Portfolio Optimization in Incomplete Markets

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Introduction-3

- Dynamic programming approach and martingale method are often used in the study of optimization problems in finance.
- The dynamic programming approach is useful for Markovian model.
- We can obtain an optimal portfolio by solving Hamilton-Jacobi-Bellman (HJB) equation.
- Martingale method is very useful when the market is complete.
  It avoids solving nonlinear (complicated) HJB equation.
- Martingale method can be applied in general (complete) diffusion models and general utility functions.

Introduction-2

In this note we want to show the possibility to use dynamic programming approach together with martingale method to solve the problem.

- We use dynamic programming approach to derive HJB equation.
- Martingale method suggests a stochastic differential game problem.
- This suggests to rewrite HJB equation as Hamilton-Jacobi-Isaacs' equation of a game problem.

Introduction-1

- An iteration procedure is used to solve the portfolio optimization problem.
- The market in each step is complete.
  Martigale method can be used.
- The limit is a solution.

The talk is based on discussions (and on going project) with Prof. Fleming (Brown University).

Optimal Consumption Problem-4

General Utility Functions

$$(C1) \qquad \sup_{(c,\pi)\in\mathcal{A}} E\left[\int_0^T e^{-\rho t} u(c_t X_t^{c,\pi})dt + e^{-\rho T} u(X_T^{c,\pi})\right]$$

u is a utility function: u is increasing and concave.

- $X_t^{c,\pi}$ is the wealth with strategy (c, π).
- π_t is a trading strategy.
  $c_t X_t^{c,\pi}$ is the rate of consumption.
- $\mathcal{A}$ is a family of admissible strategies.
- ρ is the discount factor.

Purpose: To find an optimal strategy (π_t, c_t).

Optimal Consumption Problem-3

CRRA Utility Functions:

- $u(c_t X_t^{c,\pi}) = \frac{1}{\gamma}(c_t X_t^{c,\pi})^{\gamma}, \;\; \gamma < 0, \gamma \neq 0.$
- $u(c_t X_t^{c,\pi}) = \log(c_t X_t^{c,\pi}), \gamma = 0.$

We optimize

$$(C2) \qquad \sup_{(c,\pi)\in\mathcal{A}} E\Big[\int_0^T e^{-\rho t}\frac{1}{\gamma}(c_t X_t^{c,\pi})^{\gamma} dt + e^{-\rho T}\frac{1}{\gamma}(X_T^{c,\pi})^{\gamma}\Big],$$

Optimal Consumption Problem-2

Expected Utility of Final Wealth

No consumption $c = 0$.
X_t^π is the wealth process with trading strategy π_t .

$$(E) \qquad \sup_{\pi \in \mathcal{A}} E[u(X_T^\pi)]$$

Risk Sensitive Portfolio Optimization

$$(RS) \qquad \sup_{\pi \in \mathcal{A}} E[\frac{1}{\gamma}(X_T^\pi)^\gamma],$$

The consumption problem (C2) on infinite time horizon and the problem (RS) on infinite time has close relation.

Relation between consumption problems and rsik-sensitive protfolio optimization problems can be found in the papers.

References:

H. Hata and S. J. Sheu (2012), On the HJB eqution for an optimal consumption problem: I. Existence of Solution, Vol 50, 2373-2400.

H. Hata and S. J. Sheu (2012), On the HJB eqution for an optimal consumption problem: II. Verification Theorem, Vol 50, 2401-2430.

Factor Models-4

Invest on N risky assets and a bank account.

- Prices of risky assets: $S_i(t)$, $i = 1, 2, \cdots, N$,

$$dS_i(t) = S_i(t)(\mu^i(Y(t))dt + \sigma_P^{ij}(Y(t))dB_j(t)),$$

- Bank account: interest rate is $r(Y(t))$.
- Factor process: $Y(t) = (y_1(t), \cdots, y_n(t))$.
  Dynamics:

$$dY(t) = b(Y(t))dt + \sigma_F(Y(t))dB(t).$$

Wealth Process

- Investment strategy π_t, c_t:

$$\pi(t) = (\pi_1(t), \pi_2(t), \cdots, \pi_N(t))$$

  $\pi_i(t)$ is the proportion of wealth on asset i.
  $c_t X^{\pi,c}(t)$ is the consumption rate.
  $X^{\pi,c}(t)$ is the wealth process.
- Dynamics of $X^{\pi,c}(t)$:

$$dX^{\pi,c}(t) = X^{\pi,c}(t)\{\sum_{i=1}^{N} \pi_i(t) \frac{dS_i(t)}{S_i(t)} + (1 - \sum_{i=1}^{N} \pi_i(t)) \frac{dS_0(t)}{S_0(t)} - c_t dt\}.$$

Factor Models-2

Dynamics of Wealth Process

$$dX^{\pi,c}(t) = X^{\pi,c}(t)\{(\textstyle\sum_i \pi_i(t)(\mu^i(Y(t)) - r(Y(t)) + r(Y(t)) - c(t))dt + \pi_i(t)\sigma_P^{ij}(Y_t)dB_j(t)\}.$$

Dynamics of Factor Process

$$dY(t) = b(Y(t))dt + \sigma_F(Y(t))dB(t).$$

Optimal Consumption Problem:

Find (c, π) to optimize
$(C1)'$

$$V(T, x, y) = \sup_{(c,\pi)\in\mathcal{A}} E[\int_0^T e^{-\rho t} u(c_t X_t^{c,\pi})dt + e^{-\rho T} u(X_T^{c,\pi})].$$

Factor Models-1

Thaleia Zariphopoulou(2009), Optimal asset allocation in a stochastic factor model: an overview and open problems.

