbb{R}^d$-valued function U whose components are in the domain of the form*

$$U_*[(\det\gamma[U,U^t])\cdot\nu] \ll \lambda^d$$

where det *denotes the determinant and λ^d the Lebesgue measure on* $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$.

Remark: In the case of a Lévy process, we shall apply this Lemma with ξ the identity application. We shall often consider an open domain of the form $O=\{x\in\mathbb{R}^d;\ |x|\leqslant\varepsilon\}$ which means that we "derivate" only w.r.t. small jumps and hypothesis 3. means that we do not need to assume regularity on k but only that k dominates a regular function.

2.2 The upper Dirichlet structure

We now introduce the lent particle method in this context.
One of the main point of our method, is the representation of the gradient on the upper structure $(\mathbb{D},\mathcal{E})$. To this end we consider:

- $(R,\mathcal{R},\rho)$: an auxiliary probability space s.t. $L^2(R,\mathcal{R},\rho)$ is infinite.

- D: a version of the gradient on $\mathbf{d}$ with values in the space

  $$L^2_0(R,\mathcal{R},\rho)=\{g\in L^2(R,\mathcal{R},\rho);\int_R g(r)\rho(dr)=0\},$$

  we denote it by $\flat$.

- $N\odot\rho$ the *extended marked Poisson measure*: it is a random Poisson measure on $[0,+\infty[\times X\times R$ with compensator $dt\times\nu\times\rho$ defined on the product probability space: $(\Omega_1,\mathcal{A}_1,\mathbb{P}_1)\times(R^{\mathbb{N}},\mathcal{R}^{\otimes\mathbb{N}},P^{\otimes\mathbb{N}})$.

Then, we introduce the creation operator (resp. annihilation operator) which consists in adding (resp. removing if necessary) a jump at time t with size u:

$$\begin{aligned}\varepsilon^+_{(t,u)}(w_1)&=w_1\mathbf{1}_{\{(t,u)\in supp\,w_1\}}+(w_1+\varepsilon_{(t,u)}\})\mathbf{1}_{\{(t,u)\notin supp\,w_1\}}\\ \varepsilon^-_{(t,u)}(w_1)&=w_1\mathbf{1}_{\{(t,u)\notin supp\,w_1\}}+(w_1-\varepsilon_{(t,u)}\})\mathbf{1}_{\{(t,u)\in supp\,w_1\}}.\end{aligned}$$

In a natural way, we extend these operators to the functionals by

$$\varepsilon^+H(w_1,t,u)=H(\varepsilon^+_{(t,u)}w_1,t,u)\quad\varepsilon^-H(w_1,t,u)=H(\varepsilon^-_{(t,u)}w_1,t,u).$$

we denote by $\mathbb{P}_N$ the measure $\mathbb{P}_N=\mathbb{P}_1(dw)N_w(dt,du)$.

Theorem 2. *$(\mathbb{D},\mathcal{E})$ is a local Dirichlet form which admits a carré du champ operator Γ.*
(i) The Dirichlet form $(\mathbb{D},\mathcal{E})$ admits a gradient operator that we denote by $\sharp$ and given by the following formula:

$$\forall F \in \mathbb{D}, \quad F^\sharp = \int_0^{+\infty}\int_{X\times R} \varepsilon^-((\varepsilon^+ F)^\flat)\, dN \odot \rho \in L^2(\mathbb{P}\times\hat{\mathbb{P}}). \tag{1}$$

Formula (1) *is justified by the following decomposition:*

$$F \in \mathbb{D} \quad \overset{\varepsilon^+ - I}{\longmapsto} \quad \varepsilon^+ F - F \in \underline{\mathbb{D}} \quad \overset{\varepsilon^-((.)^\flat)}{\longmapsto} \quad \varepsilon^-((\varepsilon^+ F)^\flat) \in L^2_0(\mathbb{P}_N \times \rho) \quad \overset{d(N\odot\rho)}{\longmapsto} \quad F^\sharp \in L^2(\mathbb{P}\times\hat{\mathbb{P}})$$

where each operator is continuous on the range of the preceding one and where $L^2_0(\mathbb{P}_N\times\rho)$ is the closed set of elements G in $L^2(\mathbb{P}_N\times\rho)$ such that $\int_R G d\rho = 0$ $\mathbb{P}_N$-a.e.
Moreover, we have for all $F\in\mathbb{D}$

$$\Gamma[F] = \hat{\mathbb{E}}(F^\sharp)^2 = \int_0^{+\infty}\int_X \varepsilon^-(\gamma[\varepsilon^+ F])\, dN, \tag{2}$$

where $\hat{\mathbb{E}}$ denotes the expectation with respect to probability $\hat{\mathbb{P}}$.
(ii) The upper Dirichlet structure $(\Omega_1,\mathcal{A}_1,\mathbb{P}_1,\mathbb{D},\Gamma)$ satisfies (EID).

