Robustness on large deviation estimates for controlled semi-martingale

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1. Introduction

"Market model"

Riskless asset:

(1.1) $$dS^0(t) = r(X_t)S^0(t)dt, \quad S^0(0) = s^0.$$

Risky assets:

(1.2) $$\begin{cases} dS^i(t) = S^i(t)\{\alpha^i(X_t)dt + \sum_{k=1}^{n+m} \sigma^i_k(X_t)dW^k_t\}, \\ S^i(0) = s^i, \ \ i = 1, ..., m \end{cases}$$

Factors:

(1.3) $$\begin{cases} dX_t = \beta(X_t)dt + \lambda(X_t)dW_t, \\ X(0) = x \in R^n, \end{cases}$$

Total wealth:

$$V_t = \sum_{i=0}^{m} N_t^i S_t^i$$

N_t^i : Number of the shares

$h_t^i = \frac{N_t^i S_t^i}{V_t}$: Portfolio proportion $\quad i = 0, 1, 2, \ldots, m.$

$h_t = (h_t^1, \ldots, h_t^m)$

$$\frac{dV_t}{V_t} = r(X_t)dt + h(t)^*(\alpha(X_t) - r(X_t)\mathbf{1})dt + h(t)^*\sigma(X_t)dW_t,$$

$$\log V_T = \log V_0$$

$$+\int_0^T \{-\frac{1}{2}h_s^*\sigma\sigma^*(X_s)h_s + h_s^*\widehat{\alpha}(X_s) + r(X_s)\}dt + \int_0^T h_s^*\sigma(X_s)dW_s,$$

$$\widehat{\alpha}(x) = \alpha(x) - r(x)\mathbf{1}.$$

Problem at the level of the law of large number

$$\frac{1}{T}\log V_T(h) = \frac{1}{T}\log V_T(h)$$

$$= -\frac{1}{2T}\int_0^T \{h_t - (\sigma\sigma^*)^{-1}\widehat{\alpha}(X_t)\}^*\sigma\sigma^*\{h_t - (\sigma\sigma^*)^{-1}\widehat{\alpha}(X_t)\}dt$$

$$+\frac{1}{2T}\int_0^T \{\widehat{\alpha}(X_t)^*(\sigma\sigma^*)^{-1}\widehat{\alpha}(X_t) + r(X_t)\}dt + \frac{1}{T}\int_0^T h_t^*\sigma(X_t)dW_t$$

$h_t^K := (\sigma\sigma^*)^{-1}\widehat{\alpha}(X_t)$ maximizes pathewise the growth rate of $V_T(h)$ on a long run and it is called "Kelly portfolio" (log utility portfolio) or "numéraire portfolio". If X_t is ergodic, then

$$\lim_{T\to\infty}\frac{1}{T}\log V_T(h^K) = \frac{1}{2}\int\{\widehat{\alpha}(x)^*(\sigma\sigma^*)^{-1}\widehat{\alpha}(x) + r(x)\}m(dx)$$

While, we are interested in **the large deviation estimate** related to downside risk minimization

$$\inf_{h\in\mathcal{H}_{\mathcal{F}}(T)} P(\frac{1}{T}\log V_T(h) \le \kappa) \sim e^{-TI(\kappa)}, \quad T \to \infty$$

κ : a given target growth rate

Problems

- find the rate function $I(\kappa)$
- asymptotically optimal strategy ?

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cf. upside chance maximization

$$\sup_{h\in\mathcal{H}_{\mathcal{F}}}\underline{\lim}_{T\to\infty}\frac{1}{T}\log P(\frac{1}{T}\log V_T(h)\ge\kappa)=-\inf_{[\kappa,\infty)}\sup_{\theta\in[0,\theta^*)}\{\theta k-\chi_+(\theta)\}$$

Pham '03; Stettner '04; Hata-Sekine '05; Hata-Iida '06; Sekine '06; Knispel '12; Sekine '12 , etc,..

Results on downside risk minimization $\chi_0'(-\infty) < \kappa < \chi_0'(0-)$

$$(1.4)\quad \begin{aligned} J_0(\kappa) &:= \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{H}(T)} \log P(\frac{1}{T}\log V_T(h) \le \kappa) \\ &= \lim_{T\to\infty} \frac{1}{T} \log P(\frac{1}{T}\log V_T(h^{(\theta(\kappa),T)}) \le \kappa) \end{aligned}$$

$$\begin{aligned} J_0(\kappa) = -I(\kappa) &\equiv -\inf_{k\in(\chi_0'(-\infty),\kappa]} \sup_{\theta<0}\{\theta k - \chi_0(\theta)\} \\ &= -\{\theta(\kappa)\kappa - \chi_0(\theta(\kappa))\}, \end{aligned}$$

$$(1.5)\qquad \chi_0(\theta) := \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h.} \log E[e^{\theta \log V_T(h)}], \quad \theta < 0.$$

cf. Hata - N. - Sheu '10, AAP; N. '11 QF ; Hata '11 APFM; N. '12 AAP ; Watanabe '13 SPA; Hata-Sekine '10 AMO

Complete market case:
Assume that the solution of the SDE

$$dX_t^i = \{\alpha^i(X_t) - \frac{1}{2}(\sigma\sigma^*(X_t))^{ii}\} + \sum_{j=1}^m \sigma_j^i(X_t)dW_t^j, \quad X_0^i = 0$$

is given, and set $X_t^i = \log S_t^i$, $i = 1, 2, \ldots, m$.
The solution to this SDE is regarded as "factors" and $S_t^i = s^i e^{X_t^i}$ satisfies the equation (1.2) of the dynamics of the security prices.
The "factors" are governed by

$$dX_t = \beta(X_t)dt + \lambda(X_t)dW_t, \quad X_0 = x \in R^m,$$

with $$\beta(x)^i = \alpha(x)^i - \frac{1}{2}(\sigma\sigma^*)^{ii}(x), \quad \lambda(x) = \sigma(x).$$

and security prices: $S^0(t) = s^0 e^{\int_0^t r(X_s)ds}$,

(1.2) $$\begin{cases} dS^i(t) = S^i(t)\{\alpha^i(X_t)dt + \sum_{k=1}^m \sigma_j^i(X_t)dW_t^j\}, \\ S^i(0) = s^i, \ i = 1, ..., m \end{cases}$$

2. Large deviation estimates for controlled semi-martingales

(2.1) $$dX_t = \beta(X_t)dt + \lambda(X_t)dW_t, \quad X_0 = x \in R^N,$$

W_t : M- dim. $\mathcal{F}_t$ B.M., $\lambda(x)$: $R^N \mapsto N \otimes M$, $\beta(x)$: $R^N \mapsto R^N$

(2.2) $$J(\kappa) := \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h.} \log P\left(\frac{1}{T} F_T(X_., h_.) \leq \kappa\right).$$

$$F_T(X_., h_.) = \int_0^T f(X_s, h_s)ds + \int_0^T \varphi(X_s, h_s)^* dW_s$$

h_s: $\mathcal{F}_t$ - prog. m'ble, R^m-valued, $m, N \leq M$

$$f(x,h) := -\frac{1}{2}h^* S(x)h + h^* g(x) + U(x), \quad \varphi(x,h) = \delta(x)h,$$

$$S(x) : R^N \mapsto R^m \otimes R^m, \ \ g(x) : R^N \mapsto R^m, \ \ \delta(x) : R^N \mapsto R^M \otimes R^m,$$

Risk-sensitive control and its H-J-B equation

Consider averaging limit of the portfolio optimization

$$\hat{\chi}(\theta) := \lim_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{A}(T)} J(x;h;T), \quad \theta < 0, \tag{2.3}$$

where

$$J(x;h;T) := \log E[e^{\theta\{\int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s\}}], \tag{2.4}$$

and h ranges over the set $\mathcal{A}(T)$ of all admissible investment strategies defined by

$$\begin{aligned}\mathcal{A}(T) \ = \ \{h; h : [0,T] \times R^N \mapsto R^m;\ \text{Borel},\ |h(t,x)| \le C(1+|x|), \\ h(t,X_t)\ \text{is progressively m'ble}\}\end{aligned}$$

Then, we shall see that (2.3) could be considered the dual problem to our current problem (2.2).

