Asymptotics of forward implied volatility

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Based on joint works with Patrick Roome (Imperial College London):

- The small-maturity Heston forward smile. To appear in *SIAM Journ on Fin Math*
- Asymptotics of forward implied volatility. *Submitted, arxiv 1212.0779*.
- Large-maturity regimes of the Heston forward smile. *In progress*.

(Spot) implied volatility

- Asset price process: $(S_t = \mathrm{e}^{X_t})_{t \geq 0}$, with $X_0 = 0$.
- No dividend, no interest rate.
- Black-Scholes-Merton (BSM) framework:

$$C_{\mathrm{BS}}(\tau, k, \sigma) := \mathbb{E}_0 \left(\mathrm{e}^{X_\tau} - \mathrm{e}^k\right)_+ = \mathcal{N}(d_+) - \mathrm{e}^k \mathcal{N}(d_-),$$

$d_\pm := -\frac{k}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}$.

- Spot implied volatility $\sigma_\tau(k)$: the unique (non-negative) solution to

$$C_{\mathrm{observed}}(\tau, k) = C_{\mathrm{BS}}(\tau, k, \sigma_\tau(k)).$$

- Spot implied volatility: unit-free measure of option prices.
- However not available in closed form for most models.

Spot implied volatility ($\sigma_\tau(k)$) asymptotics as $|k| \uparrow \infty$, $\tau \downarrow 0$ or $\tau \uparrow \infty$:

- Berestycki-Busca-Florent (2004): small-τ using PDE methods for diffusions.
- Henry-Labordère (2009): small-τ asymptotics using differential geometry.
- Forde et al.(2012), Jacquier et al.(2012): small-and large-τ using large deviations and saddlepoint methods.
- Lee (2003), Benaim-Friz (2009), Gulisashvili (2010-2012), De Marco-Jacquier-Hillairet (2013): $|k| \uparrow \infty$.
- Laurence-Gatheral-Hsu-Ouyang-Wang (2012): small-τ in local volatility models.
- Fouque et al.(2000-2011): perturbation techniques for slow and fast mean-reverting stochastic volatility models.

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Related works:

- Kim, Kunitomo, Osajima, Takahashi (1999-...) : asymptotic expansions based on Kusuoka-Yoshida-Watanabe method (expansion around a Gaussian).
- Deuschel-Friz-Jacquier-Violante (to appear in CPAM): small-noise expansions using Laplace method on Wiener space (Ben Arous-Bismut approach).
- Sørensen, Yoshida: for statistical estimation / maximum likelihood (small diffusion coefficient).

Note: from expansions of densities to implied volatility asymptotics is 'automatic' (Gao-Lee (2013)).

Forward implied volatility

- Fix $t > 0$: forward-starting date; $\tau > 0$: remaining maturity.
- Forward-start call option is a European call option with payoff

$$\left(\frac{S_{t+\tau}}{S_t} - \mathrm{e}^k\right)^+ = \left(\mathrm{e}^{X_{t+\tau} - X_t} - \mathrm{e}^k\right)^+,$$

  and value today

$$\mathbb{E}_0 \left(\mathrm{e}^{X_{t+\tau} - X_t} - \mathrm{e}^k\right)^+.$$

- BSM model: its value today is simply worth $C_{\mathrm{BS}}(\tau, k, \sigma)$ (stationary increments).
- Forward implied volatility $\sigma_{t,\tau}(k)$: the unique solution to

$$C_{\mathrm{observed}}(t, \tau, k) = C_{\mathrm{BS}}(\tau, k, \sigma_{t,\tau}(k)).$$

- Obviously, $\sigma_{0,\tau}(k) = \sigma_\tau(k)$.

Motivation

Calibration:

- Forward-start options serve as natural hedging instruments for many exotic securities and it is therefore important for a model to be able to calibrate to liquid forward smiles.

Model Risk:

- Calibrate two different models to some observed spot implied volatility smiles: perfect calibration. Use these calibrated models to price some 'exotic' options, say barrier options: two different prices. One of the reasons: subtle dependence on the dynamics of implied volatility smiles.
- One metric that can be used to understand the dynamics of implied volatility smiles (Bergomi(2004) calls it a 'global measure' of the dynamics of implied volatilities) is to use the forward smile defined above.
- Analytical comparison of models and check the realism of model-generated forward smiles.

Existing literature on forward smiles

- Glasserman and Wu (2011): different notions of forward volatilities to assess their predictive values in determining future option prices and future implied volatility.
- Keller-Ressel (2011): when the forward-start date t becomes large (τ fixed).
- Empirical results: Bergomi (2004), Bühler (2002), Gatheral (2006).

