Utility Maximization with Floor Constraint

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$$\sup_{(\pi,c)} \mathbb{E}\left[\int_0^T \mathrm{e}^{-\rho t} U_1(c_t)dt + \mathrm{e}^{-\rho T} U_2\left(X_T^{x,\pi,c}\right)\right],$$

where $X^{x,\pi,c}$ is the sol. to

$$dX_t = \sum_{i=1}^n \pi_t^i \frac{dS_t^i}{S_t^i} + \left(X_t - \sum_{i=1}^n \pi_t^i\right) \frac{dS_t^0}{S_t^0} - c_t dt,$$

$X_0 = x$: budget constraint

Here,

- x: initial wealth,
- $\pi := (\pi_t)_{t\in[0,T]}$: dynamic investment strategy,
- $c := (c_t)_{t\in[0,T]}$: consumption plan,
- $\rho(\geq 0)$: discount rate, U_1, U_2: utility functions.

Merton Prob. subject to

- floor constraint:

$$X_t^{x,\pi,c} \geq K_t \quad \text{for all } t \in [0,T],$$

  - $(K_t)_{t\in[0,T]}$: floor process (given, adapted).

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- (generalized) drawdown constraint:

$$X_t \geq g\left(X_t, \bar{X}_t\right) \quad \text{for all } t \in [0,T],$$

where

  - $\bar{X}_t := \sup_{s\in[0,t)} X_s$,
  - $g(\cdot,\cdot) : \mathbb{R}_+^2 \to \mathbb{R}$.

- Asset Management
  - portfolio insurance + optimization
  - protected drawdown-based performance + optimization

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  - protected drawdown-based performance + optimization
- Mathematical
  - stochastic control with state constraint
    - e.g., Ishii and Loreti (2002): analysis of HJB eq. via viscosity approach
    - difficulty in constructing optimal control
  - El Karoui, Jeanblanc and Lacoste (2005), El Karoui and Meziou (2008): $U_1 \equiv 0$ (no consumption) + Complete market case.
    - probabilistic approach,
    - optimality from stochastic order relation.

- Simplified long-term asymptotic criterion:

$$\sup_{\pi} \overline{\lim_{T\to\infty}} \frac{1}{T} \log \mathbb{E} U(X_T^{x,\pi}) \quad (\log x = -\log(-x) \text{ if } x < 0)$$

- Long-term risk-sensitive criterion:

$$\text{(RSPO)} \qquad \sup_{\pi} \overline{\lim_{T\to\infty}} C_T^{x,\pi}, \quad C_T^{x,\pi} := \frac{1}{\gamma T} \log \mathbb{E}(X_T^{x,\pi})^{\gamma}.$$

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"Risk-sensitized" modification of

$$\sup_{\pi} \overline{\lim_{T\to\infty}} \mathbb{E} G_T^{x,\pi}, \quad G_T^{x,\pi} := \frac{1}{T} \log X_T^{x,\pi},$$

recalling cumulant expansion:

$$C_T^{x,\pi} = \frac{1}{\gamma T} \log \mathbb{E} e^{\gamma T G_T^{x,\pi}} = \mathbb{E} G_T^{x,\pi} + \frac{\gamma}{2} \mathrm{var}[\sqrt{T} G_T^{x,\pi}] + O(\gamma^2).$$

- The scale-invariant property,

$$\overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}(xX)^\gamma = \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}X^\gamma, \quad \forall x \in \mathbb{R}_{++},$$

$$\overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}(NX)^\gamma = \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}(X)^\gamma \quad \forall N(>0)\text{: bdd r.v.},$$

disregarding a bounded-factor.

- The translation-invariant property,

$$\overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}e^{\gamma(\log x + \log X)} = \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}e^{\gamma \log X} \quad \forall \log x \in \mathbb{R}.$$

§1 Introduction

§2 RSPO with floor constraint

- PI techniques + optimization

§3 RSPO with drawdown constraint

- Azéma-Yor technique works for *discounted* problem

§4 A dual approach to Merton problem with floor constraint

- To overcome RSPO's drawbacks..
- To treat consumption maximization via a systematic approach..

$$\sup_{\pi} \varlimsup_{T\to\infty} \frac{1}{\gamma T} \log \mathbb{E}\left(X_T^{x,\pi}\right)^{\gamma} \quad \text{s.t.} \quad X_t^{x,\pi} \geq K_t \quad \text{for all } t \geq 0$$

- to reduce computational difficulties.
- to consider applications of other PI techniques.

RSPO: Review

- Without Floor: Bielecki and Pliska (1999), Fleming and Sheu (2000, 2002), Kuroda and Nagai (2002), Nagai (2003), Davis and Lleo (2008), Hata and S (2013), etc.
- With Floor: S (2012), Cherny and Obloj (2013).
- Large Deviations Controls:
  (Heuristics: Bielecki and Pliska, Stutzer).
  Pham (2003a-b), Hata and S (2005, 2010), Hata, Nagai and Sheu (2010), Nagai (2012a-b), Knispel (2012), etc.

- $S^0 := (S^0_t)_{t\geq 0}$: bank account process, nondecreasing adapted s.t. $S^0_0 \equiv 1$.
- $S := (S^1, \dots, S^n)^\top$: price process of n-risky assets,

$$dS^i_t = S^i_{t-}\left(dR^i_t + \frac{dS^0_t}{S^0_t}\right), \quad S^i_0 \in \mathbb{R},$$

with the excess return, $R := (R^1, \dots, R^n)^\top$, a given n-dim. semimartingale s.t. $R_0 \equiv 0$.

$$S^i_t = S^i_0 S^0_t \mathcal{E}(R^i)_t, \quad \text{where}$$

$$\mathcal{E}(R^i)_t := \exp\left(R^i_t - \frac{1}{2}[R^i]^c_t\right)\prod_{s\leq t}(1+\Delta R^i_s)\mathrm{e}^{-\Delta R^i_s}.$$

$$\begin{aligned} dX_t^{x,\pi} =& X_{t-}^{x,\pi}\left[\sum_{i=1}^{n}\pi_t^i\frac{dS_t^i}{S_{t-}^i}+\left(1-\sum_{i=1}^{n}\pi_t^i\right)\frac{dS_t^0}{S_t^0}\right] \\ X_0^{x,\pi} =& x. \end{aligned}$$

- $x \in \mathbb{R}_{>0}$: initial wealth,
- $\pi := (\pi_t)_{t\geq 0}$, $\pi_t := (\pi_t^1, \ldots, \pi_t^n)^\top$: dynamic investment policy.

$$\begin{aligned} dX_t^{x,\pi} =& X_{t-}^{x,\pi}\left(\pi_t^\top dR_t + \frac{dS_t^0}{S_t^0}\right), \quad X_0^{x,\pi} = x, \\ X_t^{x,\pi} =& x S_t^0 \mathcal{E}\left(\int \pi^\top dR\right)_t. \end{aligned}$$

First, consider

(RSPO) $$\bar{\Gamma} := \sup_{\pi \in \mathscr{A}_0} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}\left(X_T^{x,\pi}\right)^\gamma$$

without floor, assuming

Hypothesis:

(1) $\mathscr{A}_0 \ni 0$, *and (predictably) convex,*

(2) (RSPO) *has an x-independent sol., that is,* $^\exists\hat{\pi} \in \mathscr{A}_0$ *so that* $\hat{X} := X^{1,\hat{\pi}}$ *satisfies*

$$\bar{\Gamma} := \sup_{\pi \in \mathscr{A}_0} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}\left(X_T^{x,\pi}\right)^\gamma = \sup_{\pi \in \mathscr{A}_0} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}\left(X_T^{1,\pi}\right)^\gamma$$
$$= \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}\left(\hat{X}_T\right)^\gamma.$$

