Okinawa, Japan

Simulation of Diffusion Bridges with Application to Statistical Inference for SDEs

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Joint work with Fernando Baltazar-Larios and Mogens Bladt, Universidad Nacional Autonoma de Mexico and Samuel Finch, University of Copenhagen

Discrete time sampling of SDE

$$dX_t = b(X_t; \alpha)dt + \sigma(X_t; \beta)dW_t \qquad \theta = (\alpha, \beta) \in \Theta \subseteq \mathbb{R}^p$$

Data: $X_{t_0}, X_{t_1}, \ldots X_{t_n}$

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Data: $X_{t_0}, X_{t_1}, \ldots X_{t_n}$

Likelihood-function:

$$L_n(\theta) = \prod_{i=1}^{n} p(\Delta, X_{t_{i-1}}, X_{t_i}; \theta),$$

$y \mapsto p(\Delta, x, y; \theta)$ is the transition density, i.e. probability density function of the conditional distribution of $X_{t+\Delta}$ given that $X_t = x$

Likelihood inference

- Approximate p by the density of $N(x + b(x;\theta)\Delta, \sigma(x;\theta)\Delta)$

Gaussian pseudo-likelihood function:

Dorogovcev (1976), Prakasa Rao (1988), Florens-Zmirou (1989), Yoshida (1992), Chan et al. (1992), Kloeden et al. (1996), Kessler (1997), Sørensen & Uchida (2003), Uchida & Yoshida (2012)

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• Approximate p by a Gaussian density obtained by local linearization:

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- Approximate p by a Gaussian density obtained by local linearization:

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- Approximate p by the density of $N(F(x;\theta), \Phi(x;\theta))$, where

$$F(x;\theta) = E_\theta(X_\Delta | X_0 = x) \quad \text{and} \quad \Phi(x;\theta) = \text{Var}_\theta(X_\Delta | X_0 = x).$$

Quadratic martingale estimating functions:

Bibby & Sørensen (1995, 1996), Jacobsen (2002)

Likelihood inference

- Approximate p by solving the Fokker-Planck equation / the forward Kolmogorov equation numerically:

Lo (1988), Poulsen (1999), Hurn, Jeisman & Lindsay (2007)

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• Approximate p by eigenfunction expansions:

Forman & Sørensen (2008)

Likelihood inference

- Approximate p by Monte Carlo methods, where the diffusion is simulated a large number of times:

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- Approximate p by Markov chain Monte Carlo methods:

Elerian, Chib & Shephard (2001), Eraker (2001), Roberts & Stramer (2001), Golightly & Wilkinson (2005), Beskos, Papaspiliopoulos, Roberts & Fearnhead (2006), Delyon & Hu (2006), Lin, Chen & Mykland (2010)

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- Stochastic EM-algorithm:

Beskos, Papaspiliopoulos, Roberts, and Fearnhead (2006), Bladt and Sørensen (2012)

The EM-algorithm

Observation: X_{obs}

The likelihood function for X_{obs} is intractable or unknown

Suppose we can augment X_{obs} by some missing data X_{mis} such that the full (but partly unobserved) data set

$$Y = (X_{\text{obs}}, X_{\text{mis}})$$

has a tractable likelihood function

$$L(\theta) = p(Y;\theta) \qquad \theta \in \Theta$$

$$Q(\theta, \theta') = E_{\theta'}(\log p(Y;\theta) \mid X_{\text{obs}})$$

The EM-algorithm

EM-algorithm:

1. Chooce a $\hat{\theta}_0 \in \Theta$, $i := 0$
2. (E-step) Calculate $Q(\theta, \hat{\theta}_i) = E_{\hat{\theta}_i}(\log p(Y;\theta) \mid X_{\text{obs}})$ for all $\theta \in \Theta$
3. (M-step) Find a maximum $\hat{\theta}_{i+1}$ of $\theta \mapsto Q(\theta, \hat{\theta}_i)$
4. i:=i+1 go to 2