Assumptions

(2.3) $\lambda,\ \beta,\ S,\ g,\ \delta$ are smooth and globally Lipschitz, U is smooth and bounded below

$$|U(x)|,\ |DU| \le M_1|x|^2 + M_2$$

(2.4) $$c_0\delta^*\delta(x) \le S(x) \le c_1\delta^*\delta(x), \quad x \in R^N,\ c_0,\ c_1 > 0$$

(2.5) $$c_\delta I \le \delta^*\delta(x) \le c'_\delta I, \quad c_\delta,\ c'_\delta > 0$$

(2.6) $$c_2|\xi|^2 \le \xi^*\lambda\lambda^*(x)\xi \le c_3|\xi|^2,\ c_2,\ c_3 > 0,\ \xi \in R^n,$$

$$F_T(X_\cdot, h_\cdot) = \int_0^T f(X_s, h_s)ds + \int_0^T \varphi(X_s, h_s)^* dW_s$$

$$f(x,h) := -\frac{1}{2}h^*S(x)h + h^*g(x) + U(x), \quad \varphi(x,h) = \delta(x)h,$$

Note that, when setting

$$Q_\theta := S(x) - \theta\delta^*\delta(x), \qquad \theta < 0,$$

Q_θ satisfies

$$(c_0 - \theta)\delta^*\delta(x) \le Q_\theta(x) \le (c_1 - \theta)\delta^*\delta(x) \tag{2.7}$$

and

$$\theta Q_\theta^{-1}(x) \le \frac{\theta}{c_1 - \theta}(\delta^*\delta(x))^{-1}, \qquad \frac{\theta}{c_0 - \theta}(\delta^*\delta(x))^{-1} \le \theta Q_\theta^{-1}(x) \tag{2.8}$$

Moreover, we have

$$\frac{c_0}{c_0 - \theta} I \le I + \theta\delta Q_\theta^{-1}\delta^* \le I \tag{2.9}$$

$\Big($• In the case of the above market model $Q_\theta^{-1} = \frac{1}{1-\theta}(\sigma\sigma^*)^{-1}\Big)$

Transformation to **Risk-sensitive control**

(2.10) $v_*(0,x;T) = \inf_{h_\cdot \in \mathcal{A}(T)} \log E[e^{\theta\{\int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s\}}].$

Introduce a probability measue

$$P^h(A) = E[e^{\theta \int_0^T h_s^* \delta^*(X_s) dW_s - \frac{\theta^2}{2} \int_0^T h_s^* \delta^* \delta(X_s) h_s ds} : A]$$

Then, under the measure X_t satisfies

$$dX_t = \{\beta(X_t) + \theta\lambda\delta(X_t)h_t\}dt + \lambda(X_t)dW_t^h, \quad X_0 = x$$

with B. M. W_t^h defined by

$$W_t^h := W_t - \theta \int_0^t \delta(X_s) h_s ds$$

and the value $v_*(0,x;T)$ is described as

(2.11) $v_*(0,x;T) = \inf_{h_\cdot \in \mathcal{A}(T)} \log E^h[e^{\theta \int_0^T \{f(X_s,h_s) + \frac{\theta}{2} h_s^* \delta^* \delta(X_s) h_s\} ds}]$

The H-J-B equation :

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \frac{1}{2}(Dv)^*\lambda\lambda^* Dv \\ \qquad + \inf_h\{[\beta + \theta\lambda\delta h]^* Dv + \theta f(x,h) + \frac{\theta^2}{2}h^*\delta^*\delta(x)h\} = 0, \\ v(T,x) = 0, \end{cases}$$

which is written as

$$(2.12) \qquad \begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \beta_\theta^* Dv + \frac{1}{2}(Dv)^*\lambda N_\theta \lambda^* Dv \\ \qquad + \frac{\theta}{2}g^* Q_\theta^{-1} g + \theta U = 0, \\ v(T,x) = 0, \end{cases}$$

where

$$\beta_\theta = \beta + \theta\lambda\delta Q_\theta^{-1} g, \quad N_\theta = I + \theta\delta Q_\theta^{-1}\delta^*, \quad Q_\theta = S - \theta\delta\delta^*.$$

Note that

$$(2.7) \qquad (c_0-\theta)\delta^*\delta(x) \le Q_\theta(x) \le (c_1-\theta)\delta^*\delta(x)$$

and that

$$(2.9) \qquad \frac{c_0}{c_0-\theta} I \le N_\theta = I + \theta\delta Q_\theta^{-1}\delta^* \le I$$

• Under our assumptions we can see that H-J-B equation (2.12) has a sufficiently smooth solution $v(t,x)$ satisfying the nice gradient estimates.

cf. Bensoussan-Frehse-N '98 AMO, N. '96, '03 SICON, Then, we have the following verification theorem.

Proposition 1 *Assume assumptions (2.3) - (2.6) and let* $v(t,x;T)$ *be a solution to (2.12). Then, setting*

$$\hat{h}(t,x) := Q_\theta^{-1}(\delta^*\lambda^* Dv(t,x) + g(x)),$$

$\hat{h}_t^{(T)} \equiv \hat{h}_t^{(\theta,T)} := \hat{h}(t,X_t)$ *is an optimal strategy:*

$$\begin{aligned} v(0,x;T) &= \log E[e^{\theta\{\int_0^T f(X_s,\hat{h}_s^{(T)})ds + \int_0^T \varphi(X_s,\hat{h}_s^{(T)})^* dW_s\}}] \\ &= \inf_{h_\cdot \in \mathcal{A}(T)} \log E[e^{\theta\{\int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s}] \end{aligned}$$

H-J-B equation of ergodic type

Now let us consider the infinite horizon counterpart of (2.12), called H-J-B equation of ergodic type:

$$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta_\theta^* Dw + \frac{1}{2}(Dw)^*\lambda N_\theta \lambda^* Dw$$

(2.13)

$$+\frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U.$$

Owing to Bensoussan-Frehse '92, Reine Angew. Math. and Proposition 3.2 in N. '12 AAP we have the following proposition concerning (2.13).