Getting some intuition: Gärtner-Ellis theorem

Rescaled cumulant generating function h continuous with $\lim_{\tau\downarrow 0} h(\tau) = 0$):

$$\Lambda_\tau(u,h) := h(\tau)\log\mathbb{E}\left(S_\tau^{u/h(\tau)}\right) = h(\tau)\log\mathbb{E}\left(\mathrm{e}^{(u/h(\tau))X_\tau}\right), \quad u \in \mathcal{D}_{\tau,h} \subset \overline{\mathbb{R}}.$$

Theorem (Gärtner-Ellis)

If $\Lambda(u) := \lim_{\tau\downarrow 0}\Lambda_\tau(u,h)$ exists in $\overline{\mathbb{R}}$ for $u \in \mathcal{D}_0$, and Λ strictly convex and differentiable on $\mathcal{D}_0^o$, with $\lim_{u\in\partial\mathcal{D}_0}\Lambda'(u) = +\infty$, then $(X_\tau)_{\tau>0}$ satisfies a large deviations principle (LDP) (with speed $h(\tau)$) as $\tau \downarrow 0$:

$$\mathbb{P}(X_\tau \in A) \sim \exp\left(-\frac{1}{h(\tau)}\inf\{\Lambda^*(x) : x \in A\}\right), \qquad A \subset \mathbb{R}.$$

Lemma: S is a continuous-time diffusion (Black-Scholes). Then $h(\tau) \equiv \tau$.

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Lemma: S is a continuous-time diffusion (Black-Scholes). Then $h(\tau) \equiv \tau$.

Forward problem: S: Heston model, and $\Lambda_\tau^{(t)}$ the rescaled forward cgf, then

Lemma

If $h(\tau) \equiv \sqrt{\tau}$ then $\lim_{\tau \downarrow 0} \Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for $u \in (\underline{u}, \bar{u})$ and ∞ otherwise;

Getting some intuition: direct computation?

Consider the Stein-Stein / Schöbel-Zhu model:

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}\sigma_t^2\mathrm{d}t + \sigma_t\mathrm{d}W_t, \qquad X_0 = 0,\\
\mathrm{d}\sigma_t &= (a + b\sigma_t)\mathrm{d}t + \xi\mathrm{d}Z_t, \quad \sigma_0 = \sigma_0 > 0,
\end{aligned}
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Define $X_\tau^{(t)} := X_{t+\tau} - X_t$, then

$$\begin{aligned} \mathrm{d}X_\tau^{(t)} &= -\frac{1}{2}(\sigma_\tau^{(t)})^2 \mathrm{d}\tau + \sigma_\tau^{(t)} \mathrm{d}W_\tau, & X_0^{(t)} &= 0, \\ \mathrm{d}\sigma_\tau^{(t)} &= (a + b\sigma_\tau^{(t)})\mathrm{d}\tau + \xi \mathrm{d}Z_\tau, & \sigma_0^{(t)} &\sim \sigma_t, \end{aligned}$$

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Pricing Fwd-start options:

$$\mathbb{E}_0(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k)^+ = \mathbb{E}_0\left\{\mathbb{E}_t(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k|\sigma_t)^+\right\}$$

Problem: Known expansions (as $\tau \downarrow 0$) are NOT uniform in space.
Easy case: σ_t has compact support, e.g. finite-state Markov chain.

More intuition: via sample paths

Consider the "forward" model:

$$\mathrm{d}X_\tau^{(t)} = -\frac{1}{2}(\sigma_\tau^{(t)})^2\mathrm{d}\tau + \sigma_\tau^{(t)}\mathrm{d}W_\tau, \quad X_0^{(t)} = 0,$$
$$\mathrm{d}\sigma_\tau^{(t)} = (a + b\sigma_\tau^{(t)})\mathrm{d}\tau + \xi\mathrm{d}Z_\tau, \quad \sigma_0^{(t)} \sim \sigma_t,$$