Under weak regularity conditions, the sequence $\hat{\theta}_i$ will converge in probability to a local maximum of the likelihood function for X_{obs}

Dempster, Laird & Rubin (1977)

McLachlan & Krishnan (1997)

EM-algorithm for a diffusion model

$$dV_t = b(V_t;\alpha)dt + \sigma(V_t;\beta)dW_t \qquad \theta = (\alpha,\beta) \in \Theta \subseteq \mathbb{R}^p$$

Data: $V_{t_0}, \cdots, V_{t_n}, \quad 0 = t_0 < \cdots < t_n$.

$$X_{\text{obs}} = (V_{t_0}, \cdots, V_{t_n})$$

With which X_{mis} should we augment the observations to get a useful full data set

$$Y = (X_{\text{obs}}, X_{\text{mis}})?$$

EM-algorithm for a diffusion model

$$dV_t = b(V_t;\alpha)dt + \sigma(V_t;\beta)dW_t \qquad \theta = (\alpha,\beta) \in \Theta \subseteq \mathbb{R}^p$$

Data: $V_{t_0}, \cdots, V_{t_n}, \quad 0 = t_0 < \cdots < t_n$.

$$X_{\text{obs}} = (V_{t_0}, \cdots, V_{t_n})$$

With which X_{mis} should we augment the observations to get a useful full data set

$$Y = (X_{\text{obs}}, X_{\text{mis}})?$$

$$\eta_\beta(v) = \int_{v^*}^{v} \frac{1}{\sigma(u;\beta)} du, \qquad X_t = \eta_\beta(V_t)$$

$$dX_t = a_\theta(X_t)dt + dW_t, \qquad a_\theta(x) = \frac{b(\eta_\beta^{-1}(x);\alpha)}{\sigma(\eta_\beta^{-1}(x);\beta)} - \tfrac{1}{2}\sigma'(\eta_\beta^{-1}(x;\beta))$$

The missing data

$$X_{\text{mis}} = (\dot{X}^1, \ldots, \dot{X}^n)$$

$$\dot{X}_t^i = X_t^{*i} - \frac{t_i - t}{\Delta_i}\eta_\beta(V_{t_{i-1}}) - \frac{t - t_{i-1}}{\Delta_i}\eta_\beta(V_{t_i}) \qquad t \in [t_{i-1}, t_i]$$

X^{*i} is a diffusion bridge that solves $dX_t = a_\theta(X_t)dt + dW_t$ in $[t_{i-1}, t_i]$ starting at $X_{t_{i-1}} = \eta_\beta(V_{t_{i-1}})$ and ending at $X_{t_i} = \eta_\beta(V_{t_i})$

$X^{*1}, \ldots, X^{*n}$ are independent (given the data)

Define the operator

$$g_{\beta,i}(X)_t = X_t + \frac{t_i - t}{\Delta_i}\eta_\beta(V_{t_{i-1}}) + \frac{t - t_{i-1}}{\Delta_i}\eta_\beta(V_{t_i}) \qquad t \in [t_{i-1}, t_i]$$

Then

$$g_{\beta,i}(\dot{X}^i) = X^{*i}$$

Full likelihood

The full likelihood, i.e. the density of $(V_{t_0}, \cdots, V_{t_n}, \dot{X}^1, \ldots, \dot{X}^n)$ w.r.t. $\lambda^n \times \mathbf{W}^{(0,0,t_1)} \times \mathbf{W}^{(0,0,t_2-t_1)} \times \cdots \times \mathbf{W}^{(0,0,t_n-t_{n-1})}$:

$$\exp\Bigg(H_\theta(\eta_\theta(V_{t_n})) - H_\theta(\eta_\beta(V_0)) - \sum_{i=1}^n \Bigg[\frac{1}{2\Delta_i}(\eta_\beta(V_{t_i}) - \eta_\beta(V_{t_{i-1}}))^2 \\ + \log(\sigma(V_{t_i};\beta)) + \int_{t_{i-1}}^{t_i} \frac{1}{2}(a_\theta^2 + a_\theta')(g_{\beta,i}(\dot{X}^i)_s)ds \Bigg] \Bigg)$$