HJB Equations-9

- Dynamic programming approach has been extensively used in control theory.
- HJB (Hamilton-Jacobi-Bellman) equation can de derived. HJB is a nonlinear partial differential equation.
- A solution of HJB equation gives a candidate of optimal control.
- Merton (1970) implements the idea in continuous time models to derive an optimal trading strategy for an investment problem.

HJB Equation-8

- Hamilton-Jacobi-Bellman (HJB) equation for $V(\cdot)$:
  (HJB)
  $$\begin{aligned}\frac{\partial V}{\partial t} = \quad & \frac{1}{2}\sum_{ij} a_F^{ij}(y) D_{ij} V + b(y)^* D_y V + \sup_{c,\pi}\left[u(cx) - \rho V\right. \\ & + x\pi^* \sigma_F(y)\sigma_P(y)^* D_{xy} V + \frac{1}{2} x^2 \pi^* \sigma_P(y)\sigma_P^*(y)\pi D_{xx} V \\ & \left. + x D_x V(x,y)\{r(y) + \pi^*(\mu(y) - r(y)\mathbf{1}) - c\}\right],\end{aligned}$$
  c, π are taken $c \in [0, \infty), \pi \in R^m$.
- From $V(t,x,y)$, we define $c^*(t,x,y), \pi^*(t,x,y)$ by
  $$u'(cx) = D_x V(t,x,y)$$
  $$\begin{aligned} x^2 D_{xx} V \sigma_P(y)\sigma_P(y)^* \pi + x(\sigma_F(y)\sigma_P(y) D_{xy} V(t,x,y) \\ + D_x V(t,x,y)(\mu(y) - r(y)\mathbf{1}) = 0.\end{aligned}$$
- $c^*(T-t, X(t), Y(t)), \pi^*(T-t, X(t), Y(t))$ gives a candidate of optimal strategy.

HJB Equation-7

- The verification theorem is a theorem to verify this optimality.

The final form of the HJB Equation becomes:

$$\begin{aligned}\frac{\partial V}{\partial T} = \quad & \frac{1}{2} a_F^{ij}(y) D_{y_i y_j} V + g(y)^* D_{y_i} V + rx D_x V + u^*(D_x V) - \rho V \\ & - \frac{1}{2} \frac{1}{D_{xx} V} (D_x V(\mu(y) - r(y)\mathbf{1}) + \sigma_P(y)\sigma_F^*(y) D_{xy} V)^* \\ & \qquad \cdot a_P(y)^{-1} (D_x V(\mu(y) - r(y)\mathbf{1}) + \sigma_P(y)\sigma_F^*(y) D_{xy} V).\end{aligned}$$

The initial condition is $V(0, x, y) = u(x)$.

HJB Equation-6

CRRA Utility Function

- We consider $u(x) = \frac{1}{\gamma}x^\gamma, \gamma < 1,\ \gamma \neq 0$.
- Assuming

$$V(T, x, y) = \frac{x^\gamma}{\gamma} e^{W(T,y)}.$$

The equation for $W(T, y)$:

$$(HJB') \quad \frac{\partial W}{\partial t} = \frac{1}{2} \sum_{ij} a_F^{ij}(y) D_{ij} W + H_\gamma(y, W(y), DW(y)).$$

- Candidate of optimal strategy: $\pi^*(T-t, Y(t)), c^*(T-t, Y(t))$,

$$\pi^*(t, y) = \tfrac{1}{1-\gamma} a_P^{-1}(y)((\mu(y) - r(y)\mathbf{1}) + \sigma_P(y)\sigma_F^*(y)\nabla W(t, y)),$$

$$c^*(t, y) = \exp(-\tfrac{W(t,y)}{1-\gamma}),$$

HJB Equation-5

Notations:

$$H_\gamma(y,w,p) = \hat{H}_\gamma(y,p) + (1-\gamma)\exp(-\frac{w}{1-\gamma}) - \rho.$$

$$\hat{H}_\gamma(y,p) = \frac{1}{2}\sum_{ij=1}^{m} a_\gamma^{ij}(y)p_ip_j + \sum_{i=1}^{m} b_\gamma^{i}(y)p_i + U_\gamma(y),$$

$$a_F(y) = \sigma_F(y)\sigma_F^*(y), \quad a_P(y) = \sigma_P(y)\sigma_P^*(y).$$

$$b_\gamma(y) = b(y) + \frac{\gamma}{1-\gamma}\sigma_F(y)\sigma_P(y)^* a_P(y)^{-1}(\mu(y) - r(y)\mathbf{1}),$$

$$a_\gamma(y) = a_F(y) + \frac{\gamma}{1-\gamma}\sigma_F(y)\sigma_P(y)^* a_P(y)^{-1}\sigma_P(y)\sigma_F(y)^*,$$

$$U_\gamma(y) = \frac{\gamma}{2(1-\gamma)}(\mu(y) - r(y)\mathbf{1})^* a_P(y)^{-1}(\mu(y) - r(y)\mathbf{1}) + \gamma r(y)$$

HJB Equation-4

General Untility Functions

- HJB equations for general utility functions are very complicated.
- How to study the equation is not well understood.
- Existence of the solution? Regularity of the solution?
- The candidate of an optimal strategy is a function of the derivatives of the solution.
  The regularity of the solution is an important issue.
- For the existence of the solution, we may need to use the concept of viscosity solution.