Ex. 1: Black-Scholes-type Model

$S^0 \equiv 1,$

$$dS_t = \mathrm{diag}(S_t)\left(\mu dt + \sigma dw_t\right),$$

- w: n-dim. BM,
- $\mu \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^{n\times n}$.
- S is a multi-dimensional GBM.
- $\mathscr{A} := \mathscr{L}_{2,n}$, where

$$\mathscr{L}_{2,n} := \left\{ (\pi_t)_{t\geq 0} \;\middle|\; \begin{array}{l} n\text{-dim. prog. m'ble,} \\ \int_0^T |\pi_t|^2 dt < \infty \text{ for } {}^\forall T > 0 \end{array} \right\}.$$

- $\hat{\pi}^{(\gamma)} \equiv \dfrac{1}{1-\gamma}(\sigma\sigma^\top)^{-1}\mu.$

Ex. 2: Linear-Gaussian Factor Model

$S^0 \equiv 1,$

$$\begin{aligned} dS_t =& \mathrm{diag}(S_t)\left\{\mu(Y_t)dt + \Sigma dW_t\right\}, \\ dY_t =& (b_0 + B_1 Y_t)dt + \Lambda dW_t, \quad Y_0 \in \mathbb{R}^m, \end{aligned}$$

- W: $m+n$-dim. BM.
- $\Sigma \in \mathbb{R}^{n\times(m+n)}$, $\Lambda \in \mathbb{R}^{m\times(m+n)}$, $b_0 \in \mathbb{R}^m$, $B_1 \in \mathbb{R}^{m\times m}$.
- $\mu(y) := m_0 + M_1 y$, $m_0 \in \mathbb{R}^n$, $M_1 \in \mathbb{R}^{n\times m}$.
- $\mathscr{A} := \mathscr{L}_{2,n}$.
- Long-term RSPO $=$ LEQG with infinite horizon.
- $\hat{\pi}_t^{(\gamma)} := \dfrac{1}{1-\gamma}(\Sigma\Sigma^\top)^{-1}\left\{\mu(Y_t) + \gamma(\Sigma\Lambda^\top)\left(\hat{q} + \hat{Q}Y_t\right)\right\}.$

Ex. 3: Wishart-Autoregressive Factor Model

$S^0 \equiv 1,$

$$dS_t = \text{diag}(S_t)\left\{\mu(Y_t)dt + \Sigma\sqrt{Y_t}dw_t\right\},$$
$$dY_t = \left(LL^\top + KY_t + Y_tK^\top\right)dt + \sqrt{Y_t}dB_t\Lambda^\top + \Lambda dB_t^\top\sqrt{Y_t}.$$

- (B, z) is a $(d \times d + d)$-dim. BM.
  $w_t := B_t\rho + \sqrt{1 - |\rho|^2}z_t$: d-dim. BM. $\rho \in \mathbb{R}^d$ s.t. $|\rho| \leq 1$.
- $Y_0 \in \mathbb{S}^d_{++} := \{M \in \mathbb{S}^d;\ M > 0\}$.
- $\Sigma \in \mathbb{R}^{n\times d}$, $L, K, \Lambda \in \mathbb{R}^{d\times d}$.
- $\mu(y) := (\Sigma y \Sigma^\top)\lambda$, $\lambda \in \mathbb{R}^n$.
- $\mathscr{A} := \{\pi \in \mathscr{L}_{2,n};\ \text{bounded}\}$.
- $\hat{\pi}^{(\gamma)} :\equiv \dfrac{1}{1-\gamma}(\Sigma\Sigma^\top)^{-1}\left(\lambda + 2\gamma\hat{Q}\Sigma\Lambda\rho\right)$.

Upgraded Problem: RSPO with Floor

Next, admitting Hypothesis, consider

$$\bar{\Gamma}^K := \sup_{\pi \in \mathscr{A}^K(x)} \varlimsup_{T \to \infty} \frac{1}{\gamma T} \log \mathbb{E}\left(X_T^{x,\pi}\right)^\gamma$$

where

- $\mathscr{A}^K(x) := \{\pi \in \mathscr{A}_0;\ X_t^{x,\pi} \geq K_t$ for all $t \geq 0\}$,
- $K := (K_t)_{t \geq 0}$: given nonnegative adapted floor.

Key Observation

Find some $\check{\pi} \in \mathscr{A}^K(x)$ so that

$$\check{X} := X^{x,\check{\pi}} \geq \epsilon \hat{X} \quad \text{with some } \epsilon > 0.$$

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Then,

$$\varlimsup_{T\to\infty} \frac{1}{\gamma T} \log \mathbb{E}\left(\check{X}_T\right)^\gamma \geq \varlimsup_{T\to\infty} \frac{1}{\gamma T} \log \mathbb{E}\left(\hat{X}_T\right)^\gamma =: \bar{\Gamma},$$

which implies

$$\bar{\Gamma}^K = \bar{\Gamma} = \varlimsup_{T\to\infty} \frac{1}{\gamma T} \log \mathbb{E}\left(\check{X}_T\right)^\gamma,$$

the optimality of $\check{\pi} \in \mathscr{A}^K(x) (\subset \mathscr{A}_0)$ for RSPO with floor.

Let

- $\bar{K} := (\bar{K}_t)_{t\geq 0}$: superhedging of the floor $K := (K_t)_{t\geq 0}$,
- $x > \bar{K}_0$.

$$\begin{aligned}\hat{X}^{(\text{CPPI-1})} &:= (x - \bar{K}_0)\hat{X} + \bar{K} \\ &\geq (x - \bar{K}_0)\hat{X} + K\end{aligned}$$

is an RSOP with floor.

Generalized CPPI (2)

Let $x > K_0$ and K/S^0 be nonincreasing. $\hat{X}^{(\text{CPPI-2})}$, the sol. to SDE,

$$\begin{aligned} dY_t =& (Y_{t-} - K_{t-}) \sum_{i=1}^{n} \hat{\pi}_t^i \frac{dS_t^i}{S_{t-}^i} \\ &+ \left\{ \left(1 - \sum_{i=1}^{n} \hat{\pi}_t^i \right) (Y_{t-} - K_{t-}) + K_{t-} \right\} \frac{dS_t^0}{S_t^0}, \\ Y_0 =& x, \end{aligned}$$

which has the expression,

$$\begin{aligned} \hat{X}_t^{(\text{CPPI-2})} =& \left\{ (x - K_0) - \int_0^t \frac{S_u^0}{\hat{X}_u} d\left(\frac{K_u}{S_u^0} \right) \right\} \hat{X}_t + K_t \\ \geq& (x - K_0)\hat{X}_t + K_t, \end{aligned}$$

defines an RSOP with floor.