$\mathbf{W}^{(0,0,t)}$: probability measure on $C([0,t])$ induced by the standard Brownian bridge on $[0,t]$ starting at 0 and ending at 0

$$H(x;\theta) = \int^x a_\theta(y)dy$$

Roberts and Stramer (2001)

E-step

$$Q(\theta, \theta') =$$

$$H_\theta(\eta_\beta(V_{t_n})) - H_\theta(\eta_\beta(V_0)) - \sum_{i=1}^{n} \left[\frac{1}{2\Delta_i} (\eta_\beta(V_{t_i}) - \eta_\beta(V_{t_{i-1}}))^2 + \log(\sigma(V_{t_i}; \beta)) \right]$$

$$-\frac{1}{2} \sum_{i=1}^{n} E_{\theta'} \left(\int_{t_{i-1}}^{t_i} (a_\theta^2 + a'_\theta)(g_{\beta,i}(\dot{X}^i)_s) ds \right)$$

Diffusion bridge simulation

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t$$

A solution of in the interval $[t_1, t_2]$ such that $X_{t_1} = a$ and $X_{t_2} = b$ is called a (t_1, a, t_2, b)-bridge.

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• Exact simulation of one-dimensional diffusion bridges:

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• Straightforward diffusion bridge simulation using e.g. the Euler scheme:

Bladt & Sørensen (2009, 2012)

Diffusion bridge simulation

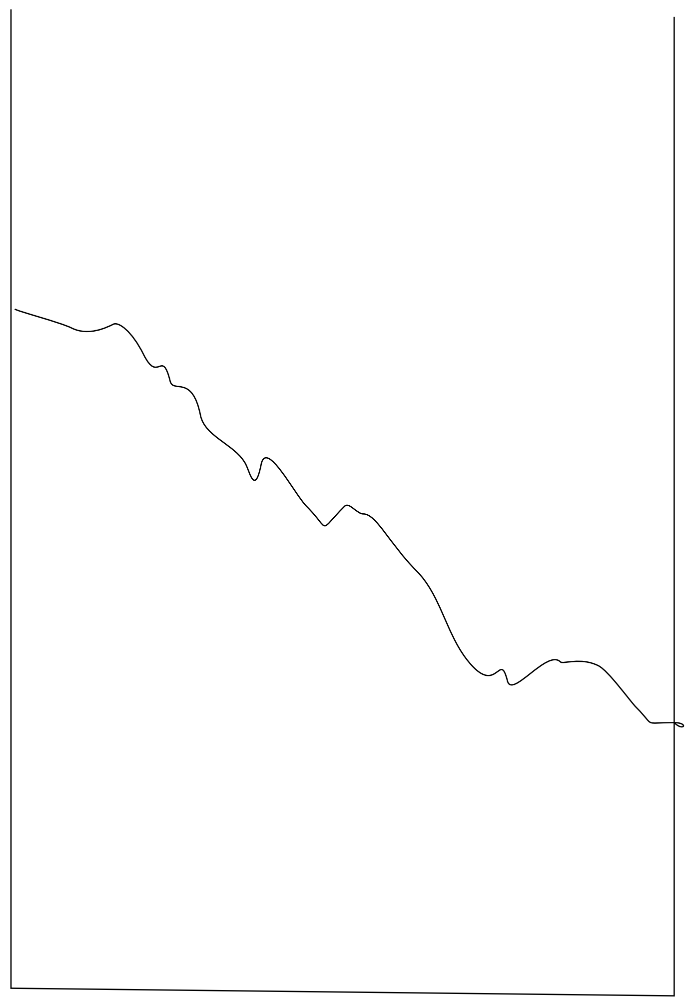
$$dX_t^i = \alpha(X_t^i)dt + \sigma(X_t^i)dW_t^i, \quad X_0^1 = a \quad \text{and } X_0^2 = b$$