Proposition 2 *i) Assume that*

$$\lim_{r\to\infty} \inf_{|x|\geq r} \{g^*(\delta^*\delta)^{-1}g(x) + U(x)\} = \infty \tag{2.14}$$

besides assumptions (2.3) - (2.6). Then, we have a solution $(\chi(\theta), w)$ *of (2.13) such that* $w(x)$ *is bounded above. Moreover, such a solution* (χ, w) *is unique up to additive constants with respect to* w *and satisfies the following estimate*

$$|\nabla \bar{w}(x)|^2 \leq C_w |x|^2 + C'_w \tag{2.15}$$

Furthermore, if we assume stronger assumption

$$c_4|x|^2 - c_5 \leq \frac{1}{c_1 - \theta} g^*(\delta^*\delta)^{-1}g(x) + U(x) \tag{2.16}$$

than (2.14), then we have

$$-c_w|x|^2 + c'_w \geq w(x), \quad c_w,\ c'_w > 0. \tag{2.17}$$

ii) Assume that

(2.18) $$\beta(x)^* x \leq -c_\beta |x|^2 + c'_\beta, \quad c_\beta > 0\ , c'_\beta > 0$$

besides assumptions (2.3) - (2.6). Then, there exists a positive constant $b_* > 0$ *such that* $\psi_{b_*}(x) := b_*|x|^2$ *satisfies*

$$F(\psi_{b_*})(x) \to -\infty, \text{ as } |x| \to \infty,$$

where

$$F(\psi) = \frac{1}{2} tr[\lambda\lambda^* D^2\psi] + \beta_\theta^* D\psi + \frac{1}{2}(D\psi)^* \lambda N_\theta \lambda^* D\psi + \frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U$$

and we have a solution $(\chi(\theta), w)$ *to (2.13) such that* $w - \psi_b(x)$ *with* $0 < b \leq b_*$ *is bounded above. Moreover, such solution is unique up to additive constants.*

ii) can be reduced to i) when considering $w - \psi_b$ in place of w

$$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2(w - \psi_b)] + (\beta_\theta + \lambda N_\theta \lambda^* D\psi_b)^* D(w - \psi_b)$$
$$+\frac{1}{2}(Dw - \psi_b)^* \lambda N_\theta \lambda^* D(w - \psi_b) + F(\psi_b),$$

cf. also Ichihara '11, SICON

• In what follows we shall proceed assuming the assumptions of Proposition 2 i) with (2.16). We can develop parallel arguments as well in the case of ii) of the proposition.

Large time asymptotics of the solution

Theorem 1 *Under the assumptions of Proposition 2 i) with (2.16), as* $T \to \infty$, $v(0, x; T) - \{w(x) + \chi(\theta)T\}$ *converges to a constant* $c_\infty \in R$ *uniformly on each compact set.*

Corollary 1 *Under the assumptions of Theorem 1 we have*

$$\lim_{T \to \infty} \frac{v(0, x; T)}{T} = \chi(\theta),$$

where $(\chi(\theta), w(x))$ *is the solution to H-J-B equation of ergodic type:*

$$\chi(\theta) = \frac{1}{2} tr[\lambda\lambda^* D^2 w] + \beta_\theta^* Dw + \frac{1}{2}(Dw)^* \lambda N_\theta \lambda^* Dw + \frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U, \tag{2.13}$$

• As for the proof of Theorem 1, cf. Ichihara and Sheu, '13 SIMA
, and also N. '12 preprint.

• A direct proof of Cor. 1 is seen in N. ' 12 AAP.

Convexity

• The solution $v(0, x; T)$ to H-J-B equation (2.12) of parabolic type chatacterize the value of another stochstic control problem, which can be seen to be convex with respect to θ

• Owing to Corollary 1, we have also convexity of $\chi(\theta)$.

Ergodicity and exponential integrability

$$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta_\theta^* Dw + \frac{1}{2}(Dw)^*\lambda N_\theta \lambda^* Dw \tag{2.13}$$
$$+\frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U,$$

We can see that the diffusion process governed by

$$d\bar{X}_t = \lambda(\bar{X}_t)dW_t + \{\beta_\theta + \lambda N_\theta \lambda^* Dw\}(\bar{X}_t)dt$$

with the generator

$$L^w\psi := \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\psi] + \beta_\theta^* D\psi + (Dw)^*\lambda N_\theta \lambda^* D\psi$$

turns out to be ergodic. Further, for each $\theta_1 \leq \theta \leq \theta_0$ there exist positive constants $k > 0$ and $C > 0$ independent of T and $\theta \in [\theta_1, \theta_0]$ such that

$$E[e^{k|\bar{X}_T|^2}] \leq C \tag{2.19}$$

Differentiability of H-J-B equation with respect to θ
We obtain
$(EE)'$
$$\chi'(\theta) = L^w w' + U + \frac{1}{2}(\lambda^* Dw + \delta(\delta^*\delta)^{-1}g)^* \frac{\partial N_\theta}{\partial \theta}(\lambda^* Dw + \delta(\delta^*\delta)^{-1}g),$$

where $w' = \frac{\partial w}{\partial \theta}$ (cf. Lemma 6.4 in N. '12, AAP), after seeing that Poisson equation :

(2.20) $$\gamma(\theta) = L^w u(x) + f(x)$$

has a unique solution $(u, \gamma(\theta))$ for $f \in F_K$ defined by

$$F_K = \{f \in L^\infty_{loc};\ \mathrm{esssup}_{x \in B^c_{R_0}} \frac{|f(x)|}{K(x;\bar{w})} < \infty\},\ \ \bar{w} = -w$$

$$K(x;\bar{w}) := \frac{1}{2}(D\bar{w})^* \lambda N_\theta \lambda^* D\bar{w} - \frac{\theta}{2} g^* Q_\theta^{-1} g - \theta U$$

and $u \in F_{\bar{w}}$:

$$F_{\bar{w}} = \{u \in W^{2,p}_{loc};\ \text{esssup}_{x \in B^c_{R_0}} \frac{|u(x)|}{\bar{w}(x)} < \infty\}.$$

When setting

$$f(x) = U + \frac{1}{2}(\lambda^* Dw + \delta(\delta^*\delta)^{-1}g)^* \frac{\partial N_\theta}{\partial \theta}(\lambda^* Dw + \delta(\delta^*\delta)^{-1}g)$$

we have (EE)'.

Duality theorem

Theorem 2 *Under the assumptions of Theorem 1, we have for* $\kappa \in (\chi'(-\infty), \chi'(0-))$

$$\underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{A}(T)} \log P\left(\frac{1}{T}F_T(X_\cdot, h_\cdot) \le \kappa\right) = -\inf_{k\in(\chi'(-\infty),\kappa]} I(k) = -I(\kappa)$$

$$\textit{rate function}: \qquad I(k) := \sup_{\theta<0}\{\theta k - \chi(\theta)\}$$

Moreover, for $\theta(\kappa)$ *such that* $\chi'(\theta(\kappa)) = \kappa \in (\chi'(-\infty), \chi'(0-))$ *take a strategy* $\hat{h}_t^{(\theta(\kappa),T)}$. *Then,*

$$\lim_{T\to\infty} \frac{1}{T} \log P\left(\frac{1}{T}F_T(X_\cdot, h^{(\theta(\kappa),T)}) \le \kappa\right) = -\inf_{k\in(\chi'(-\infty),\kappa]} I(k) = -I(\kappa)$$

Linear Gaussian case (can be independently discussed)

$\lambda(x) \equiv \lambda$, $\beta(x) = Bx + b$;

$S(x) \equiv S$, $g(x) = Ax + a$, $U(x) = \frac{1}{2}x^* V x + m$, $\delta(x) \equiv \delta$