In a complete market, American perpetual:

$$V_{\mathsf{OBPI}}(t,\lambda) := \sup_{t\le\tau\le\infty} \tilde{\mathbb{E}}\left[\frac{S_t^0}{S_\tau^0}\left(K_\tau \vee \lambda \hat{X}_t\right) \,\middle|\, \tilde{\mathcal{F}}_t\right]$$

$$:= \sup_{t\le\tau\le\infty} \tilde{\mathbb{E}}\left[\frac{S_t^0}{S_\tau^0}\left(K_\tau - \lambda \hat{X}_t\right)^+ \,\middle|\, \tilde{\mathcal{F}}_t\right] + \lambda \hat{X}_t.$$

Superhedging portfolio $\check{X}^{\mathrm{OBPI}}$, which satisfies

$$\check{X}_t^{\mathrm{OBPI}} \ge V_{\mathsf{OBPI}}\left(t, \hat{\lambda}(x)\right) \ge K_t \vee \hat{\lambda}(x)\hat{X}_t$$

and $\check{X}_0^{\mathrm{OBPI}} = x$, is an RSOP with floor.

Russian-type option (American lookback perpetual):

$$V_{\mathsf{DFP}}(t,\lambda) := \sup_{t\le\tau\le\infty} \tilde{\mathbb{E}}\left[\frac{S_t^0}{S_\tau^0}\left\{\lambda \vee \max_{s\in[0,\tau]}\left(\frac{K_s}{\hat{X}_s}\right)\right\}\hat{X}_\tau \,\middle|\, \tilde{\mathcal{F}}_t\right].$$

Superhedging portfolio $\check{X}^{\mathrm{DFP}}$, which satisfies

$$\check{X}_t^{\mathrm{DFP}} \ge V_{\mathsf{DFP}}\left(t,\hat{\lambda}(x)\right) \ge \hat{\lambda}(x)\hat{X}_t \vee K_t$$

and $\check{X}_0^{\mathrm{DFP}} = x$, is an RSOP with floor.

Another OBPI with "Gittins-type" index

Apply EJL (2005) and EM (2006, 2008): Let

$$G_t := \sup_{u \in [0,t)} L_u, \quad L_t := \sup \left\{ \lambda \in \mathbb{R}_{>0};\ V_{\mathsf{OBPI}}(t, \lambda) = K_t \vee \lambda \hat{X}_t \right\}.$$

Then,

$$\check{X}_t^{\mathsf{OBPI\text{-}G}} := V_{\mathrm{OBPI}} \left(t, \hat{\lambda}(x) \vee G_t \right)$$

is an self-financing wealth process.
It defines an RSOP with floor, since

$$\begin{aligned} \check{X}_t^{\mathsf{OBPI\text{-}G}} &\geq K_t \vee (\hat{\lambda}(x) \vee G_t) \hat{X}_t \\ &\geq K_t \vee \hat{\lambda}(x) \hat{X}_t \end{aligned}$$

and $\hat{X}_0^{\mathsf{OBPI\text{-}G}} = x$.

Example: Constant market price of risk, uncorrelated interest rates, and GBM floor

Explicit representations are obtained for OBPI/DFP-based methods in the following situation:

- Market model:

$$\begin{aligned} dS_t^0 &= r_t S_t^0 dt, \\ dS_t &= \text{diag}(S_t)\left(\mu_t dt + \sigma_t dw_t\right) \end{aligned}$$

  on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ with n-dim. BM w, where

  - σ: $\mathcal{F}_t$-adapted,
  - $\mu_t = r_t \mathbf{1} + \sigma_t \theta$ with the constant market price of risk $\theta \in \mathbb{R}^n$,
  - $r := (r_t)_{t\geq 0}$: inedpendent of w.

- Floor:

$$K_t = K_0 S_t^0 \times \mathrm{GBM}(w_t).$$

$$\sup_{\pi} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}(X_T^{x,\pi})^{\gamma}$$

subject to

$$X_t^{x,\pi} \geq f\left(\sup_{s\in[0,t)} X_s^{x,\pi}\right) \quad \text{for all } t \geq 0.$$

- $0 < f(x) < x.$

(Non-)Constant Constraint on Relative Drawdown

- Relative drawdown:

$$\mathrm{RDD}_t := \frac{\bar{X}_t - X_t}{\bar{X}_t}, \quad \text{where} \quad \bar{X}_t := \sup_{s\in[0,t)} X_s,$$

a measure of “riskiness” of fund wealth.

- DD constraint:

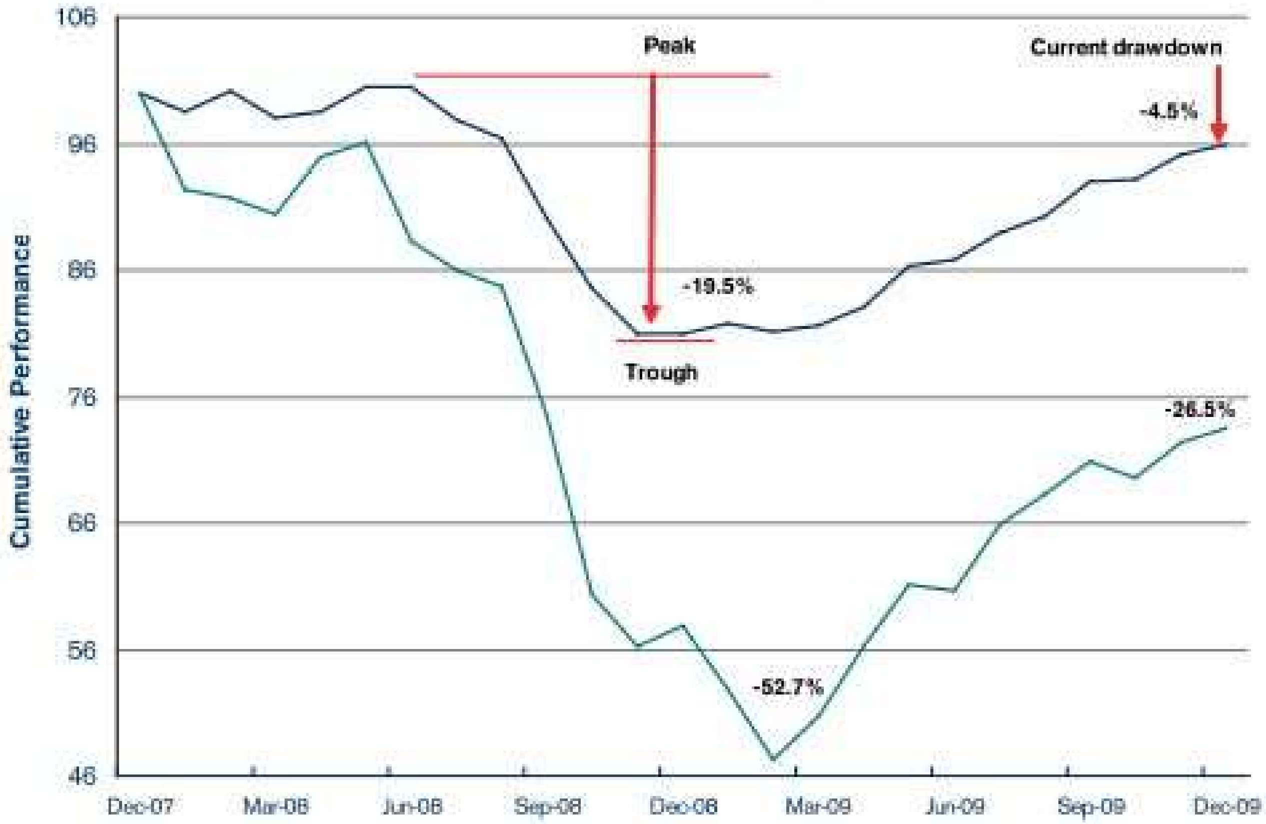
$$X_t \geq f(\bar{X}_t) \quad \Leftrightarrow \quad \mathrm{RDD}_t \leq 1 - \frac{f(\bar{X}_t)}{\bar{X}_t}.$$