W^1 and W^2 independent standard Wiener processes

Define $\tau = \inf\{0 \leq t \leq \Delta | X_t^1 = X_{\Delta - t}^2\}$ ($\inf \emptyset = +\infty$) and

$$Z_t = \begin{cases} X_t^1 & \text{if } 0 \leq t \leq \tau \\ \\ X_{\Delta-t}^2 & \text{if } \tau < t \leq \Delta. \end{cases}$$

Diffusion bridge simulation



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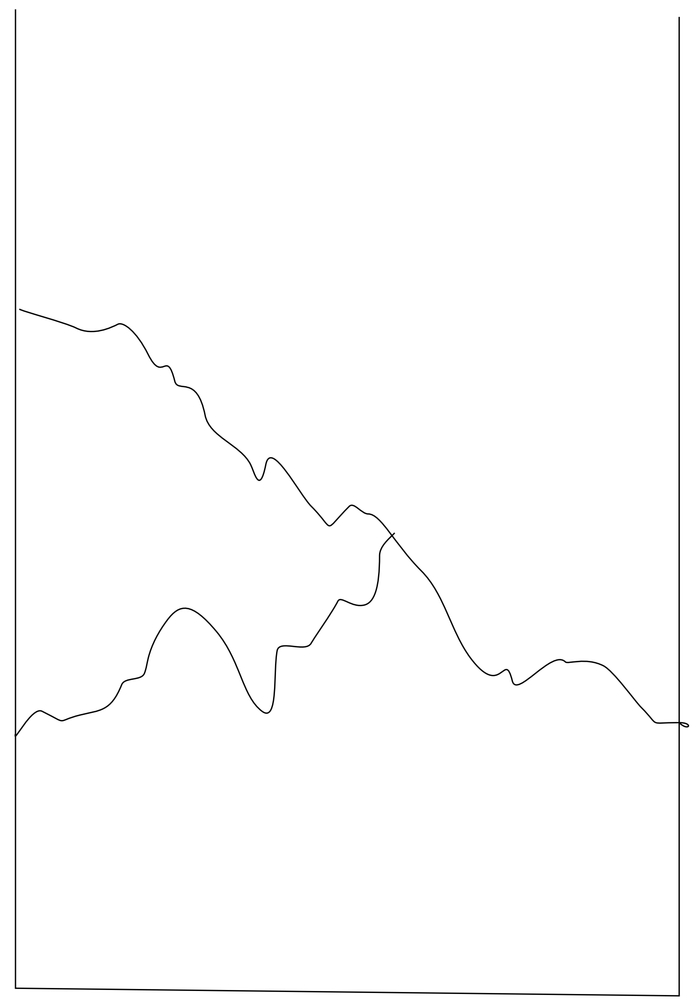

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$$dX_t^i = \alpha(X_t^i)dt + \sigma(X_t^i)dW_t^i, \quad X_0^1 = a \quad \text{and } X_0^2 = b, \qquad \text{ergodic}$$

$$Z_t = \begin{cases} X_t^1 & \text{if } 0 \leq t \leq \tau \\ \\ X_{\Delta - t}^2 & \text{if } \tau < t \leq \Delta. \end{cases}$$

The distribution of $\{Z_t\}_{0 \leq t \leq \Delta}$, conditional on the event $\{\tau \leq \Delta\}$, equals the distributions of a $(0, a, \Delta, b)$-bridge conditional on the event that the bridge is hit by an independent diffusion with the same stochastic differential equation as X_t and initial distribution with density $p_\Delta(b, \cdot)$

Let us call such a diffusion a $p_\Delta(b)$-diffusion

Two probabilities

$(0, a, \Delta, b)$-bridge

$p(\Delta) = P(\tau > \Delta)$ (rejection probability)