λ, B, S, A, V, δ are constant matrices

b, a are constant vectors and m is a constant

$\delta^*\delta > 0$, $S > 0$, $V \geq 0$

In this case, the solution $v(t,x)$ to H-J-B equation (2.12) has an explicit representation such that

$$v(t,x) = \frac{1}{2}x^* P(t)x + q(t)^* x + l(t),$$

where

$$\dot{P}(t) + K_1^* P(t) + P(t) K_1 + P(t) \lambda N_\theta \lambda^* P(t) + \theta A^* Q_\theta^{-1} A + \theta V = 0$$

$$K_1 = B + \theta \lambda \delta Q_\theta^{-1} A$$

$$\dot{q}(t) + (K_1 + \lambda N_\theta \lambda^* P(t))^* q(t) + P(t) k_1 + \theta A^* Q_\theta^{-1} a = 0$$

$$k_1 = b + \theta \lambda \delta Q_\theta^{-1} a$$

$$\dot{l}(t) + \frac{1}{2} \mathrm{tr}[\lambda \lambda^* P(t)] + k_1^* q(t) + \frac{1}{2} q(t)^* \lambda N_\theta \lambda^* q(t) + \frac{\theta}{2} a^* Q_\theta^{-1} a + \theta m = 0$$

with the terminal conditions $P(T) = 0,\ q(T) = 0,\ l(T) = 0.$

If we assume that

(I) $\lambda\lambda^* > 0, \;\; A^*A > 0,$

or

(II) B is stable,

then we can see that as $T \to \infty$,

$$P(t;T) \to \overline{P},\; q(t;T) \to \overline{q},\; \frac{l(t;T)}{T} \to \chi(\theta),$$

where $\overline{P}$ is the unique non-positive definite solution to the stationary Riccati equation:

$$K_1^*\overline{P} + \overline{P}K_1 + \overline{P}\lambda N_\theta \lambda^* \overline{P} + \theta A^* Q_\theta^{-1} A + \theta V = 0$$

such that $K_1 + \lambda N_\theta \lambda^* \overline{P}$ is stable, $\overline{q}$ is the one of the algebraic equation:

$$(k_1 + \lambda N_\theta \lambda^* \overline{P})^* \overline{q} + \overline{P}K_1 + \theta A^* Q_\theta^{-1} a = 0,$$

and $\chi(\theta)$ is the constant given by

$$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^*\overline{P}] + k_1^*\overline{q} + \frac{1}{2}\overline{q}^*\lambda N_\theta \lambda^*\overline{q} + \frac{\theta}{2}a^* Q_\theta^{-1} a + \theta m.$$

Further, the pair of a function $w(x) = \frac{1}{2}x^*\overline{P}x + \overline{q}^* x$ and a constant $\chi(\theta)$ turns out to be a solution to the H-J-B equation of ergodic type. Differentiability of the solution to the H-J-B equation with respect to θ is also seen through independent arguments of the above general case and thus, we can deduce the same statement as Theorem 2 under assumption (I) or (II)

3. Robust estimates of the large deviation probability under drift uncertainty

$$\left.\frac{dP^\zeta}{dP}\right|_{\mathcal{F}_T} := e^{\int_0^T \zeta_s^* dW_s - \frac{1}{2}\int_0^T |\zeta_s|^2 ds}$$

$$W_t^\zeta := W_t - \int_0^t \zeta_s ds \ : \ \text{B.M. under } P^\zeta$$

(3.1) $$dX_t = \{\beta(X_t) + \lambda(X_t)\zeta_t\}dt + \lambda(X_t)dW_t^\zeta.$$

(3.2)
$$J_1(\kappa) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h.} \sup_{\zeta.} \log P^\zeta\left(\frac{1}{T}\{F_T(X_., h_.) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\} \le \kappa\right).$$

ζ_t: an uncertainty parameter process, $\mu > 0$: certainty level of β

We are going to see the following duality relationship

$$J_1(\kappa) \;:=\; \underline{\lim}_{T\to\infty} \tfrac{1}{T} \inf_{h.} \sup_{\zeta.} \log P^{\zeta}\left(\tfrac{1}{T}\{F_T(X.,h.) + \tfrac{\mu}{2}\int_0^T |\zeta_s|^2 ds\} \le \kappa\right)$$

$$= \; -\sup_{\theta<0}\{\theta\kappa - \widehat{\chi}_1(\theta)\}.$$

$$\widehat{\chi}_1(\theta) := \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h.} \sup_{\zeta.} \log E^{\zeta}[e^{\theta\{F_T(X.,h.)+\frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\}}].$$

(3.3) $$F_T(X.,h.) := \int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s$$

$$f(x,h) := -\frac{1}{2}h^* S(x) h + h^* g(x) + U(x), \quad \varphi(x,h) = \delta(x)h,$$

Formulation of the game

Lower value function:

$$u_*(0,x;T) := \inf_{h\in\Delta_{\mathcal{H}}} \sup_{\zeta\in\mathcal{Z}} \log E^{\zeta}[e^{\theta\{F_T(X_\cdot,h_\cdot)+\frac{\mu}{2}\int_0^T|\zeta_s|^2ds\}}]$$

$\mathcal{Z} = \{\zeta_t; \zeta_t = \zeta(t, X_t)$ is prog. m'ble, $|\zeta(t,x)| \le C(1+|x|)\ \}$

ζ_t: an uncertainty parameter process

$\Delta_{\mathcal{H}} = \{h_t; h_t = h(t, X_t, \zeta_t)$ is prog. m'ble, $h(t,x,\zeta) \in \mathbf{H}\}$,

$\mathbf{H}$: the totality of Borel functions $h(t,x,\zeta) : [0,T] \times R^N \times R^M \mapsto R^m$

such that $|h(t,x,\zeta)| \le C(1+|x|+|\zeta|)$, $\exists C > 0$.

Transformation to **Risk sensitive stochastic differential game**

$$\left.\frac{dP^{\zeta,h}}{dP^{\zeta}}\right|_{\mathcal{F}_T} = e^{\theta \int_0^T \varphi(X_s,h_s)^* dW_s^{\zeta} - \frac{\theta^2}{2}\int_0^T |\varphi(X_s,h_s)|^2 ds}$$

$$W^{\zeta,h} = W^{\zeta} - \theta \int_0^t \varphi(X_s,h_s)ds$$

$$dX_t = \lambda(X_t)dW_t^{\zeta,h} + (\beta(X_t) + \lambda(X_t)\zeta_t + \theta\delta(X_t)h_t)dt$$

$$\begin{aligned} u_*(0,x;T) &= \inf_{h\in\Delta_{\mathcal{H}}} \sup_{\zeta\in\mathcal{Z}} \log E^{\zeta}[e^{\theta\{F_T(X_\cdot,h_\cdot)+\frac{\mu}{2}\int_0^T |\zeta|^2 ds\}}] \\ &= \inf_{h\in\Delta_{\mathcal{H}}} \sup_{\zeta\in\mathcal{Z}} \log E^{\zeta,h}[e^{\theta\int_0^T \eta(X_s,h_s,\zeta_s)}] \end{aligned}$$

$$\begin{aligned} \eta(x,h,\zeta) &:= f(x,h) + h^*\delta(x)^*\zeta + \frac{\mu}{2}|\zeta|^2 + \frac{\theta}{2}|\delta(x)h|^2 \\ &= -\frac{1}{2}h^*Q_\theta h + h^*(\delta\zeta + g) + U + \frac{\mu}{2}|\zeta|^2 \end{aligned}$$

Lower Isaacs equation

(3.4)

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + \beta^* Du + \frac{1}{2}(Du)^*\lambda\lambda^* Du + H_-(x, Du) = 0 \\ u(T, x) = 0, \end{cases}$$

$$\begin{aligned} H_-(x,p) &= \sup_{\zeta\in R^M} \inf_{h\in R^m} \Lambda(x,p,\zeta,h) \\ &= -\frac{1}{2\theta\mu}(N_\theta\lambda^* p + \theta\delta Q_\theta^{-1} g)^* R_{\theta,\mu}^{-1}(N_\theta\lambda^* p + \theta\delta Q_\theta^{-1} g)] \\ &\qquad +\frac{\theta}{2}(g + \delta^*\lambda^* p)^* Q_\theta^{-1}(g + \delta^*\lambda^* p) + \theta U, \end{aligned}$$

$$\Lambda(x,p,\zeta,h) := \{\zeta + \theta\delta(x)h\}^*\lambda(x)^* p + \theta\eta(x,\zeta,h)$$

$$N_\theta = I + \theta\delta Q_\theta^{-1}\delta^*, \quad R_{\theta,\mu} = I + \frac{1}{\mu}\delta Q_\theta^{-1}\delta^*,$$