- Example: $f(x) := x \sum_{i=1}^{m} \alpha_i 1_{[K_{i-1},K_i)}(x)$ with $x_0 = K_0 < \cdots < K_{m-1} < K_m = +\infty$ and $\alpha_i \in [0,1)$ $(i \in \{1,\ldots,m\})$. Piecewise const. constraint on RDD:

$$\mathrm{RDD}_t \leq 1 - \alpha_i 1_{[K_{i-1},K_i)}(\bar{X}_t) \quad \text{for all } t \geq 0.$$

From a Report by Credit Suisse on Hedge Fund Indexes

Figure 3: Drawdown Analysis, Credit Suisse/Tremont Hedge Fund Index and MSCI World Index (data as of December 31, 2009)



Source: Credit Suisse / Tremont Hedge Fund Index, Blomberg. Data based on a sample set of funds reflective of the Credit Suisse / Tremont Hedge Fund index as of December 31, 2009. Credit Suisse has not sought to independently verify information obtained from public and third party sources and makes no representations or warranties as to accuracy, completeness or reliability of such information.

Performance Comparisons

Performance Comparison With Other Strategies

The tables below allows you to compare the key performance statistics of one Strategy with another, and can be sorted by column headings:

Strategy Name	Av Annual Return	Total Profit	Monthly CROR	Av Monthly Profit	Sharpe Ratio	Strike Rate Ratio	Profit Factor	Winning Months
Happy Days	45.3%	$0	3.1%	$0	1.66	1.57	3.29	73.1%
NASDAQ 100 Leveraged	67.2%	$0	4.8%	$0	1.46	2.05	6.12	76.5%
Russell 2000 Leveraged	29.6%	$0	2.4%	$0	1.01	1.48	1.94	58.1%
S&P 500 Leveraged	70.3%	$0	4.6%	$0	1.41	1.58	3.39	74.4%
S&P 500 Non-Leveraged	24.0%	$0	2.1%	$0	2.19	1.98	8.12	82.1%

Strategy Name	Maximum Drawdown	Longest Drawdown	Standard Deviation	Largest Monthly Gain	Largest Monthly Loss	Av % Return/Trade	Av Gain/ Win Trade	Av Loss/ Loss Trade
Happy Days	-6.1%	2.0	6.0%	25.5%	-6.2%	1.7%	3.1%	-2.9%
NASDAQ 100 Leveraged	-14.5%	3.0	12.8%	91.0%	-24.9%	10.2%	16.9%	-11.4%
Russell 2000 Leveraged	-27.2%	2.0	11.0%	30.4%	-16.1%	3.1%	13.0%	-6.4%
S&P 500 Leveraged	-26.4%	2.0	12.2%	44.2%	-26.4%	4.9%	13.0%	-8.5%
S&P 500 Non-Leveraged	-6.5%	2.0	2.8%	7.1%	-6.5%	3.2%	4.1%	-3.3%

Strategy Name	Maximum Con Wins	Maximum Con Losses	Drawdown Ratio	Av Drawdown Length	Av Wins/ Year	Largest Win Trade	Largest Loss Trade	Number of Components
Happy Days	23	5	2.36	1.2	148.2	31.4%	-35.9%	5
NASDAQ 100 Leveraged	23	5	3.89	1.6	45.9	434.4%	-72.1%	5
Russell 2000 Leveraged	4	5	2.73	1.6	19.2	93.4%	-17.7%	4
S&P 500 Leveraged	7	4	3.59	1.3	30.6	92.2%	-123.0%	5
S&P 500 Non-Leveraged	23	4	3.00	1.2	48.6	22.1%	-13.8%	4

(http://www.pro-trading-profits.com/risk_reduction/compare.asp)

- Calmar Ratio:

$$\mathrm{CAL}_T := \frac{R_T}{\mathrm{MDD}_T} := \frac{R_T}{\max_{t\in[0,T]} \mathrm{DD}_t},$$

- Sterling Ratio:

$$\mathrm{STE}_T := \frac{R_T}{\mathrm{ADD}_T} := \frac{R_T}{\frac{1}{T}\int_0^T \mathrm{DD}_t dt},$$

- Burke Ratio:

$$\mathrm{BUR}_T := \frac{R_T}{\sqrt{\int_0^T \mathrm{DD}_t^2 dt}},$$

where $(R_t)_{t\geq 0}$ is the cumulative (excess) return and

$$\mathrm{DD}_t := \bar{R}_t - R_t, \quad \bar{R}_t := \bar{R}_0 \vee \max_{s\in[0,t]} R_s.$$

- Let $R_t := \dfrac{X_t - X_0}{X_0}$.

$$X_t \geq \frac{\beta}{1+\beta}\bar{X}_t + \frac{X_0}{1+\beta} \quad {}^{\forall}t \geq 0$$

$$\Rightarrow \ \mathrm{CAL}_T \geq \beta, \ \mathrm{STE}_T \geq \beta, \ \mathrm{BUR}_T \geq \frac{\beta}{\sqrt{T}}, \quad {}^{\forall}T \geq 0.$$

- Let $R_t := \log \dfrac{X_t}{X_0}$.

$$X_t \geq X_0^{\frac{1}{1+\beta}} \bar{X}_t^{\frac{\beta}{1+\beta}} \quad {}^{\forall}t \geq 0$$

$$\Rightarrow \mathrm{CAL}_T \geq \beta, \ \mathrm{STE}_T \geq \beta, \ \mathrm{BUR}_T \geq \frac{\beta}{\sqrt{T}}, \quad {}^{\forall}T \geq 0.$$

Related Works

- $f(x) = \alpha x$ $(0 < \alpha < 1)$:
  - Grossman and Zhou (1993), Cvitanic and Karatzas (1995), S (2006)
- Nonlinear f:
  - Cherney and Obloj (2013), S (2013).
- Cosumption optimization with BS model under linear DD const:
  - Roche(2006), Rogers(2006), Elie and Touzi (2008), Elie (2008).

Long-term RSPO with Drawdown Constraint

$$\Gamma_\alpha(x,\gamma) := \sup_{\pi\in\mathscr{A}_\alpha(x)} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}(X_T^{x,\pi})^\gamma,$$

where discounted wealth is governed by

$$\frac{dX_t^{x,\pi}}{X_t^{x,\pi}} = \sum_{i=1}^n \pi_t^i \frac{dS_t^i}{S_t^i}, \qquad X_0^{x,\pi} = x,$$

with continuous S^i, and $\mathscr{A}_\alpha(x)$ is a suitable subset of

$$\left\{\pi;\ X_t^{x,\pi} > f_\alpha\left(M_0 \vee \max_{s\in[0,t]} X_s^{x,\pi}\right) \text{ for all } t\geq 0\right\}.$$

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with continuous S^i, and $\mathscr{A}_\alpha(x)$ is a suitable subset of

$$\left\{\pi;\ X_t^{x,\pi} > f_\alpha\left(M_0 \vee \max_{s\in[0,t]} X_s^{x,\pi}\right) \text{ for all } t\geq 0\right\}.$$

- $x \geq M_0$ with $^\exists M_0 > 0$, the current running maximum,
- $f_\alpha : [M_0,\infty) \to \mathbb{R}_{>0}$ satisfies, with $^\exists\alpha \in (0,1)$ and $^\exists\beta < 1$, $0 < {}^\exists\delta < f_\alpha(x) < x$ for $^\forall x \geq M_0$ and $f_\alpha(x) = \alpha x + O(x^\beta)$ as $x \to \infty$.