HJB Equation-3

CRRA Utility Functions

There are several recent studies:

Nagai [1996, 2003], Fleming-Sheu[1999, 2000, 2002], Fleming-Hernandez[2003], Fleming-Pang[2004], Kaise-Sheu[2004,2006], Hata-Sheu[2012a, b], Hata-Nagai-Sheu[2013], Nagai[2013].

HJB Equation-2

There is a particular case that the equation is very simple.

- $\sigma_P(y)$ is invertible. The market is complete.
- HJB equation in the case of CRRA utility function becomes

$$\begin{aligned}\frac{\partial W}{\partial t} = \quad & \frac{1}{2}\sum_{ij} a_F^{ij}(y) D_{ij} W + b_\gamma(y) DW + \frac{1}{2(1-\gamma)} DW^* a_F(y) DW \\ & +(1-\gamma) e^{-\frac{W}{1-\gamma}} + (U_\gamma(y) - \rho).\end{aligned}$$

- $\psi(t,y) = \exp(\frac{W}{1-\gamma})$ satisfies a linear equation:

$$\begin{aligned}\frac{\partial \psi}{\partial t} = \quad & \frac{1}{2}\sum_{ij} a_F^{ij}(y) D_{ij}\psi + b_\gamma(y) D\psi \\ & + \frac{1}{1-\gamma}(U_\gamma - \rho)\psi + 1.\end{aligned}$$

HJB Equation-1

The solution of this linear equation is given by:

$$\begin{aligned}\psi(T,y) = \quad & E_y\left[\int_0^T \exp\left(\frac{1}{1-\gamma}\int_0^t (U_\gamma(Y_\gamma(s)) - \rho)ds\right)dt \right.\\ & \left. + \exp\left(\frac{1}{1-\gamma}\int_0^T (U_\gamma(Y_\gamma(s)) - \rho)ds\right)\right].\end{aligned}$$

$$dY_\gamma(t) = b_\gamma(Y_\gamma(t))dt + \sigma_F(Y_\gamma(t))dB(t).$$

Market Completion-10

- The problem is difficult to solve when the market is incomplete.
- The idea of the market completion is developed in Pages[1987], He-Pearson[1990], Karatzas-Lehoczky-Shreve-Xu[1991].
- The idea of ”fictitious” completion is discussed in Karatzas-Lehoczky-Shreve-Xu[1991].

Market Completion-10

Consider the wealth process

$$\begin{aligned}\frac{dX^{\pi,c}(t)}{X^{\pi,c}(t)} = \quad & \{r(Y(t)) + \pi(t)^*(\mu(Y(t)) - r(Y(t))\mathbf{1}) \\ & -c(t))dt\} + \pi(t)^*\sigma_P(Y_t)dB_j(t).\end{aligned}$$

- We write $\bar{\pi}(t) = \sigma_P(Y(t))^*\pi(t)$.
- We define $\bar{\mu}(y) - r(y)\mathbf{1} = \sigma_P(y)^* a_P(y)^{-1}(\mu(y) - r(y)\mathbf{1})$.
- The wealth process has equation

$$\begin{aligned}\frac{dX^{\pi,c}(t)}{X^{\pi,c}(t)} = \quad & \{r(Y(t)) + \bar{\pi}(t)^*(\bar{\mu}(Y(t)) - r(Y(t))\mathbf{1}) \\ & -c(t)\}dt + \bar{\pi}(t)^* dB(t).\end{aligned}$$

Market Completion-9

From

$$\bar{\pi}(t) = \sigma_P(Y(t))^* \pi(t),$$

we have

$$\sigma_P(Y(t))\bar{\pi}(t) = \sigma_P(Y(t))\sigma_P(Y(t))\pi(t) = a_P(Y(t))\pi(t).$$

That is,

$$\pi(t) = a_P(Y(t))^{-1}\sigma_P(Y(t))\bar{\pi}(t).$$

We also have

$$\begin{aligned}
&\pi(t)^*(\mu(Y(t)) - r(Y(t))\mathbf{1}) \\
&= \bar{\pi}(t)^*\sigma_P(Y(t))^* a_P(Y(t))^{-1}(\mu(Y(t)) - r(Y(t))\mathbf{1}) \\
&= \bar{\pi}(t)^*(\bar{\mu}(Y(t)) - r(Y(t))\mathbf{1}).
\end{aligned}$$

Market Completion-8

- The weath process

$$(1) \qquad \frac{dX(t)}{X(t)} = \begin{aligned}[t] &\{r(Y(t)) + \bar{\pi}(t)^*(\bar{\mu}(Y(t)) - r(Y(t))\mathbf{1}) \\ &-c(t)\}dt + \bar{\pi}(t)^* dB(t). \end{aligned}$$

- This corresponds to market with the stocks

$$(2) \qquad \frac{d\bar{S}^j(t)}{\bar{S}^j(t)} = \bar{\mu}^j(Y(t))dt + dB_j(t), \ j = 1, 2, \cdots, d.$$

- The market with price of stocks in (2) is complete.
- The original price processes are special cases of (1) with $\bar{\pi}(t) = \sigma_P(Y(t))^* \pi(t)$.
- In some sense, we complete the market from (S) to $(\bar{S})$. Then there are more choices of the porfolio.

Market Completion-7

- We show such complete market is equivalent to a market completion by Karatzas-Lehoczky-Shreve-Xu[1991].