For a given solution u to Lower Isaacs equation (3.4), $\widehat{\zeta}(t,x)$ and $\widehat{h}(t,x,\zeta)$ defined by

$$\begin{aligned}
\widehat{\zeta}(t,x) &= -\frac{1}{\theta\mu} R_{\theta,\mu}^{-1}(N_\theta \lambda^* Du + \theta\delta Q_\theta^{-1} g) \\
\widehat{h}(t,x,\zeta) &= Q_\theta^{-1}(g + \delta^*\zeta + \delta^*\lambda^* Du)
\end{aligned}$$

satisfy

$$\widehat{h}(t,x,\zeta) = \arg\min_{h\in R^m} \Lambda(x, Du(t,x), \zeta, h)$$

$$\widehat{\zeta}(t,x) = \arg\max_{\zeta\in R^M} \Lambda(x, Du(t,x), \zeta, \widehat{h}(t,x,\zeta))$$

and

$$\begin{aligned}
H_-(x,Du) &= \sup_{\zeta\in R^M} \inf_{h\in R^m} \Lambda(x, Du(t,x), \zeta, h) \\
&= \Lambda(x, Du(t,x), \widehat{\zeta}(t,x), \widehat{h}(t,x,\widehat{\zeta}(t,x)))
\end{aligned}$$

Upper Isaacs equation

(3.5)

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + \beta^* Du + \frac{1}{2}(Du)^*\lambda\lambda^* Du + H_+(x, Du) = 0 \\ u(T,x) = 0, \end{cases}$$

$$\begin{aligned} H_+(x,p) \quad &:= \quad \inf_{h\in R^m} \sup_{\zeta\in R^M} \Lambda(x,p,\zeta,h) \\ &\equiv \quad \inf_{h\in R^m} \sup_{\zeta\in R^M} [\{\zeta + \theta\delta(x)h\}^*\lambda(x)^* p + \theta\eta(x,\zeta,h)] \\ &= \quad \frac{\theta}{2}\{(\frac{1}{\theta\mu} - 1)\delta^*\lambda^* p - g\}^* Q^{-1}_{\theta - \frac{1}{\mu}}\{(\frac{1}{\theta\mu} - 1)\delta^*\lambda^* p - g\} \\ &\qquad\qquad - \frac{1}{2\theta\mu} p^*\lambda\lambda^* p + \theta U \end{aligned}$$

For a given solution u to Upper Isaacs equation (3.5), $\check{\zeta}(t,x)$ and $\check{h}(t,x,\zeta)$ defined by

$$\begin{aligned}\check{\zeta}(t,x,h) &= -\tfrac{1}{\theta\mu}(\lambda^* Du + \theta\delta h)\\ \check{h}(t,x) &= -\tfrac{1}{\theta\mu}Q^{-1}_{\theta-\frac{1}{\mu}}((1-\theta\mu)\delta^*\lambda^* Du - \theta\mu g)\end{aligned}$$

satisfy

$$\check{h}(t,x) = \arg\min_{h\in R^m} \Lambda(x, Du(t,x), \check{\zeta}(t,x,h), h)$$

$$\check{\zeta}(t,x,h) = \arg\max_{\zeta\in R^M} \Lambda(x, Du(t,x), \zeta, h)$$

and

$$\begin{aligned}H_+(x,Du) &= \inf\nolimits_{h\in R^m}\sup\nolimits_{\zeta\in R^M} \Lambda(x, Du(t,x), \zeta, h)\\ &= \Lambda(x, Du(t,x), \check{\zeta}(t,x,\check{h}(t,x)), \check{h}(t,x))\end{aligned}$$

Lemma 1 *The Isaacs condition holds :*

$$H_-(x,p) = H_+(x,p)$$

and also

$$\widehat{h}(t,x,\widehat{\zeta}(t,x)) = \check{h}(t,x)$$

for given a solution u to (3.4).

Isaacs equation (3.4) can be written as

$$(3.6) \quad \begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + \beta^*_{\theta,\mu} Du \\ \qquad + \frac{1}{2}(Du)^*\lambda N_{\theta,\mu}\lambda^* Du + \frac{\theta}{2} g^* Q^{-1}_{\theta,\mu} g + \theta U = 0 \\ u(T,x) = 0, \end{cases}$$

$$\beta_{\theta,\mu} = \beta + (\theta - \frac{1}{\mu})\lambda\delta Q^{-1}_{\theta-\frac{1}{\mu}} g, \quad N_{\theta,\mu} = (1 - \frac{1}{\theta\mu}) N_{\theta-\frac{1}{\mu}}, \quad Q^{-1}_{\theta,\mu} = Q^{-1}_{\theta-\frac{1}{\mu}}$$

Proposition 3 *Under the assumptions (2.3) - (2.6) Isaacs equation (3.6) has a solution such that*

$$u(t,x) \leq K_0$$

$$u,\ \frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x_i},\ \frac{\partial^2 u}{\partial x_i \partial x_j}, \in L^p(0,T;L^p_{loc}(R^n))$$

$$\frac{\partial u}{\partial t} \geq -C$$

$$\frac{\partial^2 u}{\partial^2 t},\ \frac{\partial^2 u}{\partial x_i \partial t},\ \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k},\ \frac{\partial^3 u}{\partial x_i \partial x_j \partial t} \in L^p(0,T;L^p_{loc}(R^n))$$

$$|Du|^2 + \frac{(c_0-\theta)(1+c)}{c_0 c_2}(\frac{\partial u}{\partial t} + C) \leq c'(|DN_{\theta,\mu}|^2_{2r} + |N_{\theta,\mu}|^2_{2r} + |D(\lambda\lambda^*)|^2_{2r}$$
$$+|D\beta_{\theta,\mu}|_{2r} + |\beta_{\theta,\mu}|^2_{2r} + |\theta U|_{2r} + |\theta DU|_{2r}$$
$$+|\theta g^* Q^{-1}_{\theta,\mu} g|_{2r} + |D(\theta g^* Q^{-1}_{\theta,\mu} g)|_{2r} + 1)$$
$$x \in B_r,\ \ t \in [0,T),$$

where, $c > 0$ *is an arbitrary positive constant,* c' *is a positive constant depending on* c_0, c_2, c, C, θ *and* n *but not on* r*, and* $-C$ *is the lower bound of* U*.*

- cf. Bensoussan-Frehse-N '98 AMO, N. '96, '03 SICON,

Remark Uniqueness results in viscosity sense are seen in
F. Da Lio - O. Ley, SICON '06,

Saddle point

Let us set

$$J(\zeta, h(\zeta)) := \log E^{\zeta, h(\zeta)}[e^{\theta \int_0^T \eta(X_s, h(s, X_s, \zeta_s), \zeta_s)}],$$

where

$$h(\zeta) = h(t, x, \zeta), \qquad \zeta \in \mathcal{Z}.$$

Then, we can see that for the solution $u(t, x; T)$ to (3.4),

$$u(0, x; T) = J(\widehat{\zeta}, \widehat{h}(\widehat{\zeta})),$$

where

$$\widehat{\zeta} = \widehat{\zeta}(t, X_t),\ \widehat{h}(\widehat{\zeta}) = \widehat{h}(t, X_t, \widehat{\zeta}(t, X_t)). \tag{3.7}$$

Further, $(\widehat{\zeta}, \widehat{h}(\widehat{\zeta}))$ turns out to be a saddle point of the game:

$$J(\zeta, \widehat{h}(\zeta)) \le J(\widehat{\zeta}, \widehat{h}(\widehat{\zeta})) \le J(\widehat{\zeta}, h(\widehat{\zeta}))$$

and hence

$$J(\widehat{\zeta}, \widehat{h}(\widehat{\zeta})) = \inf_{h \in \Delta_{\mathcal{H}}} \sup_{\zeta \in \mathcal{Z}} \log E^{\zeta,h}[e^{\theta \int_0^T \eta(X_s,h_s,\zeta_s)}] \equiv u_*(0,x;T)$$

Thus, we have the following proposition.