Small-time estimates: Let $\tau \to \varepsilon^2\tau$ and define $\bar{X}_\tau^\varepsilon := X_{\varepsilon^2\tau}^{(t)}$ and $\bar{\sigma}_\tau^\varepsilon := \sigma_{\varepsilon^2\tau}^{(t)}$, then

$$\mathrm{d}\bar{X}_\tau^\varepsilon = -\frac{1}{2}(\bar{\sigma}_\tau^\varepsilon)^2\mathrm{d}\tau + \varepsilon\bar{\sigma}_\tau^\varepsilon\mathrm{d}W_\tau, \quad \bar{X}_0^\varepsilon = 0,$$
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Tail estimates: Let $\bar{X}_\tau^\varepsilon := \varepsilon^2 X_\tau^{(t)}$ and $\bar{\sigma}_\tau^\varepsilon := \varepsilon\sigma_\tau^{(t)}$, then

$$\begin{aligned} \mathrm{d}\bar{X}_\tau^\varepsilon &= -\frac{1}{2}(\bar{\sigma}_\tau^\varepsilon)^2 \mathrm{d}\tau + \varepsilon\bar{\sigma}_\tau^\varepsilon \mathrm{d}W_\tau, \quad & \bar{X}_0^\varepsilon &= 0, \\ \mathrm{d}\bar{\sigma}_\tau^\varepsilon &= (a\varepsilon + b\bar{\sigma}_\tau^\varepsilon)\mathrm{d}\tau + \varepsilon\xi \mathrm{d}Z_\tau, \quad & \bar{\sigma}_0^\varepsilon &\sim \varepsilon\sigma_t, \end{aligned}$$

Standard large deviations / Wentzell-Freidlin problem, but **random initial data**.
In progress...

General results: framework

(Y_ε): (general) stochastic process. Denote the re-normalised log moment generating function (lmgf) by $\Lambda_\varepsilon(u) := \varepsilon \log \mathbb{E}\left[\exp\left(\frac{uY_\varepsilon}{\varepsilon}\right)\right]$, for all $u \in \mathcal{D}_\varepsilon \subseteq \mathbb{R}$.

We require the following assumptions (**Assumption OA**) on the behaviour of Λ_ε:

(i) **Expansion property:** $\Lambda_\varepsilon(u) = \sum_{i=0}^{2} \Lambda_i(u)\varepsilon^i + \mathcal{O}(\varepsilon^3)$ holds for $u \in \mathcal{D}_0^o$ as $\varepsilon \downarrow 0$;

(ii) **Differentiability:** The map $(\varepsilon, u) \mapsto \Lambda_\varepsilon(u)$ is of class $\mathcal{C}^\infty$ on $(0, \varepsilon_0) \times \mathcal{D}_0^o$;

(iii) **Non-degenerate interior:** $0 \in \mathcal{D}_0^o$;

(iv) **Essential smoothness:** Λ_0 is strictly convex and essentially smooth on $\mathcal{D}_0^o$;

(v) **Tail error control:** For any fixed $p_r \in \mathcal{D}_0^o \backslash \{0\}$,
   - (a) $\Re\left(\Lambda_\varepsilon\left(\mathrm{i}p_i + p_r\right)\right) = \Re\left(\Lambda_0\left(\mathrm{i}p_i + p_r\right)\right) + \mathcal{O}(\varepsilon)$, for any $p_i \in \mathbb{R}$;
   - (b) $L : \mathbb{R} \ni p_i \mapsto \Re\left(\Lambda_0\left(\mathrm{i}p_i + p_r\right)\right)$ has a unique maximum at zero and is bounded away from $L(0)$ as $|p_i|$ tends to infinity;
   - (c) $\Re\left[\Lambda_\varepsilon(\mathrm{i}p_i + p_r) - \Lambda_0(\mathrm{i}p_i + p_r)\right] \leq M\varepsilon$, for some $M > 0$, for large $|p_i|$ and small ε.

Note: (i)-(iv) are Gärtner-Ellis assumptions for large deviations:

$$\mathbb{P}(Y_\varepsilon \in A) \sim \exp\left(-\frac{1}{\varepsilon}\inf\{\Lambda^*(x) : x \in A\}\right),$$

for $A \subset \mathbb{R}$, as $\varepsilon \downarrow 0$, Λ^*: dual of Λ_0.

Main result: Option price asymptotics

Theorem (J-Roome, 2013)

Let (Y_ε) satisfy OA and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying $f(\varepsilon)\varepsilon = c + \mathcal{O}(\varepsilon)$, for some constant $c \geq 0$ as $\varepsilon \downarrow 0$. For $k > \Lambda_0'(c)$, as $\varepsilon \downarrow 0$,

$$\mathbb{E}\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)} - \mathrm{e}^{kf(\varepsilon)}\right)^+ = \psi(k,c,\varepsilon)\exp\left\{-\frac{\Lambda^*(k)}{\varepsilon} + kf(\varepsilon)\right\}\left[1 + \alpha_1(k,c)\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\right],$$

where $\psi(k,c,\varepsilon) \equiv \alpha(k,c)\left(c\sqrt{\varepsilon}\mathbb{1}_{\{c>0\}} + \varepsilon^{3/2}f(\varepsilon)\mathbb{1}_{\{c=0\}}\right)$, and $\Lambda^* : \mathbb{R} \to \mathbb{R}_+$ is the Fenchel-Legendre transform of Λ_0:

$$\Lambda^*(k) := \sup_{u\in\mathcal{D}_0}\left\{uk - \Lambda_0(u)\right\}, \qquad \text{for all } k \in \mathbb{R}.$$

We shall denote $u^*(k)$ the corresponding saddlepoint: $\Lambda^*(k) = u^*(k)k - \Lambda_0(u^*(k))$.