Remarks
(i) The ideas and the scheme of the proofs are given in [5], in this volume.
(ii) This Theorem gives a method for obtaining $\Gamma[F]$ for $F\in\mathbb{D}$ or $F\in\mathbb{D}^n$, then with the hypotheses giving (EID) it suffices to prove $\det\Gamma[F] > 0$ $\mathbb{P}$-a.s. to assert that F has a density on $\mathbb{R}^n$.
(iii) We can interpret (1) in the following manner: to calculate the gradient of F, first add a particle at time t with size u then derivate w.r.t. the size (i.e. calculate the gradient w.r.t. the bottom structure); remove the particle and finally integrate w.r.t. to $dN\odot\rho$.
Let us mention some other interesting properties:

Proposition 3. *If $h\in L^2(\mathbb{R}^+,dt)\otimes\mathbf{d}$, then $\tilde{N}(h) = \int_0^{+\infty}\int_X h(t,u)\tilde{N}(ds,du)$ belongs to $\mathbb{D}$ and*

$$\Gamma[\tilde{N}(h)] = \int_0^{+\infty}\int_X \gamma[h(t,\cdot)](u) N(dt,du). \tag{3}$$

$$\left(\tilde{N}(h)\right)^\sharp = \int_0^{+\infty}\int_{X\times R} h^\flat(t,u,r) N\odot\rho(dt,du,dr). \tag{4}$$

3 Application to SDE's driven by a Poisson measure

3.1 The equation we study

We consider another probability space $(\Omega_2,\mathcal{A}_2,\mathbb{P}_2)$ on which an $\mathbb{R}^n$-valued semimartingale $Z=(Z^1,\cdots,Z^n)$ is defined, $n\in\mathbb{N}^*$. We adopt the following assumption on the bracket of Z and on the total variation of its finite variation part. It is satisfied if both are dominated by the Lebesgue measure uniformly:

Assumption on Z: There exists a positive constant C such that for any square integrable $\mathbb{R}^n$-valued predictable process h:

$$\forall t \geqslant 0,\ \mathbb{E}[(\int_0^t h_s dZ_s)^2] \leqslant C^2 \mathbb{E}[\int_0^t |h_s|^2 ds]. \tag{5}$$

We shall work on the product probability space: $(\Omega, \mathcal{A}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \times \mathbb{P}_2)$. For simplicity, we fix a finite terminal time $T > 0$.
Let $d \in \mathbb{N}^*$, we consider the following SDE:

$$X_t = x + \int_0^t \int_X c(s, X_{s^-}, u) \tilde{N}(ds, du) + \int_0^t \sigma(s, X_{s^-}) dZ_s \tag{6}$$

where $x \in \mathbb{R}^d$, $c : \mathbb{R}^+ \times \mathbb{R}^d \times X \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d \times n}$ satisfy the set of hypotheses below denoted (R).

Hypotheses (R):
1. There exists $\eta \in L^2(X, \nu)$ such that:
a) for all $t \in [0, T]$ and $u \in X$, $c(t, \cdot, u)$ is differentiable with continuous derivative and

$$\forall u \in X, \quad \sup_{t \in [0,T], x \in \mathbb{R}^d} |D_x c(t, x, u)| \leqslant \eta(u),$$

b) $\forall (t, u) \in [0, T] \times U,\ |c(t, 0, u)| \leqslant \eta(u)$,
c) for all $t \in [0, T]$ and $x \in \mathbb{R}^d$, $c(t, x, \cdot) \in \mathbf{d}$ and

$$\sup_{t \in [0,T], x \in \mathbb{R}^d} \gamma[c(t, x, \cdot)](u) \leqslant \eta(u),$$

d) for all $t \in [0, T]$, all $x \in \mathbb{R}^d$ and $u \in X$, the matrix $I + D_x c(t, x, u)$ is invertible and

$$\sup_{t \in [0,T], x \in \mathbb{R}^d} \left| (I + D_x c(t, x, u))^{-1} \right| \leqslant \eta(u).$$

2. For all $t \in [0, T]$, $\sigma(t, \cdot)$ is differentiable with continuous derivative and

$$\sup_{t \in [0,T], x \in \mathbb{R}^d} |D_x \sigma(t, x)| < +\infty.$$

3. As a consequence of hypotheses 1. and 2. above, it is well known that equation (6) admits a unique solution X such that $\mathbb{E}[\sup_{t \in [0,T]} |X_t|^2] < +\infty$. We suppose that for all $t \in [0, T]$, the matrix $(I + \sum_{j=1}^n D_x \sigma_{\cdot,j}(t, X_{t^-}) \Delta Z_t^j)$ is invertible and its inverse is bounded by a deterministic constant uniformly with respect to $t \in [0, T]$.