Proposition 4 *Let $u(t,x;T)$ be a solution to (3.4). Then, under the assumptions (2.3) -(2.6), the pair $(\widehat{\zeta}, \widehat{h}(\widehat{\zeta}))$ of the strategies defined by (3.7) satisfies $\widehat{\zeta} \in \mathcal{Z}$, $\widehat{h}(\widehat{\zeta}) \in \Delta_{\mathcal{H}}$ and attains the value of the game:*

$$u(0,x;T) = J(\widehat{\zeta}, \widehat{h}(\widehat{\zeta})) = u_*(0,x;T)$$

We further have the following proposition.

Proposition 5 $\widehat{h}(t, X_t, \widehat{\zeta}(t, X_t)) = \check{h}(t, X_t)$.

H-J-B equation of ergodic type

Now let us consider the infinite horizon counterpart of (3.6), called H-J-B equation of ergodic type:

$$
(3.8)\quad \begin{aligned} \chi_1(\theta) = \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta^*_{\theta,\mu} Dw + \tfrac{1}{2}(Dw)^*\lambda N_{\theta,\mu}\lambda^* Dw \\ + \tfrac{\theta}{2} g^* Q^{-1}_{\theta,\mu} g + \theta U, \end{aligned}
$$

$$
\beta_{\theta,\mu} = \beta + (\theta - \frac{1}{\mu})\lambda\delta Q^{-1}_{\theta-\frac{1}{\mu}} g, \;\; N_{\theta,\mu} = (1 - \frac{1}{\theta\mu}) N_{\theta-\frac{1}{\mu}}, \;\; Q^{-1}_{\theta,\mu} = Q^{-1}_{\theta-\frac{1}{\mu}}
$$

Proposition 6 *i) Assume that*

$$\lim_{r\to\infty} \inf_{|x|\geq r} \{g^*(\delta^*\delta)^{-1}g(x) + U(x)\} = \infty \tag{3.9}$$

besides assumptions (2.3) - (2.6). Then, we have a solution $(\chi(\theta), w)$ *of (3.8) such that* $w(x)$ *is bounded above. Moreover, such a solution* (χ, w) *is unique up to additive constants with respect to* w *and satisfies the following estimate*

$$|\nabla \bar{w}(x)|^2 \leq C_w|x|^2 + C'_w \tag{3.10}$$

Furthermore, if we assume stronger assumption

$$c_4|x|^2 - c_5 \leq \frac{1}{c_1 - \theta} g^*(\delta^*\delta)^{-1}g(x) + U(x) \tag{3.11}$$

than (3.9), then we have

$$-c_w|x|^2 + c'_w \geq w(x), \quad c_w,\ c'_w > 0. \tag{3.12}$$

ii) Assume that

$$\beta(x)^* x \le -c_\beta |x|^2 + c'_\beta, \quad c_\beta > 0\ , c'_\beta > 0 \tag{3.13}$$

besides assumptions (2.3) - (2.6). Then, there exists a positive constant $b_* > 0$ *such that* $\psi_{b_*}(x) := b_*|x|^2$ *satisfies*

$$F(\psi_{b_*})(x) \to -\infty, \text{ as } |x| \to \infty,$$

where

$$F(\psi) = \frac{1}{2} tr[\lambda\lambda^* D^2\psi] + \beta^*_{\theta,\mu} D\psi + \frac{1}{2}(D\psi)^* \lambda N_{\theta,\mu} \lambda^* D\psi + \frac{\theta}{2} g^* Q^{-1}_{\theta,\mu} g + \theta U$$

and we have a solution $(\chi(\theta), w)$ *to (3.8) such that* $w - \psi_b(x)$ *with* $0 < b \le b_*$ *is bounded above. Moreover, such solution is unique up to additive constants.*

• Thus, we have seen the situation is almost same as the non-robust case and we can proceed assuming the assumptions of Proposition 6 i) with (3.11). The case of ii) of the proposition could be similarly discussed according to the above comment.

• Similarly to the non-robust case, we can develop parallel arguments to obtain our duality theorem. Indeed,

• Under the assumptions of Proposition 6 i) with (3.11), as $T \to \infty$, $u(0, x; T) - \{w(x) + \chi_1(\theta)T\}$ converges to a constant $c_\infty \in R$ uniformly on each compact set.

• As its corollary, we have

$$\lim_{T\to\infty} \frac{u(0, x; T)}{T} = \chi_1(\theta), \tag{3.14}$$

where $(\chi_1(\theta), w(x))$ is the solution to H-J-B equation of ergodic type:

$$\chi_1(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta^*_{\theta,\mu} Dw + \frac{1}{2}(Dw)^*\lambda N_{\theta,\mu}\lambda^* Dw$$
$$+\frac{\theta}{2} g^* Q^{-1}_{\theta,\mu} g + \theta U,$$

• Convexity of $\chi_1(\theta)$ is seen in a similar manner to the non- robust case, Indeed,

• To see the convexity of the solution $u(t,x)$ to the H-J-B equation of parabolic type with respect to θ we introduce a classical stochastic control problem

$$\tilde{u}_*(0,x;T) = \sup_{Z_\cdot \in \mathbf{Z}} E[\int_0^T \Phi(X_s, Z_s)ds], \tag{3.15}$$

with the controlled process X_t governed by the stochastic differential equation

$$dX_t = \lambda(X_t)dW_t + \{G(X_t) + \lambda(X_t)Z_t\}dt, \quad X_0 = x \tag{3.16}$$

where

$$G(x) = \beta - \lambda\delta(\delta^*\delta)^{-1}g.$$

$$\begin{aligned}\Phi(x,z;\theta) &= \tfrac{\theta\mu}{2(1-\theta)} z^* N^{-1}_{\theta-\frac{1}{\mu}} z - \tfrac{\theta\mu}{1-\theta\mu} z^*\delta(\delta^*\delta)^{-1}g \\ &\quad + \tfrac{\theta\mu}{2(1-\theta\mu)} g^*(\delta^*\delta)^{-1}g + \theta U.\end{aligned}$$

Its H-J-B equation turns out to be identical to the original H-J-B-Isaacs equation for the risk-sensitive differential game. From the convexity of $\Phi(x,z;\theta)$ and the verification theorem we can see that $u(0,x;T)$ is convex. Further, Owing to (3.14) above we see the convexity of $\chi(\theta)$

- We can see that

$$\begin{aligned}\chi(\theta) = \lim_{T\to\infty}\frac{1}{T}u(0,x;T) &= \lim_{T\to\infty}\frac{1}{T}\sup_{z_\cdot} E[\int_0^T \Phi(Y_s,z_s)ds] \\ &= \sup_{z_\cdot} \underline{\lim}_{T\to\infty}\frac{1}{T}E[\int_0^T \Phi(Y_s,z_s)ds].\end{aligned}$$

the generator of the optimal diffusion process for the problem on infinite time horizon is seen to be

$$L^w\psi := \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\psi] + \beta^*_{\theta,\mu}D\psi + (Dw)^*\lambda N_{\theta,\mu}\lambda^* D\psi$$

and we see that L^w is ergodic.
The optimal diffusion is governed by:

$$d\bar{X}_t = \lambda(\bar{X}_t)dW_t + \{\beta_{\theta,\mu} + \lambda N_{\theta,\mu}\lambda^* Dw\}(\bar{X}_t)dt$$

• Further, under the assumptions of Theorem 1, for each $\theta_1 \leq \theta \leq \theta_0$ there exist positive constants $k > 0$ and $C > 0$ independent of T and $\theta \in [\theta_1, \theta_0]$ such that

$$E[e^{k|\bar{X}_T|^2}] \leq C \tag{3.17}$$

• Differentiability of H-J-B equation with respect to θ can be seen similarly to the non robust case and we have

$$\chi_1'(\theta) = L^{\bar{w}} w' + (\frac{\partial \beta_{\theta,\mu}}{\partial \theta})^* Dw + \frac{1}{2}(Dw)^* \lambda \frac{\partial N_{\theta,\mu}}{\partial \theta} \lambda^* Dw$$
$$+ \frac{1}{2} g^* \frac{\partial Q_{\theta,\mu}^{-1}}{\partial \theta} g + U,$$

where $w' = \frac{\partial w}{\partial \theta}$.