NB: analogous results hold for Put options when $k < \Lambda_0'(c)$ and for covered calls when $k \in (\Lambda_0'(0), \Lambda_0'(c))$.

Forward-start options: large maturity

Recall the fwd-start process: $X_\tau^{(t)} := X_{t+\tau} - X_t$. Let $(Y_\varepsilon) := (\varepsilon X_{\tau/\varepsilon}^{(t)})$ and $f(\varepsilon) \equiv 1/\varepsilon$:

$$\mathbb{E}\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)} - \mathrm{e}^{kf(\varepsilon)}\right)^+ = \mathbb{E}\left(\mathrm{e}^{X_{\tau/\varepsilon}^{(t)}} - \mathrm{e}^{k/\varepsilon}\right)^+$$

Corollary (Large-maturity, $t \geq 0$)

If $(\tau^{-1} X_\tau^{(t)})_{\tau>0}$ satisfies OA with $\varepsilon = \tau^{-1}$ and $1 \in \mathcal{D}_0^o$, then for $k > \Lambda_0'(1)$, as $\tau \uparrow \infty$:

$$\mathbb{E}_0\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^{k\tau}\right)^+ = \frac{\mathrm{e}^{-\tau\left(\Lambda^*(k)-k\right)+\Lambda_1(u^*(k))}\tau^{-1/2}}{u^*(k)\left(u^*(k)-1\right)\sqrt{2\pi\Lambda_0''(u^*(k))}}\left[1 + \frac{\alpha_1(k)}{\tau} + \mathcal{O}\left(\frac{1}{\tau^2}\right)\right]$$

Corollary: $(\tau^{-1} X_\tau^{(t)})_\tau$ satisfies a LDP with speed τ^{-1} and rate function $k \mapsto \Lambda^*(k) - k$.

Note: encompasses the case $t = 0$ (see Jacquier, Keller-Ressel, Mijatović (2013)); makes sense intuitively.

Forward-start options: diagonal small-maturity

Recall the fwd-start process: $X_\tau^{(t)} := X_{t+\tau} - X_t$. Let $(Y_\varepsilon) := (X_{\varepsilon\tau}^{(\varepsilon t)})$ and $f(\varepsilon) \equiv 1$:

$$\mathbb{E}\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)} - \mathrm{e}^{kf(\varepsilon)}\right)^+ = \mathbb{E}\left(\mathrm{e}^{X_{\varepsilon\tau}^{(\varepsilon t)}} - \mathrm{e}^k\right)^+$$

Corollary (Diagonal small-maturity, $t, \tau > 0$)

If $(X_{\varepsilon\tau}^{(\varepsilon t)})_{\varepsilon>0}$ satisfies OA, then the following holds for $k > \Lambda_0'(0)$, as $\varepsilon \downarrow 0$:

$$\mathbb{E}_0\left(\mathrm{e}^{X_{\varepsilon\tau}^{(\varepsilon t)}} - \mathrm{e}^k\right)^+ = \frac{\exp\left\{-\frac{\Lambda^*(k)}{\varepsilon} + k + \Lambda_1(u^*(k))\right\}\varepsilon^{3/2}}{u^*(k)^2\sqrt{2\pi\Lambda_0''(u^*(k))}}\left[1 + \alpha_1(k)\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\right].$$

Corollary: $(X_{\varepsilon\tau}^{(\varepsilon t)})_{\varepsilon>0}$ satisfies a LDP with speed ε and rate function Λ^*.

Forward smile asymptotics: large maturity

Corollary (Large-maturity forward smile asymptotics)

If $(\tau^{-1}X_\tau^{(t)})_{\tau>0}$ satisfies OA with $\varepsilon = \tau^{-1}$ and $\Lambda_0(1) = 0$ with $1 \in \mathcal{D}_0^o$, then for all $k \in \mathbb{R}$, as $\tau \uparrow \infty$:

$$\sigma_{t,\tau}^2(k\tau) = v_0^\infty(k,t) + \frac{v_1^\infty(k,t)}{\tau} + \frac{v_2^\infty(k,t)}{\tau^2} + \mathcal{O}\left(\frac{1}{\tau^3}\right),$$

where $v_0^\infty(\cdot,t)$, $v_1^\infty(\cdot,t)$ and $v_2^\infty(\cdot,t)$ are continuous functions on $\mathbb{R}$.