Remark: We have defined a Dirichlet structure $(\mathbb{D}, \mathcal{E})$ on $L^2(\Omega_1, \mathbb{P}_1)$. Now, we work on the product space, $\Omega_1 \times \Omega_2$. Using natural notations, we consider from now on that $(\mathbb{D}, \mathcal{E})$ is a Dirichlet structure on $L^2(\Omega, \mathbb{P})$. In fact, it is the product structure of $(\mathbb{D}, \mathcal{E})$ with the trivial one on $L^2(\Omega_2, \mathbb{P}_2)$ (see [2]). Of course, all the properties remain true. In other words, we only differentiate w.r.t. the Poisson noise and not w.r.t. to the one introduced by Z.

3.2 Spaces of processes and functional calculus

We denote by $\mathcal{P}$ the predictable sigma-field on $[0,T]\times\Omega$ and we define the following sets of processes:

- $\mathcal{H}$: the set of real valued processes $(X_t)_{t\in[0,T]}$, defined on $(\Omega,\mathcal{A},\mathbb{P})$, which belong to $L^2([0,T]\times\Omega)$.
- $\mathcal{H}_{\mathcal{P}}$: the set of predictable processes in $\mathcal{H}$.
- $\mathcal{H}_{\mathbb{D}}$: the set of real valued processes $(H_t)_{t\in[0,T]}$, which belong to $L^2([0,T];\mathbb{D})$ i.e. such that
$$\|H\|^2_{\mathcal{H}_{\mathbb{D}}}=\mathbb{E}[\int_0^T|H_t|^2dt]+\int_0^T\mathcal{E}(H_t)dt<+\infty.$$
- $\mathcal{H}_{\mathbb{D},\mathcal{P}}$: the subvector space of predictable processes in $\mathcal{H}_{\mathbb{D}}$.
- $\mathcal{H}_{\mathbb{D}\otimes\mathbf{d},\mathcal{P}}$: the set of real valued processes H defined on $[0,T]\times\Omega\times X$ which are predictable and belong to $L^2([0,T];\mathbb{D}\otimes\mathbf{d})$ i.e. such that
$$\|H\|^2_{\mathcal{H}_{\mathbb{D}\otimes\mathbf{d},\mathcal{P}}}=\mathbb{E}[\int_0^T\int_X|H_t|^2\nu(du)dt]+\int_0^T\int_X\mathcal{E}(H_t(\cdot,u))\nu(du)dt+\mathbb{E}[\int_0^T e(H_t)dt]<+\infty.$$

The main idea is to derivate equation (6), to do that we need some functional calculus. It is given by the next Proposition that we prove by approximation:

Proposition 4. *Let $H\in\mathcal{H}_{\mathbb{D}\otimes\mathbf{d},\mathcal{P}}$ and $G\in\mathcal{H}^n_{\mathbb{D},\mathcal{P}}$, then:*

1. *The process*
$$\forall t\in[0,T],\ X_t=\int_0^t\int_X H(s,w,u)\tilde{N}(ds,du)$$
is a square integrable martingale which belongs to $\mathcal{H}_{\mathbb{D}}$ and such that the process $X^-=(X_{t^-})_{t\in[0,T]}$ belongs to $\mathcal{H}_{\mathbb{D},\mathcal{P}}$. The gradient operator satisfies for all $t\in[0,T]$:
$$X_t^\sharp(w,\hat{w})=\int_0^t\int_X H^\sharp(s,w,u,\hat{w})d\tilde{N}(ds,du)+\int_0^t\int_{X\times R}H^\flat(s,w,u,r)N\odot\rho(ds,du,dr). \tag{7}$$

2. *The process*
$$\forall t\in[0,T],\ Y_t=\int_0^t G(s,w)dZ_s$$
is a square integrable semimartingale which belongs to $\mathcal{H}_{\mathbb{D}}$, $Y^-=(Y_{t^-})_{t\in[0,T]}$ belongs to $\mathcal{H}_{\mathbb{D},\mathcal{P}}$ and
$$\forall t\in[0,T],\ Y_t^\sharp(w,\hat{w})=\int_0^t G^\sharp(s,w,\hat{w})dZ_s. \tag{8}$$