- They introduce the "fictitious" stocks:

$$(3)\quad \frac{d\tilde{S}^j(t)}{\tilde{S}^j(t)} = r(Y(t))dt + \sigma_f^{jk}(Y(t))dB_k(t),\ j = 1, 2, \cdots, d,$$

- $\sigma_f(y) = I - \sigma_P(y)^*(\sigma_P(y)\sigma_P(y)^*)^{-1}\sigma_P(y)$.
- Special properties of $\sigma_f(y)$:

$$\begin{aligned} (i) &\quad \sigma_f(y)\sigma_f(y) = \sigma_f(y). \\ (ii) &\quad \sigma_p(y)\sigma_f(y) = 0,\ \sigma_f(y)\sigma_P(y)^* = 0. \end{aligned}$$

- We have m stocks $S^i(t)$ and d stocks $\tilde{S}^j(t)$.
- A portfolio is given by $c, \pi, \tilde{\pi}$.
  $\pi^i(t)$ is the proportion of the wealth in $S^i(t)$.
  $\tilde{\pi}^j(t)$ is the proportion of the wealth in $\tilde{S}^j(t)$.

Market Completion-6

- The wealth process $\tilde{X}(t) = \tilde{X}^{x,c,\pi,\tilde{\pi}}(t)$ has the dynamics,

$$\begin{aligned}\frac{d\tilde{X}(t)}{\tilde{X}(t)} = & \ \{r(Y(t)) + \pi(t)^*(\mu(Y(t)) - r(Y(t))\mathbf{1}) - c(t)\}\, dt \\ & +(\sigma_P(Y(t))^*\pi(t) + \sigma_f(Y(t))\tilde{\pi}(t))^* dB_t\end{aligned}$$

- There is one-one correspondence for trading strategies for $(\overline{S}^i(t))$ and for $(S^i(t), \tilde{S}^j(t))$ with the same wealth process.
- Hence two markets are equivalent.
- For each $(\pi(t), \tilde{\pi}(t))$, we have

$$\bar{\pi}(t) = \sigma_P(Y(t))^*\pi(t) + \sigma_f(Y(t))\tilde{\pi}(t).$$

- Using properties of σ_f above, we have

$$\sigma_P(Y(t))\bar{\pi}(t) = \sigma_P(Y(t))\sigma_P(Y(t))^*\pi(t) = a_P(Y(t))\pi(t).$$

  Then

$$\pi(t) = a_P(Y(t))^{-1}\sigma_P(Y(t))\bar{\pi}(t).$$

- We can see

$$\tilde{X}^{x,c,\pi,\tilde{\pi}}(t) = \bar{X}^{x,c,\bar{\pi}}(t).$$

Market Completion-5

The First Observation:

- Market with price $(\bar{S})$ (or $(S, \tilde{S})$) is complete.
  The wealth process is $\bar{X}^{\bar{\pi},c}$ (or $\bar{X}^{\pi,\tilde{\pi},c}$).
- Market with price (S) is not complete.
  The wealth process is $X^{\pi,c}$.
- For each π, we take $\bar{\pi}(t) = \sigma_P(Y(t))\pi(t)$, we have

$$X^{\pi,c} = \bar{X}^{\bar{\pi},c}.$$

- The value function satisfies
  $V(T,x,y) \leq \bar{V}(T,x,y) = \tilde{V}(T,x,y)$.

Market Completion-4

The Second Observation:

- $(S, \tilde{S})$ is a special case of "fictitious" completion of the market in Karatzas-Lehoczky-Shreve-Xu[1991].
- Let $\theta(t)$ be a R^d-valued progressive measurable process.
- Using this, we can construct additional stocks for trading.

$$\frac{d\tilde{S}^j_\theta(t)}{\tilde{S}^j_\theta(t)} = (r(Y(t)) + (\sigma_f(Y(t))\theta(t))^j)dt + \sigma_f(Y(t))^{jk}dB_k(t)),$$

$j = 1, 2, \cdots, d$.

- $\pi^i(t)$ is the proportion of wealth on S^istock. $\tilde{\pi}^j(t)$ the proportion of the wealth on the $\tilde{S}^j_\theta$ stock.

Market Completion-3

- The wealth process $\tilde{X}_\theta(t) = \tilde{X}^{x,c,\pi,\tilde{\pi}}(t)$, has the dynamics,

$$\begin{aligned}\frac{d\tilde{X}_\theta(t)}{\tilde{X}_\theta(t)} = & \{r(Y(t)) + (\pi(t)^*(\mu(Y(t)) - r(Y(t))\mathbf{1}) + \tilde{\pi}(t)^*\sigma_f(Y(t))\theta_t \\ & -c(t)\}dt + (\sigma_P(Y(t))^*\pi_t + \sigma_f(Y(t))\tilde{\pi}(t))^*dB(t)\end{aligned}$$

- Denote $\bar{\pi}(t) = \sigma_P(Y(t))^*\pi(t) + \sigma_f(Y(t))\tilde{\pi}(t)$.
- The dynamics is rewritten as

$$\begin{aligned}\frac{d\tilde{X}_\theta(t)}{\tilde{X}_\theta(t)} = & \{r(Y(t)) + (\bar{\pi}(t)^*(\bar{\mu}(Y(t)) - r(Y(t))\mathbf{1}) \\ & +\bar{\pi}(t)^*\sigma_f(Y(t))\theta(t) - c(t)\}\,dt + \bar{\pi}(t)^*dB(t)\end{aligned}$$

Market Completion-2

- The dynamices for the wealth process $\tilde{X}_\theta(t) = \tilde{X}^{x,c,\pi,\tilde{\pi}}(t)$:

$$\frac{d\tilde{X}_\theta(t)}{\tilde{X}_\theta(t)} = \begin{aligned}[t] &\{r(Y(t)) + (\pi(t)^*(\mu(Y(t)) - r(Y(t))\mathbf{1}) + \tilde{\pi}(t)^*\sigma_f(Y(t))\theta_t \\ &-c(t)\}dt + (\sigma_P(Y(t))^*\pi_t + \sigma_f(Y(t))\tilde{\pi}(t))^* dB(t) \end{aligned}$$

- This is the wealth process for the market with stock price dynamics

$$\frac{d\bar{S}^j_\theta(t)}{\bar{S}^j_\theta(t)} = (\bar{\mu}^j(Y(t)) + (\sigma_f(Y(t))\theta(t))^j)dt + dB_j(t), \; j = 1, 2, \cdots, d.$$

- Markets with stock prices $(S, \tilde{S}_\theta)$ and $(\bar{S}_\theta)$ are equivalent.