- If $S = \mathrm{e}^X$ is a true martingale, then $\Lambda_0(1) = 0$.
- For $t = 0$ (spot smiles), we recover Jacquier, Keller-Ressel, Mijatović (2013).

Forward smile asymptotics: diagonal small maturity

Corollary (Diagonal small-maturity forward smile asymptotics)

If $(X^{(\varepsilon t)}_{\varepsilon\tau})_{\varepsilon>0}$ satisfies OA and $\Lambda_0'(0)=0$, then for all $k\in\mathbb{R}$, as $\varepsilon\downarrow 0$,

$$\sigma^2_{\varepsilon t,\varepsilon\tau}(k) = v_0(k,t,\tau) + v_1(k,t,\tau)\varepsilon + v_2(k,t,\tau)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right),$$

where $v_0(\cdot,t,\tau)$, $v_1(\cdot,t,\tau)$ and $v_2(\cdot,t,\tau)$ are continuous functions on $\mathbb{R}$.

Under the assumption $\Lambda_\varepsilon(u)=\sum_{i=0}^2\Lambda_i(u)\varepsilon^i+\mathcal{O}(\varepsilon^3)$, as $\varepsilon\downarrow 0$, v_0,v_1, and v_2 depend on the derivatives of Λ_0, Λ_1 and Λ_2 evaluated at $u^*(k)$.
When $t=0$ (spot smiles), we recover Forde-Jacquier-Lee (2012), Gao-Lee (2013), Berestycki-Busca-Florent (2004).

Examples

- S: exponential Lévy model. Stationary increment property implies $\sigma_{t,\tau}$ does not depend on t.
- S: time-changed exponential Lévy model.
- Stochastic volatility models; Schöbel-Zhu: $\mathrm{d}\sqrt{V_t} = \kappa(\theta - \sqrt{V_t})\mathrm{d}t + \xi \mathrm{d}Z_t$.
- Heston (affine stochastic volatility) model:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2} V_t \mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, && X_0 = 0, \\ \mathrm{d}V_t &= \kappa\left(\theta - V_t\right)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t, && V_0 = v > 0, \\ \mathrm{d}\left\langle W, Z\right\rangle_t &= \rho \mathrm{d}t, \end{aligned}$$

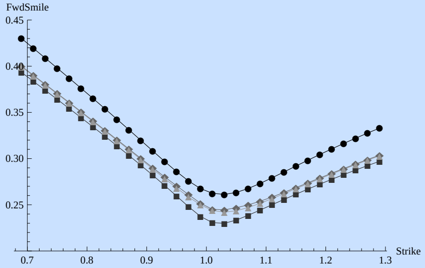
with $\kappa > 0$, $\xi > 0$, $\theta > 0$ and $|\rho| < 1$.

Heston diagonal small-maturity

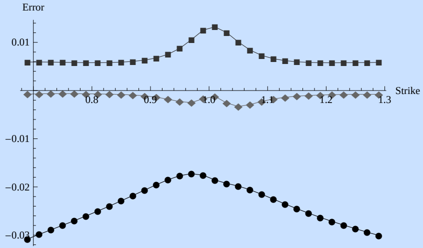
- We can compare spot and forward (diagonal) small-maturity smiles:

$$\begin{aligned}\sigma_{\varepsilon t,\varepsilon\tau}(0) &= \sigma_{0,\varepsilon\tau}(0) &-\frac{\varepsilon t}{8\sqrt{v}}\left(\xi^2+4\kappa(v-\theta)\right)+\mathcal{O}(\varepsilon^2),\\ \partial_k^2\sigma_{\varepsilon t,\varepsilon\tau}(0) &= \partial_k^2\sigma_{0,\varepsilon\tau}(0) &+\xi^2 t/(4\tau v^{3/2})+\mathcal{O}(\varepsilon).\end{aligned}$$

- At zeroth order in ε the wings of the forward smile increase to arbitrarily high levels with decreasing maturity.
- Bühler (2002): *'Heston implied forward volatility: short skew becomes U-shaped, which is inconsistent with observations.'*



(a) Heston diagonal small-maturity vs Fourier inversion.