3.3 Computation of the Carré du champ matrix of the solution

Applying the standard functional calculus related to Dirichlet forms, the previous Proposition and a Picard iteration argument, we obtain:

Proposition 5. *The equation* (6) *admits a unique solution X in $\mathcal{H}^d_{\mathbb{D}}$. Moreover, the gradient of X satisfies:*

$$\begin{aligned} X^\sharp_t &= \int_0^t \int_U D_x c(s, X_{s-}, u) \cdot X^\sharp_{s-} \tilde{N}(ds, du) \\ &\quad + \int_0^t \int_{X\times R} c^\flat(s, X_{s-}, u, r) N \odot \rho(ds, du, dr) \\ &\quad + \int_0^t D_x\sigma(s, X_{s-}) \cdot X^\sharp_{s-} dZ_s \end{aligned}$$

Let us define the $\mathbb{R}^{d\times d}$-valued processes U by

$$dU_s = \sum_{j=1}^n D_x \sigma_{.,j}(s, X_{s-}) dZ^j_s,$$

and the derivative of the flow generated by X:

$$K_t = I + \int_0^t \int_X D_x c(s, X_{s-}, u) K_{s-} \tilde{N}(ds, du) + \int_0^t dU_s K_{s-}$$

Proposition 6. *Under our hypotheses, for all $t \geqslant 0$, the matrix K_t is invertible and it inverse $\bar{K}_t = (K_t)^{-1}$ satisfies:*

$$\begin{aligned} \bar{K}_t &= I - \int_0^t \int_X \bar{K}_{s-}(I + D_x c(s, X_{s-}, u))^{-1} D_x c(s, X_{s-}, u) \tilde{N}(ds, du) \\ &\quad - \int_0^t \bar{K}_{s-} dU_s + \sum_{s \leqslant t} \bar{K}_{s-} (\Delta U_s)^2 (I + \Delta U_s)^{-1} \\ &\quad + \int_0^t \bar{K}_s d < U^c, U^c >_s . \end{aligned}$$

We are now able to calculate the carré du champ matrix. This this the aim of the next Theorem, to show how simple is the *lent particle method* we give a sketch of the proof.

Theorem 7. *For all $t \in [0, T]$,*

$$\Gamma[X_t] = K_t \int_0^t \int_X \bar{K}_s \gamma[c(s, X_{s-}, \cdot)] \bar{K}^*_s N(ds, du) K^*_t.$$

Proof. Let $(\alpha, u) \in [0, T] \times X$. We put $X^{(\alpha,u)}_t = \varepsilon^+_{(\alpha,u)} X_t$.

$$\begin{aligned} X^{(\alpha,u)}_t &= x + \int_0^\alpha \int_X c(s, X^{(\alpha,u)}_{s^-}, u') \tilde{N}(ds, du') \\ &\quad + \int_0^\alpha \sigma(s, X^{(\alpha,u)}_{s^-}) dZ_s + c(\alpha, X^{(\alpha,u)}_{\alpha^-}, u) \\ &\quad + \int_{]\alpha,t]} \int_X c(s, X^{(\alpha,u)}_{s^-}, u') \tilde{N}(ds, du') + \int_{]\alpha,t]} \sigma(s, X^{(\alpha,u)}_{s^-}) dZ_s. \end{aligned}$$

Let us remark that $X_t^{(\alpha,u)} = X_t$ if $t < \alpha$ so that, taking the gradient with respect to the variable u, we obtain:

$$\begin{aligned}(X_t^{(\alpha,u)})^\flat &= (c(\alpha, X_{\alpha^-}^{(\alpha,u)}, u))^\flat \\ &+ \int_{]\alpha,t]}\int_X D_x c(s, X_{s^-}^{(\alpha,u)}, u')\cdot (X_{s^-}^{(\alpha,u)})^\flat \tilde{N}(ds,du') \\ &+ \int_{]\alpha,t]} D_x\sigma(s, X_{s^-}^{(\alpha,u)})\cdot (X_{s^-}^{(\alpha,u)})^\flat dZ_s.\end{aligned}$$