Baseline Standard Problem: RSPO without Floor.

First, consider

(RSPO) $$\Lambda(\gamma) := \sup_{\pi \in \mathscr{A}} \varlimsup_{T \to \infty} \frac{1}{\gamma T} \log \mathbb{E} \left(X_T^{x,\pi} \right)^{\gamma}$$

without floor, where $\mathscr{A}$ is (predictably) convex, $0 \in \mathscr{A}$.

First, consider

(RSPO) $$\Lambda(\gamma) := \sup_{\pi \in \mathscr{A}} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E} \left(X_T^{x,\pi} \right)^\gamma$$

without floor, where $\mathscr{A}$ is (predictably) convex, $0 \in \mathscr{A}$.
Suppose (RSPO) has an x-independent sol., that is,
$\exists \hat{\pi}^{(\gamma)} \in \mathscr{A}$ so that $\hat{X} := X^{1,\hat{\pi}^{(\gamma)}}$ satisfies

$$\Lambda(\gamma) := \sup_{\pi \in \mathscr{A}} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E} \left(X_T^{x,\pi} \right)^\gamma = \sup_{\pi \in \mathscr{A}} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E} \left(X_T^{1,\pi} \right)^\gamma$$
$$= \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E} \left(\hat{X}_T \right)^\gamma .$$

Moreover, suppose $\Lambda(\cdot)$ is continuous on some interval.

Upgraded Problem: RSPO with GDD Constraint

Next, using the solution to Baseline problem (without floor), we construct a solution to

$$\Gamma_\alpha(x,\gamma) := \sup_{\pi\in\mathscr{A}_\alpha(x)} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log \mathbb{E}(X_T^{x,\pi})^\gamma,$$

where $\mathscr{A}_\alpha(x)$ is a suitable subset of

$$\left\{\pi;\ X_t^{x,\pi} > f_\alpha\left(\max_{s\in[0,t]} X_s^{x,\pi}\right) \text{ for all } t\geq 0\right\}.$$

Drawdown Equation

For a given $X > 0$, consider SDE,

(DD) $$\frac{dY_t}{Y_t - f_\alpha(\bar{Y}_t)} = \frac{dX_t}{X_t}, \quad \text{where} \quad \bar{Y}_t := \sup_{s\in[0,t]} Y_s.$$

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The solution is the Azéma-Yor process, given by

(AY) $$Y_t := U_\alpha(\bar{X}_t) - u_\alpha(\bar{X}_t)\left(\bar{X}_t - X_t\right) =: M^{U_\alpha}(X)_t,$$

where

- $V_\alpha : [M_0, \infty) \to [v_0^*, \infty)$ by

$$V_\alpha(y) = v_0^* \exp\left\{\int_{M_0}^{y} \frac{dx}{x - f_\alpha(x)}\right\}, \quad v_0^* \in \mathbb{R}_{>0}.$$

- $U_\alpha := V_\alpha^{-1}$, $u_\alpha := U'_\alpha$.

$$(1) \quad \bar{Y} = U_\alpha(\bar{X}),\ \bar{X} = V_\alpha(\bar{Y}).$$

Result by Carraro, El Karoui and Obłój (2010)

(1) $\bar{Y} = U_\alpha(\bar{X})$, $\bar{X} = V_\alpha(\bar{Y})$.

(2) Let $X :\equiv X^{V_\alpha(x),\pi}$, and $Y :\equiv M^{U_\alpha}(X)$. Then, $Y - f_\alpha(\bar{Y}) = u_\alpha(\bar{X})X > 0$. Moreover,

$$Y \equiv X^{x,\rho} \quad \text{with} \quad \rho_t := \left\{1 - \frac{f_\alpha(\bar{Y}_t)}{Y_t}\right\}\pi_t.$$

(1) $\bar{Y} = U_\alpha(\bar{X})$, $\bar{X} = V_\alpha(\bar{Y})$.

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$$Y \equiv X^{x,\rho} \quad \text{with} \quad \rho_t := \left\{1 - \frac{f_\alpha(\bar{Y}_t)}{Y_t}\right\} \pi_t.$$

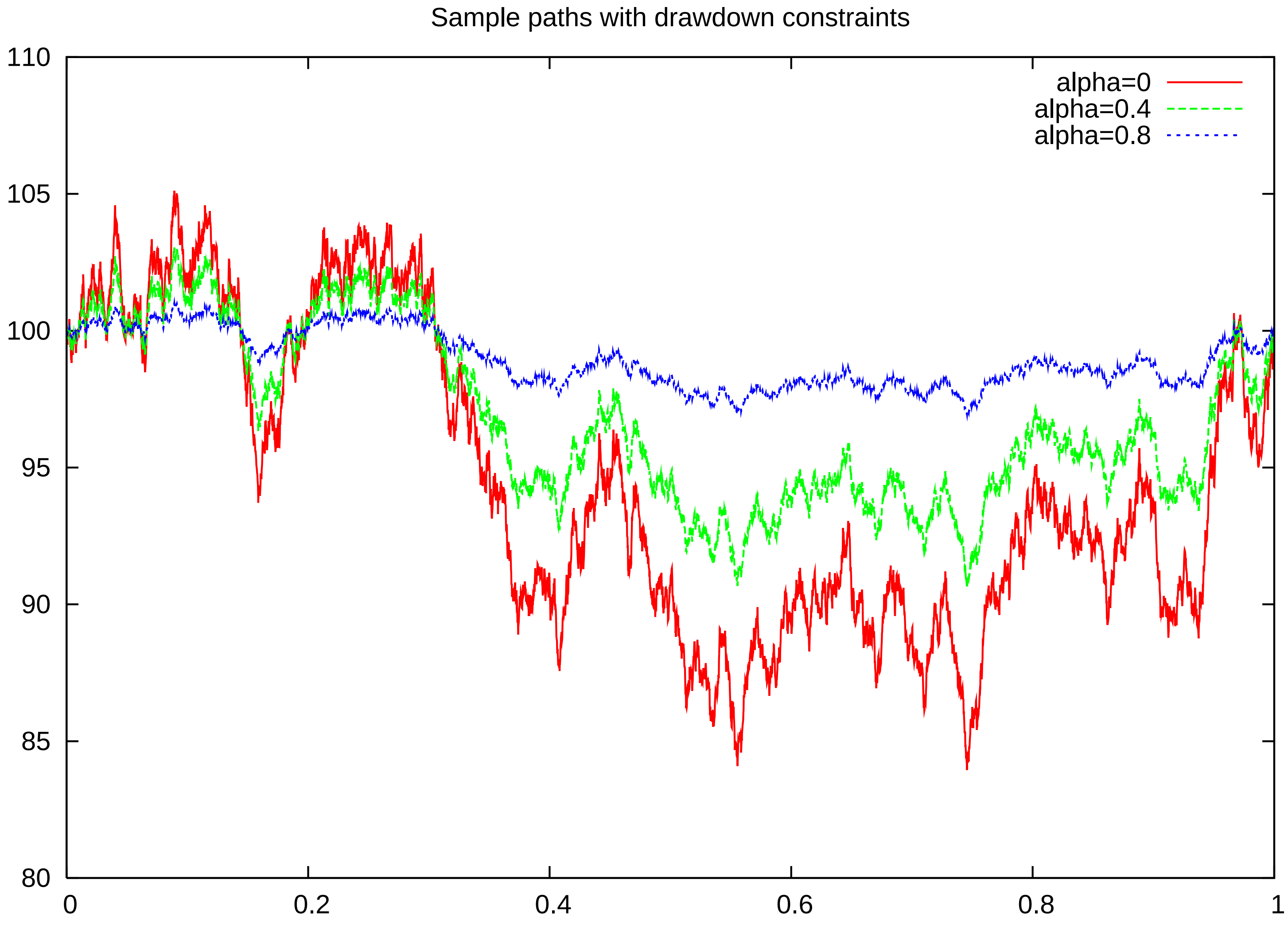
(3) Let $Y :\equiv X^{x,\rho}$ with $\rho \in \mathscr{A}_\alpha(x)$, i.e., $Y - f_\alpha(\bar{Y}) > 0$. Then, Y solves (DD):