Market Completion-1

- We denote $\bar{V}^{\theta}$ the value function for the consumption problem in θ-market.
- Karatzas-Lehoczky-Shreve-Xu[1991] uses the duality argument to derive the fundamental relation:

$$V(T,x,y) = \inf_{\theta} \bar{V}^{\theta}(T,x,y).$$

Issacs' Equation-7

The relation

$$V(T,x,y) = \inf_\theta \bar{V}^\theta(T,x,y).$$

suggests the following result.

Lemma 1. The HJB equation for $V(T,x,y)$ can be rewritten as

$$\begin{aligned}\frac{\partial V}{\partial T} = \quad & \tfrac{1}{2} a_F(y)^{ij} D_{y_i y_j} V + b(y)^* D_{y_i} V + xr(y) D_x V \\ & + \inf_\theta sup_{\bar{\pi},c}\{u(xc) - \rho V + \tfrac{1}{2}|\bar{\pi}|^2 x^2 D_{xx} V \\ & + x(\sigma_F(y)\bar{\pi})^* D_{xy} V + x(\bar{\pi}^*(\bar{\mu}(y) - r(y)\mathbf{1} + \sigma_f(y)\theta) - c) D_x V\}.\end{aligned}$$

HJB equation:

$$\begin{aligned}\frac{\partial V}{\partial t} = \quad & \tfrac{1}{2}\textstyle\sum_{ij} a_F^{ij}(y) D_{ij} V + b(y)^* D_y V + \sup_{c,\pi}\left[u(cx) - \rho V\right. \\ (HJB) \qquad & + x\pi^* \sigma_F(y)\sigma_P(y)^* D_{xy} V + \tfrac{1}{2} x^2 \pi^* \sigma_P(y)\sigma_P^*(y)\pi D_{xx} V \\ & \left. + x D_x V(x,y)\{r(y) + \pi^*(\mu(y) - r(y)\mathbf{1}) - c\}\right],\end{aligned}$$

Issacs' Equation-6

- This is the Issacs' equation for a stochastic differential game.
- We can also verify Issacs' condition. That is,

$$\inf_{\theta} \sup_{c,\pi} \{\cdots\} = \sup_{c,\pi} \inf_{\theta} \{\cdots\}.$$

Issacs' Equation-5

The θ to take inf will be important for the following analysis, we give a proof of Lemma 1.

- In $\inf_\theta \sup_{c,\pi}\{\cdots\}$, we fix θ. Take supremum on c, π. The result is

$$\frac{1}{-2D_{xx}V}|\sigma_F(y)^* D_{xy}V + ((\bar{\mu}(y) - r(y)\mathbf{1}) + \sigma_f(y)\theta)D_xV|^2$$

$$+xr(y)D_xV + u^*(D_xV) - \rho V.$$

- The $\bar{\pi}$ taking supremum is equal to

$$\bar{\pi} = -\frac{1}{D_{xx}V}(\sigma_F(y)^* D_{xy}V + ((\bar{\mu}(y) - r(y)\mathbf{1}) + \sigma_f(y)\theta)D_xV).$$

- We need condition $D_{xx}V < 0$. $V(T, x, y)$ is concave in x.

Issacs' Equation-4

- We now choose θ to minimize

$$\frac{1}{-2D_{xx}V}|\sigma_F(y)^* D_{xy}V + ((\bar{\mu}(y) - r(y)\mathbf{1}) + \sigma_f(y)\theta)D_xV|^2 \\ +xr(y)D_xV + u^*(D_xV) - \rho V.$$

- We write

$$\sigma_F(y)^* D_{xy}V + (\bar{\mu}(y) - r(y)\mathbf{1})D_xV = \sigma_P(y)^*\pi + \sigma_f(y)\tilde{\pi}.$$

  π and $\tilde{\pi}$ will be given later.

- We need to minimize

$$|\sigma_P(y)^*\pi + \sigma_f(y)(\tilde{\pi} + \theta D_xV)|^2 \\ = |\sigma_P(y)^*\pi|^2 + |\sigma_f(y)(\tilde{\pi} + \theta D_xV)|^2.$$

  Here we use $\sigma_P\sigma_f = 0 = \sigma_f\sigma_P^*$.