(b) Errors

Figure: In (a) circles, squares and diamonds represent the zeroth, first-and second-order asymptotics respectively and triangles represent the true forward smile using Fourier inversion. In (b): differences between the true forward smile and the asymptotic. We use $t = 1/2$ and $\tau = 1/12$ and the Heston parameters $v = 0.07$, $\theta = 0.07$, $\kappa = 1$, $\xi = 0.34$, $\rho = -0.8$.

Heston large-maturity

Natural conjecture: the limiting forward smile is the same as the limiting large-maturity spot smile (e.g. no dependence on the forward date t).

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Proposition: If $\rho_{-}(t) \leq \rho \leq \min\left(\rho_{+}(t), \kappa/\xi\right)$ then the conjecture holds. *Other cases: degenerate problem, in progress.*

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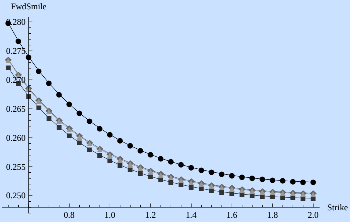
Proposition: If $\rho_-(t) \leq \rho \leq \min(\rho_+(t), \kappa/\xi)$ then the conjecture holds. *Other cases: degenerate problem, in progress.*



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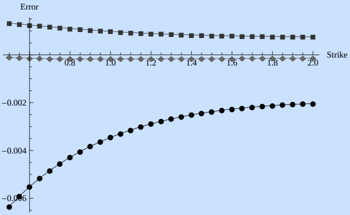

(b) Errors

Figure: (a) circles, squares and diamonds: the zeroth, first-and second-order; triangles: true forward smile. (b): differences between the true forward smile and the asymptotic. Here $t = 1$, $\tau = 5$ and the Heston parameters $v = 0.07$, $\theta = 0.07$, $\kappa = 1.5$, $\xi = 0.34$, $\rho = -0.25$.

Small-maturity asymptotics: problem overview

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Lemma

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ continuous with $\lim_{\tau\downarrow 0} h(\tau) = 0$ and $a > 0$. Then

(i) If $h(\tau) \equiv a\sqrt{\tau}$ then $\lim_{\tau\downarrow 0} \Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for $|u| < a/\sqrt{\beta_t}$ and ∞ otherwise;

(ii) if $\sqrt{\tau}/h(\tau) \uparrow \infty$ then $\lim_{\tau\downarrow 0} \Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for $u = 0$ and ∞ otherwise;

(iii) if $\sqrt{\tau}/h(\tau) \downarrow 0$ then $\lim_{\tau\downarrow 0} \Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for all $u \in \mathbb{R}$.

(i) is the only non-trivial zero limit.

From now on, define $\Lambda_\tau^{(t)}(u) := \Lambda_\tau^{(t)}(u, \sqrt{\tau})$, and $\Lambda(u) := \lim_{\tau\downarrow 0} \Lambda_\tau^{(t)}(u)$.
Define the Fenchel-Legendre transform $\Lambda^*(k) := \sup\{uk - \Lambda(u), u \in \mathbb{R}\}$.

Lemma

$\Lambda^*(k) = |k|/\sqrt{\beta_t}$, *for all* $k \in \mathbb{R}$.

Clearly, no convexity argument holds here.

Main result

Theorem (J-Roome, 2012)

Let $t > 0$. In the Heston model, the following expansion holds for all $k \in \mathbb{R}^*$ as $\tau \downarrow 0$:

$$\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k\right)^+ = \left(1 - \mathrm{e}^k\right) \mathbb{1}_{\{k<0\}}$$
$$+ \exp\left(-\frac{\Lambda^*(k)}{\sqrt{\tau}} + \frac{c_0(k)}{\tau^{1/4}} + c_1(k) + k\right) \tau^{\frac{7}{8} - \frac{\kappa\theta}{2\xi^2}} c_2(k) \left[1 + c_3(k)\tau^{1/4} + o(\tau^{1/4})\right].$$

Corollary: $(X_\tau^{(t)})_{\tau \geq 0}$ satisfies a LDP with speed $\sqrt{\tau}$ and rate function Λ^* as $\tau \downarrow 0$.

Compare with (see Forde-Jacquier-Lee (2012)), when $t = 0$:

- $$\mathbb{E}\left(\mathrm{e}^{X_\tau^{(0)}} - \mathrm{e}^k\right)^+ = (1 - \mathrm{e}^k)\mathbb{1}_{\{k<0\}} + \mathrm{e}^{\Lambda^*(k)/\tau} \tau^{3/2} c_2(k) \left(1 + \mathcal{O}(\tau)\right),$$
- $(X_\tau^{(0)})_{\tau \geq 0}$ satisfies a LDP with speed τ and good rate function Λ^* as $\tau \downarrow 0$.