$$\frac{dY_t}{Y_t - f_\alpha(\bar{Y}_t)} = \frac{dX_t^{V_\alpha(x),\pi}}{X_t^{V_\alpha(x),\pi}}$$

with $^\exists\pi$. Moreover, letting $v_\alpha := V'_\alpha$,

$$X_t^{V_\alpha(x),\pi} = V_\alpha(\bar{Y}_t) - v_\alpha(\bar{Y}_t)\left(\bar{Y}_t - Y_t\right).$$

Plotting Azéma-Yor processes: $f(x) := \alpha x$, i.e., $\mathrm{RDD} < 1 - \alpha$.



(1) $\Gamma_\alpha(x,\gamma) = (1-\alpha)\Lambda\left((1-\alpha)\gamma\right)$ for $^\forall x \geq x_0$. Here,

$$\Lambda(\lambda) := \sup_\pi \overline{\lim_{T\to\infty}} \frac{1}{\lambda T} \log \mathbb{E}(X_T^{x,\pi})^\lambda = \overline{\lim_{T\to\infty}} \frac{1}{\lambda T} \log \mathbb{E}(X_T^{x,\hat{\pi}^{(\lambda)}})^\lambda,$$

imposing *no* DD-constraint.

(1) $\Gamma_\alpha(x,\gamma) = (1-\alpha)\Lambda\left((1-\alpha)\gamma\right)$ for $^\forall x \geq x_0$. Here,

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imposing *no* DD-constraint.

(2) Let $\hat{X} := X^{V_\alpha(x),\hat{\pi}^{((1-\alpha)\gamma)}}$. $\hat{Y} := M^{U_\alpha}(\hat{X})$ is optimal for GDD constrained problem. The associated optimal investment strategy is

$$\check{\pi}_t := \left\{1 - \frac{f_\alpha\left(\sup_{s\in[0,t]} \hat{Y}_s\right)}{\hat{Y}_t}\right\} \hat{\pi}_t^{((1-\alpha)\gamma)}.$$

Optimization under

$$X_t^{x,\pi} > K_t \text{ and } X_t^{x,\pi} > f_\alpha\left(\max_{s\in[0,t]} X_s^{x,\pi}\right) \text{ for } {}^\forall t \geq 0,$$

where $(K_t)_{t\geq 0}$ is a given floor process and $f_\alpha(x) := \alpha x$.
Optimal strategy:

$$\check{\pi}_t := \left\{1 - \frac{\alpha\left(\max_{s\in[0,t]} \hat{Y}_s\right)}{\hat{Y}_t}\right\}\left\{1 - \frac{V_\alpha\left(\frac{1}{\alpha}K\right)}{M^{V_\alpha}(\hat{Y})_t}\right\}\hat{\pi}_t,$$

where $$d\hat{Y}_t = \left\{\hat{Y}_t - \alpha\left(\max_{s\in[0,t]} \hat{Y}_s\right)\right\}\frac{d\bar{X}_t}{\bar{X}_t}, \quad \hat{Y}_0 = x,$$

$$d\bar{X}_t = \left\{\bar{X}_t - V_\alpha\left(\frac{1}{\alpha}K\right)\right\}\sum_{i=1}^n \hat{\pi}_t^i \frac{dS_t^i}{S_t^i}, \quad \bar{X}_0 = V_\alpha(x)$$

with $\hat{\pi} := \hat{\pi}^{((1-\alpha)\gamma)}$ $(\hat{Y} := M^{U_\alpha}(\bar{X}))$.

$$
\begin{aligned}
dS_t^0 &= rS_t dt, \quad S_0^0 = 1, \\
dS_t &= \text{diag}(S_t)\left\{\mu(Y_t)dt + \sigma(Y_t)dW(t)\right\}, \quad S_0 \in \mathbb{R}^n_{++}, \\
dY_t &= b(Y_t)dt + a(Y_t)dW_t, \quad Y_0 \in \mathbb{R}^m,
\end{aligned}
$$

where $W := (W^1, \ldots, W^n)^\top$, $W^i := (W_t^i)_{t \geq 0}$: n-dim. BM, $r \in \mathbb{R}$, $\mu : \mathbb{R}^m \to \mathbb{R}^n$, $\sigma : \mathbb{R}^m \to \mathbb{R}^{n \times d}$ so that

$$
\sigma\sigma^\top(\cdot) > 0,
$$

$b : \mathbb{R}^m \to \mathbb{R}^m$, and $a : \mathbb{R}^m \to \mathbb{R}^{m \times n}$.

$$
\begin{aligned}
dX_t =& \sum_{i=1}^{n} \pi_t^i \frac{dS_t^i}{S_t^i} + \left(X_t - \sum_{i=1}^{n} \pi_t^i \right) \frac{dS_t^0}{S_t^0} - c_t dt, \\
X_0 =& x \in \mathbb{R}_{++}.
\end{aligned}
$$

(X, Y) satisfies

$$
\begin{aligned}
dX_t =& \left\{ rX_t - c_t + \pi_t^\top \lambda(Y_t) \right\} dt + \pi_t^\top \sigma(Y_t) dW_t, \\
X_0 =& x \in \mathbb{R}_{++}, \\
dY_t =& b(Y_t)dt + a(Y_t)dW_t, \\
Y_0 =& y \in \mathbb{R}^m,
\end{aligned}
$$

where, with $\mathbf{1} := (1, \dots, 1)^\top \in \mathbb{R}^n$,

$$
\lambda(y) := \mu(y) - r\mathbf{1}.
$$

- Floor $K := (K_t)_{t\geq 0}$

$$dK_t = K_t(r+\alpha)dt, \quad K_0 = k_0 \in \mathbb{R}_+,$$

($\alpha \geq 0$: absorbing barrier, $\alpha < 0$: reflecting barrier)

- Space of admissible strategies

$$\mathscr{A}_T^K(x) := \{(\pi, c) |\ c \geq 0,\ X_t^{x,\pi,c} \geq K(t) \text{ for all } t \in [0,T]\}\,.$$

- $U_1, U_2 : \mathbb{R}_{++} \to \mathbb{R}$ are utility functions so that, for $i = 1, 2$,

$$U_i \in C^2(\mathbb{R}_{++}),\ U_i' > 0,\ U_i'' < 0,\ U_i'(0+) = +\infty,\ U_i'(+\infty) = 0.$$

Primal HJB(1)