$\pi(\Delta)$ = the probability that a $(0, a, \Delta, b)$-bridge is hit by an independent $p_\Delta(b)$-diffusion

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For ergodic diffusions:

$$p(\Delta) \to 0 \qquad \text{as } \Delta \to \infty$$

$$\pi(\Delta) \to 1 \qquad \text{as } \Delta \to \infty$$

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For diffusions with a spectral gap, $\lambda > 0$ (geometrically ergodic):

$$p(\Delta) = O(e^{-\lambda\Delta/2}) \qquad\qquad 1 - \pi(\Delta) = O(e^{-\lambda\Delta/2}).$$

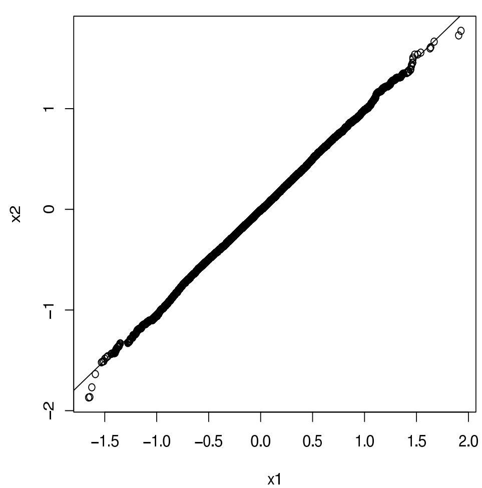
Hyperbolic diffusion bridge

$$dX_t = -\frac{X_t}{\sqrt{1+X_t^2}}dt + dW_t$$

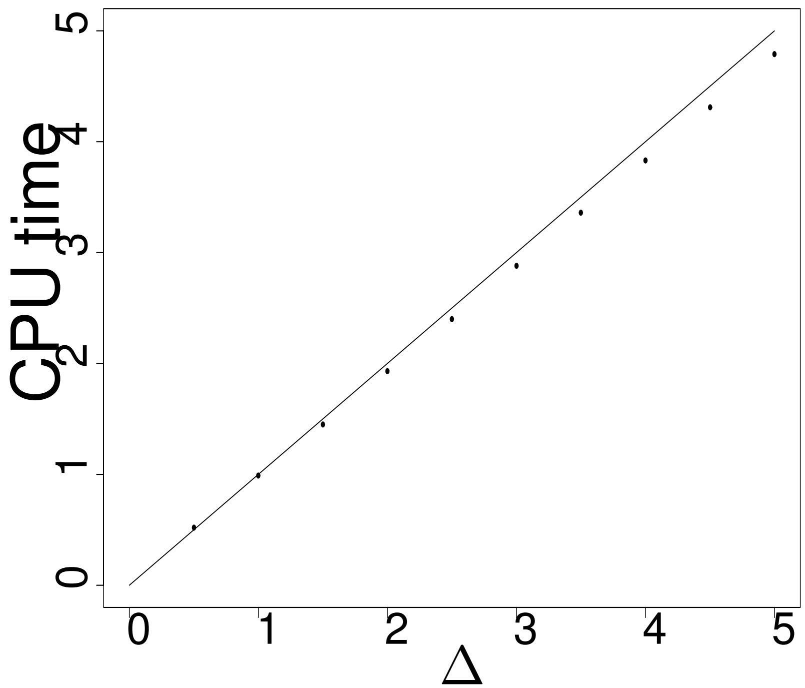
Barndorff-Nielsen (1978)

The exact EA1 algorithm of Beskos, Papaspiliopoulos and Roberts (2006) works for this diffusion