Small-maturity smile

Proposition (J-Roome, 2012), $t > 0$

The following expansion holds for the forward smile for all $k \in \mathbb{R}^*$ as $\tau \downarrow 0$:

$$\sigma^2_{t,\tau}(k) = \begin{cases} \dfrac{v_0(k,t)}{\tau^{1/2}} + \dfrac{v_1(k,t)}{\tau^{1/4}} + o\left(\dfrac{1}{\tau^{1/4}}\right), & \text{if } 4\kappa\theta \neq \xi^2, \\ \dfrac{v_0(k,t)}{\tau^{1/2}} + \dfrac{v_1(k,t)}{\tau^{1/4}} + v_2(k,t) + v_3(k,t)\tau^{1/4} + o\left(\tau^{1/4}\right), & \text{if } 4\kappa\theta = \xi^2. \end{cases}$$

Compare with the $t = 0$ case: $\sigma^2_{0,\tau}(k) = \sigma^2_0(k) + a(k)\tau + o(\tau)$, when $k \neq 0$.

Small-maturity smile

Proposition (J-Roome, 2012), $t > 0$

The following expansion holds for the forward smile for all $k \in \mathbb{R}^*$ as $\tau \downarrow 0$:

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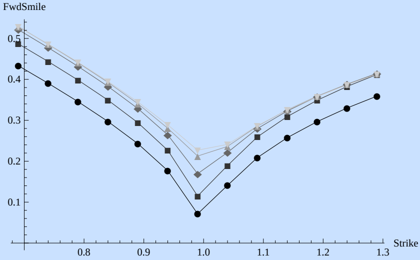
Compare with the $t = 0$ case: $\sigma_{0,\tau}^2(k) = \sigma_0^2(k) + a(k)\tau + o(\tau)$, when $k \neq 0$.

At-the-money case $k = 0$, $t > 0$.

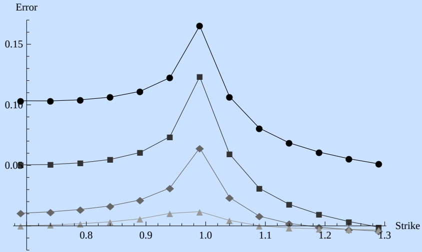
As $\tau \downarrow 0$,

$$\sigma_{t,\tau}(0) = \begin{cases} \Delta_0(t) + o(1), & \text{if } 4\kappa\theta \leq \xi^2, \\ \Delta_0(t) + \Delta_1(t)\tau + o(\tau), & \text{if } 4\kappa\theta > \xi^2. \end{cases}$$

Numerics



(a) Heston small-maturity vs Fourier inversion.



(b) Errors

Figure: Here $t = 1$ and $\tau = 1/12$. In (a) circles, squares, diamonds and triangles represent the zeroth, first, second and third-order asymptotics respectively and backwards triangles represent the true forward smile using Fourier inversion. In (b) we plot the errors.

Sketch of proof: large deviations analysis

Step 1: Find the speed of convergence

Rescaled log mgf: $\Lambda_\tau^{(t)}(u,a) := a \log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}/a\right)$.

Lemma

Let $h:\mathbb{R}_+ \to \mathbb{R}_+$ continuous with $\lim_{\tau\downarrow 0} h(\tau) = 0$ and $a > 0$. Then

(i) If $h(\tau) \equiv a\sqrt{\tau}$ then $\lim_{\tau\downarrow 0}\Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for $|u| < a/\sqrt{\beta_t}$ and ∞ otherwise;

(ii) if $\sqrt{\tau}/h(\tau) \uparrow \infty$ then $\lim_{\tau\downarrow 0}\Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for $u = 0$ and ∞ otherwise;

(iii) if $\sqrt{\tau}/h(\tau) \downarrow 0$ then $\lim_{\tau\downarrow 0}\Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for all $u \in \mathbb{R}$.

(i) is the only non-trivial zero limit and $\mathcal{D}_\Lambda = (-1/\sqrt{\beta_t}, 1/\sqrt{\beta_t})$.

From now on, define $\Lambda_\tau^{(t)}(u) := \Lambda_\tau^{(t)}(u, \sqrt{\tau})$, and $\Lambda(u) := \lim_{\tau\downarrow 0}\Lambda_\tau^{(t)}(u)$ on $\mathcal{D}_\Lambda$.
Define the Fenchel-Legendre transform $\Lambda^*(k) := \sup\{uk - \Lambda(u), u \in \mathcal{D}_\Lambda\}$.