Associated with Merton problem with floor constraint, deduce the HJB equation,

$$\begin{aligned} -V_t =& \sup_{(\pi,c)\in\mathbb{R}^n\times\mathbb{R}_{++}} H(x,y,\pi,c,V,V_x,V_y,V_{xx},V_{xy},V_{yy}), \\ & (t,x,y)\in[0,T)\times[K(t),\infty)\times\mathbb{R}^m, \\ V(T,x,y) =& U_2(x), \end{aligned}$$

where we define

$$\begin{aligned} & H(x,y,\pi,c,V_x,V_y,V_{xx},V_{xy},V_{yy}) \\ & \quad := -\rho V + U_1(c) + (rx - c + \pi^\top\lambda)V_x + b^\top V_y \\ & \qquad + \frac{1}{2}\pi^\top\sigma\sigma^\top\pi V_{xx} + \pi^\top\sigma a^\top V_{xy} + \frac{1}{2}\mathrm{tr}\left(aa^\top V_{yy}\right). \end{aligned}$$

Assuming that $V_{xx} < 0$, HJB is rewritten as

$$\begin{aligned} -V_t =& -\rho V + \tilde{U}_1(V_x) + rxV_x + b^\top V_y + \frac{1}{2}\mathrm{tr}\left(aa^\top V_{yy}\right) \\ & - \frac{1}{2V_{xx}}\left(\lambda V_x + \sigma a^\top V_{xy}\right)^\top (\sigma\sigma^\top)^{-1}\left(\lambda V_x + \sigma a^\top V_{xy}\right), \\ & (t, x, y) \in [0, T) \times [K(t), \infty) \times \mathbb{R}^m, \\ V(T, x, y) =& U_2(x). \end{aligned}$$

where

$$\tilde{U}_1(y) := \sup_{c>0}\{U_1(c) - cy\},$$

and the maximizers are given by

$$\begin{aligned} \bar{c} :=& (U_1')^{-1}(V_x), \\ \bar{\pi} :=& -\frac{1}{V_{xx}}(\sigma\sigma^\top)^{-1}\left(\lambda V_x + \sigma a^\top V_{xy}\right). \end{aligned}$$

Dual HJB (1)

Suppose $K = 0$. $\tilde{V}(t,z,y) := \sup_{x\in(0,\infty)} \{V(t,x,y) - xz\}$ satisfies

$$\tilde{V}(t,z,y) = \mathbb{E}\left[\int_0^{T-t} \mathrm{e}^{-\rho u}\tilde{U}_1(Z_u)\,du + \mathrm{e}^{-\rho(T-t)}\tilde{U}_2(Z_{T-t}) \,\middle|\, Z_0 = z, Y_0 = y\right],$$

with

$$dY_t = b(Y_t)dt + a(Y_t)dW_t, \quad dZ_t = Z_t\left\{(\rho - r)dt - \theta(Y_t)^\top dW_t\right\}$$

and $\theta := \sigma^{-1}(\mu - r\mathbf{1})$. So,

$$\tilde{V}_t + (\mathbb{L} - \rho)\tilde{V} = 0, \quad \tilde{V}(T,z,y) = \tilde{U}_2(z),$$

$$\mathbb{L} := b^\top\partial_y + (\rho - r)z\partial_z + \frac{1}{2}\mathrm{tr}(aa^\top\partial_{yy}) - \theta^\top a^\top z\partial_{yz} + \frac{1}{2}|\theta|^2 z^2\partial_{zz}.$$

For the *constrained* Fenchel-Legendre transform:

$$\hat{V}(t,z,y) := \sup_{x \in (K(t),\infty)} \{V(t,x,y) - xz\},$$

we deduce the variational inequality,

$$\max\left\{\hat{V}_t + (\mathbb{L} - \rho)\hat{V}, \hat{V} - \mathcal{K}\right\} = 0, \quad \hat{V}(T,z,y) = \hat{U}_2(T,z)$$

where

$$\mathcal{K}(t,z,y) := V(t,K(t),y) - K(t)z,$$

$$\hat{U}_2(T,z) := \sup_{x \geq K(T)} \{U_2(x) - zx\}.$$

The associated dual stochastic control problem is

$$\inf_{\eta \in \mathscr{B}_{T-t}} \mathbb{E}\left[\int_0^{T-t} \mathrm{e}^{-\rho u} \tilde{U}_1\left(Z_u^{z,\eta}\right) du + \mathrm{e}^{-\rho(T-t)} \hat{U}_2\left(T, Z_{T-t}^{z,\eta}\right) + \int_{[0,T-t)} \mathrm{e}^{-\rho u} \mathcal{K}\left(t+u, Z_u^{z,\eta}, Y_u\right) d\eta_u \,\middle|\, Y_0 = y\right],$$

where $\mathscr{B}_T$ is the space of nondecreasing, LCRL, adapted processes on the time-interval $[0, T]$, starting from 0 at time 0, and

$$dZ_s^{z,\eta} = Z_s^{z,\eta}\left[(\rho - r)ds - \theta(Y(s))^\top dW_s\right] - d\eta_s,$$
$$Z_0^{z,\eta} = z.$$

Not described in "closed-forms". They contain the solution V to the primal.

Compute the equation for $g : [0,T] \times \mathbb{R}_+ \times \mathbb{R}^m$, which satisfies

$$\begin{aligned} &\text{(i)} \quad g := g(t,z,y) \geq K(t), \\ &\text{(ii)} \quad V_x(t,g,y) - z = 0 \quad \text{if } g(t,z,y) > K(t). \end{aligned}$$

We deduce that

$$\min\left\{-g_t - (\hat{\mathbb{L}} - r)g + \tilde{U}_1', g - K\right\} = 0,$$

where

$$\hat{\mathbb{L}} := (b - a\theta)^\top \partial_y + (|\theta|^2 + \rho - r) z \partial_z + \frac{1}{2}\mathrm{tr}(aa^\top \partial_{yy}) - \theta^\top a^\top z \partial_{yz} + \frac{1}{2}|\theta|^2 z^2 \partial_{zz}.$$

This is deduced from the optimal stopping problem associated with the differential of the dual stochastic control problem:

$$\sup_{t\le\tau\le T} \mathbb{E}\Big[-\int_t^\tau e^{-\rho(s-t)} Z(s)\tilde{U}_1'(Z(s))ds + e^{-\rho(\tau-t)} Z(\tau)\Big\{K(\tau)1_{\{\tau<T\}} - \hat{U}_{2z}(T, Z(T))1_{\{\tau=T\}}\Big\} \Big| \mathcal{F}_t\Big],$$

where $\hat{U}_{2z} := \partial_z \hat{U}_2$, or

$$Z(t)\times \sup_{t\le\tau\le T} \tilde{\mathbb{E}}\Big[-\int_t^\tau e^{-r(s-t)} \tilde{U}_1'(Z(s))ds + e^{-r(\tau-t)} \Big\{K(\tau)1_{\{\tau<T\}} - \hat{U}_{2z}(T, Z(T))1_{\{\tau=T\}}\Big\} \Big| \mathcal{F}_t\Big],$$

introducing a measure change.

$$
\begin{aligned}
-\hat{g}_t =& (\hat{\mathbb{L}} - r)\hat{g} - \tilde{U}_1' \quad \text{for } z < \beta(t,y), \\
\hat{g} >& K \qquad \text{for } z < \beta(t,y), \\
\hat{g}(t,z,y)\big|_{z=\beta(t,y)-} =& K(t), \\
\hat{g} =& K \qquad \text{for } z > \beta(t,y), \\
-\hat{g}_t >& (\hat{\mathbb{L}} - r)\hat{g} - \tilde{U}_1' \quad \text{for } z > \beta(t,y), \\
\hat{g}(T,z,y) =& -\hat{U}_{2z}(T,z)
\end{aligned}
$$

with smooth-fit conditions:

$$
g_z(t,z,y)\,\big|_{z=\beta(t,y)-} = 0, \quad g_y(t,z,y)\,\big|_{z=\beta(t,y)-} = 0,
$$

assuming that, with some $\beta : [0,T] \times \mathbb{R}^m \to \mathbb{R}_+$,

$$
\begin{aligned}
\mathcal{C} &= \{(t,z,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}^m;\ z < \beta(t,y)\}, \\
\mathcal{D} &= \{(t,z,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}^m;\ z \geq \beta(t,y)\}.
\end{aligned}
$$

Suppose that the free-boundary problem has a nice solution:

(i) The solution is $C^{1,2,2}$ in $\mathcal{C}$.
(ii) The smooth-fit condition (4.8) is satisfied at the free-boundary.
(iii) The free-boundary satisfies $\beta \in C^{1,2}([0,T] \times \mathbb{R}^m)$ (to apply Itô's calculs).