• Introduce a stochastic differential game

(3.18)

$$\bar{J}(0,x;T) = \inf_{h_\cdot} \sup_{\zeta_\cdot \nu_\cdot} E^{\zeta,h,\nu}[\theta\{\int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s\}$$

$$+\frac{\mu}{2}\int_0^T |\zeta_s|^2 ds - \frac{1}{2}\int_0^T |\nu_s + \theta\delta(\tilde{X}_s)h_s|^2 ds],$$

where X_t is a solution to the stochastic differential equation

$$dX_t = \{\beta(X_t) + \lambda(X_t)(\zeta_t + \nu_t + \theta\delta(X_t)h_t)\}dt + \lambda(X_t)d\tilde{W}_t, \quad X_0 = x,$$

$P^{\zeta,h,\nu}$ is a probability mesure:

$$P^{\zeta,h,\nu}(A) = E^{\zeta}[e^{\int_0^T (\nu_s+\theta\delta(X_s)h_s)^* dW_s^{\zeta} - \frac{1}{2}\int_0^T |\nu_s+\theta\delta(X_s)h_s|^2 ds}; A]$$

$$\tilde{W}_t = W_t^{\zeta} - \int_0^t (\nu_s + \theta\delta(X_s)h_s)ds.$$

Setting

$$\Xi_1(x,h,\nu;\theta) = -\frac{\theta}{2}h^*Q_\theta(x)h + \theta h^*(g(x)+\zeta) + \theta U(x) + \frac{\theta\mu}{2}|\zeta|^2 - \frac{1}{2}|\nu|^2,$$

(3.18) is written as

$$\bar{J}(0,x;T) = \inf_{h_\cdot} \sup_{\nu_\cdot,\zeta_\cdot} E^{\zeta,h,\nu}[\int_0^T \Xi_1(X_s,h_s,\zeta_s,\nu_s;\theta)ds].$$

The corresponding Isaacs equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u]$$
$$+ \sup_{\zeta,\nu} \inf_h [\{\beta + \lambda(\nu + \zeta + \theta\delta h)\}^* Du + \Xi_1(x,h,\zeta,\nu;\theta)] = 0,$$

which is same as

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + \beta^* Du + \frac{1}{2}(Du)^*\lambda\lambda^* Du + H_-(x,Du) = 0 \\ u(T,x) = 0, \end{cases}$$

Ergodic type equation

$$\chi_1(\theta) = \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w]$$

$$+\sup_{\nu\in R^n,\zeta\in R^M}\inf_{h\in R^m}[\{\beta+\lambda\zeta+\lambda(\nu+\theta\delta h)\}^* Dw+\Xi_1(x,h,\zeta,\nu)]$$

can be written as

$$\chi_1(\theta) = L_1^w w + \Xi_1(x,\tilde{h},\tilde{\zeta},\tilde{\nu}) \tag{3.19}$$

where

$$L_1^w\psi = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\psi] + \beta_{\theta,\mu}^* D\psi + (Dw)^*\lambda N_{\theta,\mu}\lambda^* D\psi$$

$$\tilde{h} = Q_\theta^{-1}(g+\delta^*\tilde{\zeta}+\delta^*\lambda^* Dw)$$

$$\tilde{\zeta} = -\tfrac{1}{\theta\mu}R_{\theta,\mu}^{-1}(N_\theta\lambda^* Dw+\theta\delta Q_\theta^{-1}g) = -\tfrac{1}{\theta\mu}N_{\theta-\frac{1}{\mu}}\lambda^* Dw - \tfrac{1}{\mu}\delta Q_{\theta-\frac{1}{\mu}}^{-1}g$$

$$\tilde{\nu} = \lambda^* Dw$$

$$\chi_1(\theta) = L_1^w w - \frac{1}{2}(Dw)^*\lambda N_{\theta,\mu}\lambda^* Dw + \frac{\theta}{2} g^* Q_{\theta,\mu}^{-1} g + \theta U \tag{3.19'}$$

$$\begin{aligned}\chi_1'(\theta) = L_1^w w' + (\tfrac{\partial \beta_{\theta,\mu}}{\partial\theta})^* Dw + \tfrac{1}{2}(Dw)^*\lambda \tfrac{\partial N_{\theta,\mu}}{\partial\theta}\lambda^* Dw \\ + \tfrac{1}{2} g^* \tfrac{\partial \theta Q_{\theta,\mu}}{\partial\theta}^{-1} g + U.\end{aligned} \tag{3.20}$$

(3.19)' and (3.20) lead

$$\chi_1(\theta) - \theta\chi_1'(\theta) = L_1^w(w - \theta w') - \frac{1}{2}|\tilde{\nu} + \theta\delta\tilde{h}|^2$$

and

$$\frac{1}{2}|\tilde{\nu} + \theta\delta\tilde{h}|^2 = \frac{1}{2}\{N_{\theta-\frac{1}{\mu}}\lambda^* Dw + \theta\delta Q_{\theta-\frac{1}{\mu}}^{-1} g\}^*\{N_{\theta-\frac{1}{\mu}}\lambda^* Dw + \theta\delta Q_{\theta-\frac{1}{\mu}}^{-1} g\}$$

Let us consider the worst case uncertainty $\tilde{\zeta}_t = \tilde{\zeta}(X_t)$ with

$$\tilde{\zeta}(x) = -\frac{1}{\theta\mu} R_{\theta,\mu}^{-1}(N_\theta \lambda^* Dw + \theta\delta Q_\theta^{-1} g)$$

and take a probability measure $\tilde{P}$ defined by

$$\left.\frac{d\tilde{P}}{dP^{\tilde{\zeta}}}\right|_{\mathcal{F}_T} = e^{\int_0^T \{\tilde{\nu}+\theta\delta\tilde{h}\}(X_s)^* dW_s^{\tilde{\zeta}} - \frac{1}{2}\int_0^T |\tilde{\nu}+\theta\delta\tilde{h}|^2(X_s)ds}$$