Issacs' Equation-3

- The minimum satisfies $\tilde{\pi} + \theta D_x V = 0$.

$$\theta = -\frac{1}{D_x V}\tilde{\pi}$$

- $\inf_\theta \sup_{c,\pi}\{\cdots\}$ is equal to

$$\frac{1}{-2D_{xx}V}|\sigma_P(y)^*\pi|^2 + xr(y)D_x V + u^*(D_x V) - \rho V.$$

- $\pi, \tilde{\pi}$ solves

$$\sigma_F(y)^* D_{xy} V + (\bar{\mu}(y) - r(y)\mathbf{1})D_x V = \sigma_P(y)^*\pi + \sigma_f(y)\tilde{\pi}.$$

Issacs' Equation-2

- We now solve the following relation for $\pi, \tilde{\pi}$:

$$\sigma_F(y)^* D_{xy} V + (\bar{\mu}(y) - r(y)\mathbf{1}) D_x V = \sigma_P(y)^* \pi + \sigma_f(y)\tilde{\pi}.$$

- We have

$$\pi = a_P(y)^{-1} \sigma_P(y)(\sigma_F(y)^* D_{xy} V + (\bar{\mu}(y) - r(y)\mathbf{1}) D_x V)$$

$$\begin{aligned} \sigma_f(y)\tilde{\pi} &= \sigma_f(y)(\sigma_F(y)^* D_{xy} V + (\bar{\mu}(y) - r(y)\mathbf{1}) D_x V) \\ &= \sigma_f(y)\sigma_F(y)^* D_{xy} V. \end{aligned}$$

- In the second relation, we use

$$\sigma_f(y)(\bar{\mu}(y) - r(y)\mathbf{1}) = 0.$$

This follows from the definition of

$$\bar{\mu}(y) - r(y)\mathbf{1} = \sigma_P(y) a_P(y)^{-1}(\mu(y) - r(y)\mathbf{1})$$

and $\sigma_f \sigma_P = 0 = \sigma_P^* \sigma_f$.

Issacs' Equation-1

- We use the expression of π to check Issacs' equation.
- We use the expression of $\tilde{\pi}$ to obtain

$$\theta = -\frac{1}{D_x V}\sigma_f(y)\sigma_F(y)^* D_{xy} V.$$

Iteration Procedure-10

In the following, we want to show a use of this game interpretation of HJB equation.

- In the rest we consider CRRA utility functions:

$$u(x) = u_\gamma(y) = \frac{1}{\gamma} x^\gamma,$$

  where $\gamma < 1$ and for $\gamma = 0$ we understand $u_0(x) = \log x$.
- In the following, we consider $0 < \gamma < 1$.
  The case $\gamma < 0$ can be similarly treated.

Iteration Procedure-9

- In this case the value function of the consumption problem has the form

$$V(T, x, y) = \frac{1}{\gamma} x^{\gamma} \exp(W(T, y)).$$

- The equation for $W(T, y)$ is given by

$$\begin{aligned}\frac{\partial W}{\partial T} = \quad & \tfrac{1}{2} a_F(y)^{ij} D_{y_i y_j} W + \tfrac{1}{2} Q_{\gamma}(y, DW) \\ & + U_{\gamma}(y) - \rho + (1-\gamma)\exp(-\tfrac{W}{1-\gamma}).\end{aligned}$$

$$W(0, y) = 1.$$

- Here

$$Q_{\gamma}(y, p) = p^* \sigma_F(y)(I + \tfrac{\gamma}{1-\gamma} \sigma_P(y)^* a_P(y)^{-1} \sigma_P(y)) \sigma_F(y)^* p,$$

$$U_{\gamma}(y) = \tfrac{\gamma}{2(1-\gamma)} (\mu(y) - r(y)\mathbf{1})^* a_P(y)^{-1} (\mu(y) - r(y)\mathbf{1}) + \gamma r(y).$$

Iteration Procedure-8

- We return to the fundamental relation $V(T, x, y) = \inf_\theta \overline{V}(T, x, y)$.
- Is it possible to choose θ in a smart way that the value is close to $V(T, x, y)$?
- We shall try an iteration procedure so that we can gradually reach this purpose.

Iteration Procedure-7

- We now consider the Markovian fictitious completion, $\theta(t) = \theta(t, Y(t))$.
- The stock price dynamics

$$\frac{d\bar{S}^j_\theta(t)}{\bar{S}^j_\theta(t)} = (\bar{\mu}^j(Y(t)) + (\sigma_f(Y(t))\theta(t, Y(t)))^j)dt + dB_j(t),\ j = 1, 2, \cdots, d.$$

- Factor process dynamics

$$dY(t) = b(Y(t))dt + \sigma_F(Y(t))dB(t).$$

Iteration Procedure-6

- The wealth process dynamics :

$$\frac{d\tilde{X}_\theta(t)}{\tilde{X}_\theta(t)} = \{r(Y(t)) + (\bar{\pi}(t)^*(\bar{\mu}(Y(t)) - r(Y(t))\mathbf{1}) + \bar{\pi}(t)^*\sigma_f(Y(t))\theta(t) - c(t)\}\, dt + \bar{\pi}(t)^* dB(t).$$

- The value function is

$$\overline{V}^{(\theta)}(t,x,y) = \sup_{\pi,c} E_{t,y}[\int_t^T \exp(-\rho(t-s))\frac{1}{\gamma}(c(s)X^{c,\bar{\pi}}(s))^\gamma ds + \exp(-\rho(T-t))\frac{1}{\gamma}(X^{c,\bar{\pi}}(T))^\gamma].$$

- Here $E_{t,y}[\cdots]$ indicates $Y(t) = y$.