Recovering the primal value function V directly from g (differential of the dual) is not easy ($^\exists$ambiguity of (t,y)). However, maximizers are written as

$$\bar{c}(t,x,y) := (U_1')^{-1}(V_x(t,x,y)) = (U_1')^{-1}(z),$$

$$\begin{aligned}\bar{\pi}(t,x,y) &:= -\left\{(\sigma\sigma^\top)^{-1}\frac{\lambda V_x + \sigma a^\top V_{xy}}{V_{xx}}\right\}(t,x,y)\\ &= -z g_z(t,z,y)((\sigma\sigma^\top)^{-1}\lambda)(y) + ((\sigma\sigma^\top)^{-1}\sigma a^\top)(y) g_y(t,z,y)\end{aligned}$$

with $z = h(t,x,y)$ $(h := g^{-1})$.

Let

$$
\begin{aligned}
dZ^{(z)}(s) =& Z^{(z)}(s)\left\{-\theta(Y(s))^\top dW(s) + (\rho - r)ds\right\}, \\
Z^{(z)}(0) =& z.
\end{aligned}
$$

Define the process $\bar{Y} := (Y^0, Y)$ by the SDE with (non-sticky) reflection

$$
\begin{aligned}
dY^0(t) =& dZ^{(z)}(t) - d\beta(t, Y(t)) - d\eta(t), \quad Y^0(0) = z - \beta(0, y), \\
dY(t) =& b(Y(t))dt + a(Y(t))dW(t), \quad Y(0) = y \in \mathbb{R}^m.
\end{aligned}
$$

Here, $Y^0 \leq 0$, and η is a continuous, adapted and non-decreasing process such that $\eta(0) = 0$ and that

$$
\int_0^t 1_{\{Y^0(s)=0\}} d\eta(s) = \eta(t).
$$

Constructing Optimal Strategy (3)

Write

$$\bar{Z}^{(y,z)}(t) := Y^0(t) + \beta(t, Y^{(y)}(t)), \quad Y^{(y)}(t) := Y(t),$$

emphasizing the initial value $(\bar{Z}^{(y,z)}(0), Y^{(y)}(0)) = (z, y) \in \mathbb{R} \times \mathbb{R}^m$. Note that the process $\bar{Z}$ always stays below the free-boundary, i.e.,

$$\bar{Z}(t) \leq \beta(t, Y(t), \quad {}^{\forall}t \in [0, T],$$

and (non-stickly) reflects on the free-boundary. Define

$$\begin{aligned}\tilde{c}(z) :=& (U_1')^{-1}(z),\\ \tilde{\pi}(t, z, y) :=& - z g_z(t, z, y) \left((\sigma\sigma^\top)^{-1}\lambda \right)(y) + \left((\sigma\sigma^\top)^{-1}\sigma a^\top \right)(y) g_y(t, z, y\end{aligned}$$

Define

$$\hat{\Xi}(t; y, z) := g\left(t, \bar{Z}^{(z,y)}(t), Y^{(y)}(t)\right).$$

Use Itô's formula to see that

$$\begin{aligned}
d\hat{\Xi}(t) =& g_z\left(t, \bar{Z}(t), Y(t)\right) d\bar{Z}(t) + g_y\left(t, \bar{Z}(t), Y(t)\right) dY(t) \\
&+ g_t\left(t, \bar{Z}(t), Y(t)\right) dt + \frac{1}{2} g_{zz}\left(t, \bar{Z}(t), Y(t)\right) d\langle \bar{Z}\rangle(t) \\
&+ g_{zy}\left(t, \bar{Z}(t), Y(t)\right) d\langle \bar{Z}, Y\rangle(t) + \frac{1}{2} g_{yy}\left(t, \bar{Z}(t), Y(t)\right) d\langle Y\rangle(t) \\
=& \left(-z g_z \theta^\top + g_y^\top a\right)\left(t, \bar{Z}(t), Y(t)\right)\{dW(t) + \theta(Y(t))dt\} \\
&- g_z\left(t, \bar{Z}(t), Y(t)\right) d\eta(t) + (\partial_t + \hat{\mathbb{L}}) g\left(t, \bar{Z}(t), Y(t)\right) dt.
\end{aligned}$$

Note the following:

(i) $-zg_z\theta^\top + g_y^\top a = \tilde{\pi}^\top \sigma$.

(ii) $g_z\left(t, \bar{Z}(t), Y(t)\right) d\eta(t) = 0$ a.e. (t,ω) by smooth-fit.

(iii) $\left(t, \bar{Z}(t), Y(t)\right) \in \mathcal{C}$ for a.e. (t,ω), hence, the drift $\hat{\mathbb{L}}g\left(t, \bar{Z}(t), Y(t)\right) dt$ is well-defined and we see that

$$(\partial_t + \hat{\mathbb{L}})g\left(t, \bar{Z}(t), Y(t)\right) dt = \left\{rg\left(t, \bar{Z}(t), Y(t)\right) - \tilde{c}\left(\bar{Z}(t)\right)\right\} dt.$$

So,

$$\begin{aligned} d\hat{\Xi}(t) &= \left\{r\hat{\Xi}(t) - \tilde{c} + \tilde{\pi}^\top \lambda\right\}\left(t, \bar{Z}(t), Y(t)\right) dt + (\tilde{\pi}^\top\sigma)\left(t, \bar{Z}(t), Y(t)\right) dW \\ \hat{\Xi}(0) &= g(0, z, y). \end{aligned}$$

Define the wealth process

$$\hat{X}(t) = \hat{\Xi}(t; h(0,x,y), y) = g\left(t, \bar{Z}^{(y,h(0,x,y))}(t), Y(t)\right),$$

which satisfies $\hat{X}(0) = x$. The associated portfolio isinvestment strategy is

$$\hat{\pi}(t) := \tilde{\pi}\left(t, \hat{Z}(t), Y(t)\right), \quad \hat{c}(t) := \tilde{c}\left(\hat{Z}(t)\right),$$

where

$$\hat{Z}(t) := \bar{Z}^{(y,h(0,x,y))}(t).$$

Using

$$\hat{X}(t) = g(t, \hat{Z}(t), Y(t)), \quad \hat{Z}(t) = h(t, \hat{X}(t), Y(t)),$$

we verify that $\hat{X}$ is the optimal wealth and can write down the feedback-form dynamics of $(\hat{X}, Y)$.