Let us now introduce the process $K_t^{(\alpha,u)} = \varepsilon^+_{(\alpha,u)}(K_t)$ which satisfies the following SDE:

$$K_t^{(\alpha,u)} = I + \int_0^t\int_X D_x c(s, X_{s^-}^{(\alpha,u)}, u') K_{s-}^{(\alpha,u)} \tilde{N}(ds,du') + \int_0^t dU_s^{(\alpha,u)} K_{s-}^{(\alpha,u)}$$

and its inverse $\bar{K}_t^{(\alpha,u)} = (K_t^{(\alpha,u)})^{-1}$. Then, using the flow property, we have:

$$\forall t \geqslant 0,\ (X_t^{(\alpha,u)})^\flat = K_t^{(\alpha,u)} \bar{K}_\alpha^{(\alpha,u)} (c(\alpha, X_{\alpha^-}, u))^\flat.$$

Now, we calculate the carré du champ and then we take back the particle:

$$\forall t \geqslant 0,\ \varepsilon^-_{(\alpha,u)}\gamma[(X_t^{(\alpha,u)})] = K_t\bar{K}_\alpha \gamma[c(\alpha, X_{\alpha^-}, \cdot)]\bar{K}_\alpha^* K_t^*$$

Finally integrating with respect to N we get

$$\forall t \geqslant 0,\ \Gamma[X_t] \;=\; K_t \int_0^t\int_X \bar{K}_s\gamma[c(s, X_{s^-}, \cdot)](u)\bar{K}_s^* N(ds,du) K_t^*.$$

□

3.4 First application: the regular case

An immediate consequence of the previous Theorem is:

Proposition 8. *Assume that X is a topological space, that the intensity measure $ds \times \nu$ of N is such that ν has an infinite mass near some point u_0 in X. If the matrix $(s,y,u) \to \gamma[c(s,y,\cdot)](u)$ is continuous on a neighborhood of $(0,x,u_0)$ and invertible at $(0,x,u_0)$, then the solution X_t of (6) has a density for all $t \in]0,T]$.*

3.5 Application to SDE's driven by a Lévy process

Let Y be a Lévy process with values in $\mathbb{R}^d$, independent of another variable X_0.
We consider the following equation

$$X_t = X_0 + \int_0^t a(X_{s-}, s)\, dY_s, \quad t \geqslant 0$$

where $a : \mathbb{R}^k \times \mathbb{R}^+ \to \mathbb{R}^{k\times d}$ is a given map.

Proposition 9. *We assume that:*

1. *The Lévy measure, ν, of Y satisfies hypotheses of the example given in Section 2.1 with $\nu(O) = +\infty$ and $\xi_{i,j}(x) = x_i\delta_{i,j}$. Then we may choose the operator γ to be*

$$\gamma[f] = \frac{\psi(x)}{k(x)} \sum_{i=1}^{d} x_i^2 \sum_{i=1}^{d} (\partial_i f)^2 \quad \text{for } f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$$

2. *a is $\mathcal{C}^1 \cap Lip$ with respect to the first variable uniformly in s and*

$$\sup_{t,x} |(I + D_x a \cdot u)^{-1}(x,t)| \leqslant \eta(u),$$

where $\eta \in L^2(\nu)$.

3. *a is continuous with respect to the second variable at 0, and such that the matrix $aa^*(X_0, 0)$ is invertible;*

then for all $t > 0$ the law of X_t is absolutely continuous w.r.t. the Lebesgue measure.

Proof. We just give an idea of the proof in the case $d = 1$:
Let us recall that $\gamma[f] = \frac{\psi(x)}{k(x)} x^2 f'^2(x)$.
We have the representation: $Y_t = \int_0^t \int_{\mathbb{R}} u\tilde{N}(ds,du)$, so that

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} a(s, X_{s-})u\, \tilde{N}(ds,du).$$

The lent particle method yields:

$$\Gamma[X_t] = K_t^2 \int_0^t \int_X \bar{K}_s^2 a^2(s, X_{s-}) \gamma[j](u) N(ds,du)$$

where j is the identity application: $\gamma[j](u) = \frac{\psi(u)}{k(u)} u^2$.
So

$$\begin{aligned} \Gamma[X_t] &= K_t^2 \int_0^t \int_X \bar{K}_s^2 a^2(s, X_{s-}) \frac{\psi(u)}{k(u)} u^2 N(ds,du) \\ &= K_t^2 \sum_{\alpha < t} \bar{K}_s^2 a^2(s, X_{s-}) \frac{\psi(\Delta Y_s)}{k(\Delta Y_s)} \Delta Y_s^2, \end{aligned}$$

and it is easy to conclude. □

Remarks:
(i) We refer to [4] for other examples and applications.
(ii) Let us finally remark that as easily seen, one can iterate the gradient and so obtain criteria of regularity for the density of Poisson functionals such as solutions of SDE's, this is the object of a forthcoming paper.

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