Iteration Procedure-5

- Assume
$$\overline{V}^{\theta}(t,x,y) = \frac{1}{\gamma} x^{\gamma} \exp(\overline{W}^{\theta}(t,y)).$$
- The HJB equation: **(HJB-θ)**
$$\frac{\partial \overline{W}^{\theta}}{\partial t} + \frac{1}{2} a_F(y)^{ij} D_{y_i y_j} \overline{W}^{\theta} + \frac{1}{2(1-\gamma)} a_F(y)^{ij} D_{y_i} \overline{W}^{\theta} D_{y_j} \overline{W}^{\theta}$$
$$+ b_{\gamma}^{\theta}(t,y)^* D_y \overline{W}^{\theta} + U_{\gamma}^{\theta}(t,y) - \rho + (1-\gamma) \exp(-\frac{\overline{W}^{\theta}}{1-\gamma}) = 0$$
- Here
$$b_{\gamma}^{\theta}(t,y) = b(y) + \frac{\gamma}{1-\gamma} \sigma_F(y)(\overline{\mu}(y) - r(y)\mathbf{1} + \sigma_f(y)\theta(t,y))$$
$$U_{\gamma}^{\theta}(t,y) = \frac{\gamma}{2(1-\gamma)} |\overline{\mu}(y) - r(y)\mathbf{1} + \sigma_f(y)\theta(t,y)|^2 + \gamma r(y).$$

Iteration Procedure-4

- Define
$$\overline{\psi}^{\theta}(t,y) = \exp(\frac{1}{1-\gamma}\overline{W}^{\theta}(t,y)).$$
- We have
$$\frac{\partial\overline{\psi}^{\theta}}{\partial t} + \frac{1}{2}a_F(y)^{ij}D_{y_iy_j}\overline{\psi}^{\theta} + b_{\gamma}^{\theta}(t,y)^*D_y\overline{\psi}^{\theta} + \frac{1}{1-\gamma}(U_{\gamma}^{\theta}(t,y)-\rho)\overline{\psi}^{\theta} + 1 = 0.$$
$$\overline{\psi}^{\theta}(T,y) = 1.$$
- A solution is given by
$$\begin{aligned}\overline{\psi}^{\theta}(t,y) = \quad & E_{t,y}[\int_t^T \exp(\frac{1}{1-\gamma}\int_t^s(U_{\gamma}(u,Y^{\gamma}(u))-\rho)du)ds \\ & + \exp(\frac{1}{1-\gamma}\int_t^T(U_{\gamma}(u,Y^{\gamma}(u))-\rho)du)].\end{aligned}$$
- Here
$$dY^{\gamma}(s) = b_{\gamma}^{\theta}(s,Y^{\gamma}(s))ds+\sigma_F(Y^{\gamma}(s))dB(s), t<s, \quad Y^{\gamma}(t) = y.$$

Iteration Procedure-3

- Assume a smooth solution $\overline{\psi}^{\theta}(t, y)$.
  A candidate of the optimal strategy is given by

$$\overline{c}^{\theta}(t) = \exp(-\tfrac{1}{1-\gamma}\overline{W}^{\theta}(t, Y(t))$$

$$\overline{\pi}^{\theta}(t) = \tfrac{1}{1-\gamma}((\overline{\mu}(Y(t)) - r(Y(t))\mathbf{1}) + \sigma_f(Y(t))\theta(t, Y(t)) \\ +\sigma_F(Y(t))^* D\overline{W}^{\theta}(t, Y(t))).$$

- Now we want to use the solution for θ-market to switch to another θ so that we can get a better value (for the approximation of the original problem).

Iteration Procedure-2

- Issacs' equation suggests to change to

$$\theta^{(1)}(t,y) = -\sigma_f(y)\sigma_F(y)^* D\overline{W}^{\theta}(t,y).$$

- With this particular $\theta(t,y) = \theta^{(1)}(t,y)$, the value function is denoted by

$$\overline{V}_1(t,x,y) = \frac{1}{\gamma}x^{\gamma}\exp(\overline{W}_1(t,y)).$$

- $\overline{W}_1(t,y) := \overline{W}^{\theta^{(1)}}(t,y)$.

Iteration Procedure-1

- $\overline{W}_1$ satisfies (HJB-$\theta^{(1)}$):

$$\frac{\partial \overline{W}_1}{\partial t} + \frac{1}{2} a_F(y)^{ij} D_{y_i y_j} \overline{W}_1 + \frac{1}{2(1-\gamma)} a_F(y)^{ij} D_{y_i} \overline{W}_1 D_{y_j} \overline{W}_1$$
$$+ b^{(1)}(t,y)^* D_y \overline{W}_1 + U^{(1)}(t,y) - \rho + (1-\gamma)\exp\left(-\frac{\overline{W}_1}{1-\gamma}\right) = 0.$$

- Here

$$b^{(1)}(t,y) = b_\gamma^\theta(y), \; U^{(1)}(t,y) = U_\gamma^\theta(y)$$

with $\theta = \theta^{(1)}(t,y)$.

Lemma 2. $\overline{W}^\theta(t,y)$ is a supersolution (subsolution) of (HJB-$\theta^{(1)}$) if $\gamma > 0$ (if $\gamma < 0$).

Assumption (A)

(A1) $\sigma_F^{ij}(\cdot), \sigma_P^{ij}(\cdot), b^i(\cdot), \mu^i(\cdot), r(\cdot)$ are smooth functions.

(A2) For any $r > 0$, there are $c_1(r), c_2(r) > 0$ such that for all $y \in B_r$, we have

$$c_1(r)I_m \leq \sigma_P(y)\sigma_P(y)^* \leq c_2(r)I_m,$$
$$c_1(r)I_n \leq \sigma_F(y)\sigma_F(y)^* \leq c_2(r)I_n.$$

(A3) $r(y) > 0$ for all y.