Lemma

$\Lambda^*(k) = |k|/\sqrt{\beta_t}$, *for all* $k \in \mathbb{R}$.

Step 2: Weak convergence of (rescaled) measure

Consider the saddlepoint equation $(*)$: $\partial_u \Lambda_\tau^{(t)}(u_\tau^*(k)) = k$.

Lemma

For any $k \neq 0$, $\tau > 0$, $(*)$ admits a unique solution $u_\tau^*(k)$, and $u_\tau^*(k) = a_0(k) + a_1(k)\tau^{1/4} + a_2(k)\tau^{1/2} + a_3(k)\tau^{3/4} + \mathcal{O}(\tau) \in \mathcal{D}_\Lambda^o$, as $\tau \downarrow 0$.

For small τ, introduce the time-dependent change of measure

$$\frac{\mathrm{d}\mathbb{Q}_{k,\tau}}{\mathrm{d}\mathbb{P}} := \exp\left(\frac{u_\tau^*(k)X_\tau^{(t)}}{\sqrt{\tau}} - \frac{\Lambda_\tau^{(t)}(u_\tau^*(k))}{\sqrt{\tau}}\right).$$

Define $Z_{\tau,k} := (X_\tau^{(t)} - k)/\tau^{1/8}$ and $\Phi_{\tau,k}(u) := \mathbb{E}^{\mathbb{Q}_{k,\tau}}\left(\mathrm{e}^{\mathrm{i}uZ_{\tau,k}}\right)$.

Lemma

The following expansion holds for all $k \neq 0$ as $\tau \downarrow 0$:

$$\Phi_{\tau,k}(u) = \mathrm{e}^{-\frac{1}{2}\zeta^2(k)u^2}\left[1 + \phi_1(k,u)\tau^{1/8} + \phi_2(k,u)\tau^{1/4} + \mathcal{O}(\tau^{3/8})\right]. \qquad (1)$$

Corollary: $Z_{\tau,k}$ converges weakly to $\mathcal{N}(0,\zeta(k)^2)$ under $\mathbb{Q}_{k,\tau}$.

Step 3: Wrapping up

$$\mathbb{E}\left[\mathrm{e}^{X_\tau^{(t)}}-\mathrm{e}^k\right]^+ = \mathbb{E}^{\mathbb{Q}_{k,\tau}}\left[\frac{\mathrm{d}\mathbb{Q}_{k,\tau}}{\mathrm{d}\mathbb{P}}\left\{\mathrm{e}^{X_\tau^{(t)}}-\mathrm{e}^k\right\}^+\right] = \mathrm{e}^{\frac{\Lambda_\tau^{(t)}(u_\tau^*)}{\sqrt{\tau}}}\mathbb{E}^{\mathbb{Q}_{k,\tau}}\left[\mathrm{e}^{-\frac{u_\tau^* X_\tau^{(t)}}{\sqrt{\tau}}}\left\{\mathrm{e}^{X_\tau^{(t)}}-\mathrm{e}^k\right\}^+\right]$$

$$= \mathrm{e}^{-\frac{k u_\tau^* - \Lambda_\tau^{(t)}(u_\tau^*)}{\sqrt{\tau}}}\mathrm{e}^k\mathbb{E}^{\mathbb{Q}_{k,\tau}}\left[\mathrm{e}^{-\frac{u_\tau^* Z_{\tau,k}}{\tau^{3/8}}}\left(\mathrm{e}^{Z_{\tau,k}\tau^{1/8}}-1\right)^+\right].$$

Final steps, take the Fourier transform:

$$\mathcal{F}\left(\mathrm{e}^{-\frac{u_\tau^* Z_{\tau,k}}{\tau^{3/8}}}\left(\mathrm{e}^{Z_{\tau,k}\tau^{1/8}}-1\right)^+\right)(u) = C_{k,\tau}(u)$$

use Parseval's identity (or so):

$$\mathbb{E}\left[\mathrm{e}^{-\frac{u_\tau^* Z_{\tau,k}}{\tau^{3/8}}}\left(\mathrm{e}^{Z_{\tau,k}\tau^{1/8}}-1\right)^+\right] = \frac{1}{2\pi}\int_{\mathbb{R}}\Phi_{\tau,k}(u)\overline{C}_{k,\tau}(u)\mathrm{d}u,$$

'conclude' using (1) and a control of the tails
$\left|\int_{|u|>1/\sqrt{\varepsilon}}\Phi_{\tau,k}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u\right| = \mathcal{O}(\mathrm{e}^{-\gamma/\varepsilon})$.