Then, under the probability measure $\tilde{P}$, X_t satisfies

$$dX_t = \{\beta(X_t) + \lambda(X_t)(\tilde{\zeta}_t + \tilde{\nu}_t + \theta\delta(X_t)\tilde{h}_t)\}dt + \lambda(X_t)d\tilde{W}_t, \quad X_0 = x,$$

where

$$\tilde{W}_t = W_t^{\tilde{\zeta}} - \int_0^t (\tilde{\nu}_s + \theta\delta(X_s)\tilde{h}_s)ds.$$

$$\tilde{\nu}_t = \tilde{\nu}(X_t), \;\; \tilde{h}_t = \tilde{h}(X_t)$$

Duality theorem

Theorem 3 *For* $\kappa \in (\chi_1'(-\infty), \chi_1'(0-))$*, we have*

$$\varliminf_{T\to\infty} \frac{1}{T} \inf_{h\in\Delta_{\mathcal{H}}} \sup_{\zeta\in\mathcal{Z}} \log P^{\zeta}(F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds \le \kappa T) = -I_1(\kappa)$$

$$I_1(k) := \sup_{\theta<0}\{\theta k - \chi_1(\theta)\}$$

Moreover, for $\theta(\kappa)$ *such that* $\chi_1'(\theta(\kappa)) = \kappa \in (\chi_1'(-\infty), \chi_1'(0-))$ *take a strategy* $\widehat{h}^{(\theta(\kappa))}(t, x, \tilde{\zeta})$*. Then,*

$$\lim_{T\to\infty} \frac{1}{T} \log P^{\tilde{\zeta}}(F_T(X_\cdot, \widehat{h}^{(\theta(\kappa))}(\cdot, X_\cdot, \tilde{\zeta}_\cdot)) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2 ds \le \kappa T) = -I_1(\kappa)$$

Note that

$$
\begin{aligned}
&\inf_{h\in\Delta_{\mathcal{H}}} \log P^{\tilde{\zeta}}(F_T(X_\cdot, h(\cdot, X_\cdot, \tilde{\zeta}_\cdot)) + \tfrac{\mu}{2}\textstyle\int_0^T |\tilde{\zeta}_s|^2 ds \le \kappa T) \\
&\le \log P^{\tilde{\zeta}}(F_T(X_\cdot, \hat{h}^{(\theta(\kappa))}(\cdot, X_\cdot, \tilde{\zeta}_\cdot)) + \tfrac{\mu}{2}\textstyle\int_0^T |\tilde{\zeta}_s|^2 ds \le \kappa T) \\
&\le \sup_{\zeta} \log P^{\zeta}(F_T(X_\cdot, \hat{h}^{(\theta(\kappa))}(\cdot, X_\cdot, \zeta_\cdot)) + \tfrac{\mu}{2}\textstyle\int_0^T |\zeta_s|^2 ds \le \kappa T) \\
&\le \sup_{\zeta} \log E^{\zeta}[e^{\theta\{(F_T(X_\cdot, \hat{h}^{(\theta(\kappa))}(\cdot, X_\cdot, \zeta_\cdot)) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\} - \theta\kappa T}] \\
&= \log E^{\hat{\zeta}}[e^{\theta\{(F_T(X_\cdot, \hat{h}^{(\theta(\kappa))}(\cdot, X_\cdot, \hat{\zeta}_\cdot)) + \frac{\mu}{2}\int_0^T |\hat{\zeta}_s|^2 ds\}}] - \theta\kappa T \\
&= u(0, x; T) - \theta\kappa T
\end{aligned}
$$

for $\theta < 0$.

- μ is considered as the certainty level of β since we have the following estimate:

$$\| \tilde{\zeta} \|_{L^{\infty}_{loc}} = O(\frac{1}{\mu})$$

$\tilde{\zeta}(x)$: the worst case uncertainty

Linear Gaussian case

$$\beta(x) = Bx + b, \;\; g(x) = Ax + a, \;\; U(x) = \tfrac{1}{2}x^*Vx + m,$$

$\lambda, \delta, S, A, B, V$: const. matrices

$$u(t,x) = \frac{1}{2}x^*P(t)x + q(t)^*x + l(t), \;\; \text{sol. to the Lower Isaacs eq.}$$

$$\begin{cases} \dot{P}(t) - \frac{1-\theta\mu}{\theta\mu}P(t)\lambda N_{\theta-\frac{1}{\mu}}\lambda^*P(t) + K_1^*P(t) + P(t)K_1 \\ \qquad\qquad -C^*C + \theta V = 0 \\ P(T) = 0 \end{cases}$$

$$\begin{aligned} K_1 &= B + (\theta - \tfrac{1}{\mu})\lambda\delta Q^{-1}_{\theta-\frac{1}{\mu}}A \\ C^*C &= -\theta A^*Q^{-1}_{\theta-\frac{1}{\mu}}A \end{aligned}$$

$$
\begin{cases}
\dot{q}(t) + (K_1^* - \frac{1-\theta\mu}{\theta\mu} P(t)\lambda N_{\theta-\frac{1}{\mu}}\lambda^*)q(t) + P(t)b \\
\qquad -\theta A^*\delta Q_{\theta-\frac{1}{\mu}}^{-1} a - \frac{1-\theta\mu}{\mu} P(t)\lambda\delta Q_{\theta-\frac{1}{\mu}}^{-1} a = 0 \\
q(T) = 0
\end{cases}
$$

$$
\begin{cases}
\dot{l}(t) + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* P(t)] + \frac{1}{2}(1 - \frac{1}{\theta\mu})q(t)\lambda N_{\theta-\frac{1}{\mu}}\lambda^* q(t) \\
\quad +(b + (\theta - \frac{1}{\mu})\lambda\delta Q_{\theta-\frac{1}{\mu}}^{-1} a)^* q(t) + \frac{\theta}{2} a^* Q_{\theta-\frac{1}{\mu}}^{-1} a + \theta m \\
l(T) = 0
\end{cases}
$$

In this case, if we assume that

(i) B is stable,

or

(ii) $\lambda\lambda^* > 0, \quad A^*A > 0$

then,

$$P(t;T) \to \overline{P}, \;\; T \to \infty \quad q(t;T) \to \bar{q}$$

$\overline{P}$ is a solution to the stationary equation:

$$-\frac{1-\theta\mu}{\theta\mu}\overline{P}\lambda N_{\theta-\frac{1}{\mu}}\lambda^*\overline{P} + K_1^*\overline{P} + \overline{P}K_1 - C^*C + \theta V = 0$$

such that

$$K_1 - \frac{1-\theta\mu}{\theta\mu}\lambda N_{\theta-\frac{1}{\mu}}\lambda^*\overline{P} \;\text{ is stable}$$

$\bar{q}$ is the solution to

$$(K_1^* - \tfrac{1-\theta\mu}{\theta\mu}\overline{P}\lambda N_{\theta-\frac{1}{\mu}}\lambda^*)\bar{q} + \overline{P}b$$

$$-\theta A^*\delta Q^{-1}_{\theta-\frac{1}{\mu}}a - \tfrac{1-\theta\mu}{\mu}\overline{P}\lambda\delta Q^{-1}_{\theta-\frac{1}{\mu}}a = 0.$$

Further,

$$\frac{l(0;T)}{T} \to \chi_1(\theta)$$

$$\chi_1(\theta) = \quad \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^*\overline{P}] + \tfrac{1}{2}(1 - \tfrac{1}{\theta\mu})\bar{q}\lambda N_{\theta-\frac{1}{\mu}}\lambda^*\bar{q}$$

$$+(b + (\theta - \tfrac{1}{\mu})\lambda\delta Q^{-1}_{\theta-\frac{1}{\mu}}a)^*\bar{q} + \tfrac{\theta}{2}a^*Q^{-1}_{\theta}a + \theta m$$

• differentiability of $\overline{P}$, $\bar{q}$, $\chi_1(\theta)$ is seen in a similar way to Hata-N.-Sheu.

H. Nagai, Downside risk minimization via a large deviations approach, Annals of Appl. Prob. vol. 22 (2012) 608-669

H. Nagai, Large deviation estimates for controlled semi-martingales, preprint

H. Nagai, Robust estimates of certain large deviation probabilities for controlled semi-martingales, Preprint