Theorem 2. Assume (A), $\theta(t,y)$ is a smooth function such that $\theta(T,y)=0$ for all y. Assume HJB equation has smooth subsolution $\underline{W}$ and (HJB-θ) has a smooth solution $\overline{W}^{\theta}$ such that

$$\underline{W}(T-t,y)\le \overline{W}^{\theta}(t,y),\ \ t\in[0,T].$$

We take

$$\theta^{(1)}(t,y)=-\sigma_f(y)\sigma_F(y)^* D\overline{W}^{\theta}(T-t,y).$$

Then (HJB-$\theta^{(1)}$) has a smooth solution $\overline{W}^{\theta^{(1)}}$ satisfying

$$\underline{W}(T-t,y)\le \overline{W}^{\theta^{(1)}}(t,y)\le \overline{W}^{\theta}(t,y).$$

Theorem 3. Assume (A), $\theta(t, y)$ is a smooth function such that $\theta(T, y) = 0$ for all y. Assume (HJB) has subsolution $\underline{W}$ and (HJB-θ) has a solution $\overline{W}^{\theta}$ such that

$$\underline{W}(T - t, y) \leq \overline{W}^{\theta}(t, y), \; t \in [0, T].$$

Then (HJB) has a solution

$$\underline{W}(T - t, y) \leq W(t, y) \leq \overline{W}^{\theta}(t, y).$$

Assumption (A)'

(A1)' $\sigma_F^{ij}(\cdot), \sigma_P^{ij}(\cdot), b^i(\cdot), \mu^i(\cdot), r(\cdot)$ are smooth functions.

(A2)' There are $c_1, c_2 > 0$ such that for all $y \in R^n$

$$
\begin{aligned}
c_1 I_m \le \sigma_P(y)\sigma_P(y)^* \le c_2 I_m, \\
c_1 I_n \le \sigma_F(y)\sigma_F(y)^* \le c_2 I_n.
\end{aligned}
$$

(A3)' There is $r_0 > 0$ such that $r(y) \ge r_0$ for all y.

(A4)' The functions in (A1)' are bounded with bounded first order derivatives.

Results-3

- We consider $\theta = 0$.
- The (HJB-0) is given by

$$\begin{aligned}\frac{\partial W}{\partial T} = \quad & \tfrac{1}{2} a_F(y)^{ij} D_{y_i y_j} W + \tfrac{1}{2(1-\gamma)} a_F(y)^{ij} D_{y_i} W D_{y_j} W \\ & + b_\gamma(y)^* D_y W + U_\gamma(y) - \rho(1-\gamma)\exp(-\tfrac{W}{1-\gamma}).\end{aligned}$$

- Here

$$\begin{aligned} b_\gamma(y) &= b(y) + \tfrac{\gamma}{1-\gamma}\sigma_F(y)(\overline{\mu}(y) - r(y)\mathbf{1}) \\ U_\gamma(y) &= \tfrac{\gamma}{2(1-\gamma)}|\overline{\mu}(y) - r(y)\mathbf{1}|^2 + \gamma r(y).\end{aligned}$$

Lemma 4. We assume the condition (A)'. Then (HJB-0) has a solution given by $W_0(T,y) = (1-\gamma)\log\psi_0(T,y)$, where

$$\begin{aligned}\psi_0(T,y) = \quad & E[\int_0^T \exp(\tfrac{1}{1-\gamma}\int_0^t (U_\gamma(Y(s)) - \rho)ds) \\ & + \exp(\tfrac{1}{1-\gamma}\int_0^T (U_\gamma(Y(s)) - \rho)ds)].\end{aligned}$$

$W_0(T,y)$ satisfies

$$W_0(T,y) \leq (1-\gamma)(\log(\frac{1+c}{c} + cT),$$

where $c = \sup_y\{\frac{1}{1-\gamma}U_\gamma(y)\}$.
(HJB) has a subsolution given by

$$\underline{W}(T,y) = (1-\gamma)\log\left\{\frac{1-\gamma}{\rho}(1-\exp(-\frac{\rho}{1-\gamma}T)) + \exp(-\frac{\rho}{1-\gamma}T)\right\}.$$

Moreover, we have $\underline{W}(T,y) \leq W_0(T,y),\ y \in R^n,\ T > 0.$

Results-1

Theorem 5. Assume (A)', we can define recursively $\theta_n(t, y)$ and W_n such that $\theta_0 = 0$,

$$\theta_{n+1}(t, y) = -\sigma_f(y)\sigma_F(y)^* DW_n(t, y),$$

$$\begin{aligned} W_n(t, y) = \quad & E_{t,y}[\int_t^T \exp(\tfrac{1}{1-\gamma} \int_t^s (U^{(n)}(u, Y^{(n)}(u)) - \rho)du)ds \\ & + \exp(\tfrac{1}{1-\gamma} \int_t^T (U^{(n)}(u, Y^{(n)}(u)) - \rho)du)], \end{aligned}$$

where

$$U^{(n)}(s, y) = U_\gamma^{\theta_n}(s, y), b^{(n)}(s, y) = b_\gamma^{\theta_n}(s, y)$$

$$dY^{(n)}(s) = b^{(n)}(s, Y^{(n)}(s))ds + \sigma_F(Y^{(n)}(s))dB(s).$$

Then W_n converges to W. W is the unique solution of (HJB) such that W is bounded.

Remarks

- Hata-Nagai-Sheu [2013] proves the existence and uniqueness of HJB equation under some condition of the cofficients.
- Similar ideas may be applied for general utility functions.
- In such case, the equation (HJB-θ) may not have smooth solution even if θ is Markovian and smooth.
  We may need to use martingale method (for complete market) to find solution for each Markovian θ.
- The detail analysis shall be interesting.
- The incompleteness due to the portfolio constrains or transaction costs may be also interesting cases to implement such idea.