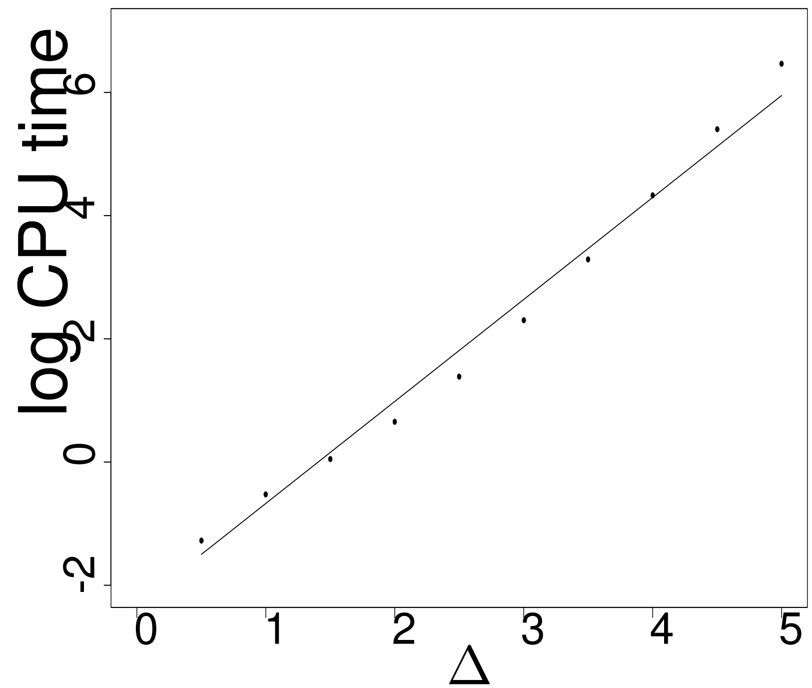
$(0, 0, 1, 0)$-hyperbolic diffusion bridge ($t = 0.5$, 25000 bridges)



Hyperbolic diffusion bridge



Approximate bridge simulation



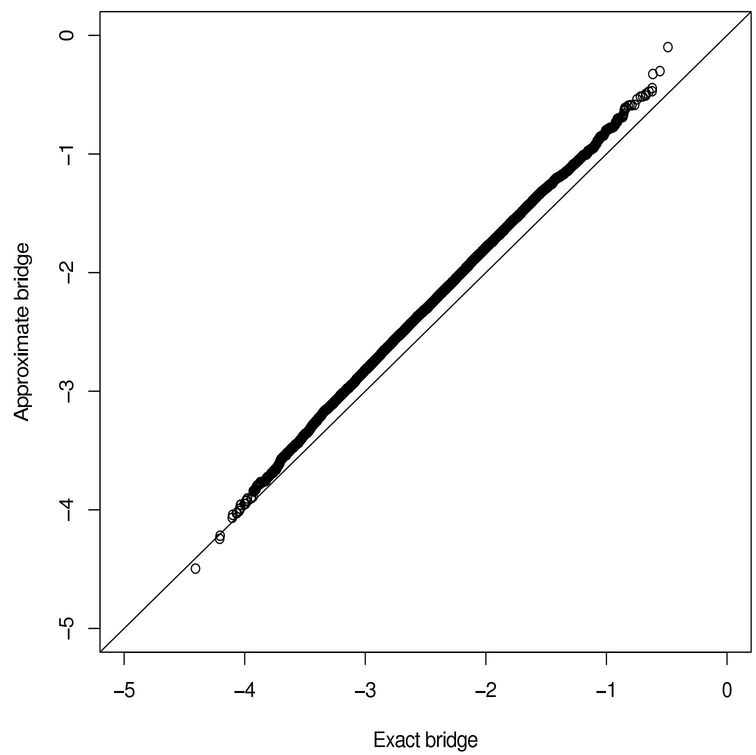
Beskos et al. (2006)

The CPU execution time to simulate 10,000 hyperbolic $(0, 0, \Delta, 0)$-bridges

Ornstein-Uhlenbeck bridge

$$dX_t = -0.5X_t dt + dW_t$$

$(0, -3, 1, -2)$ O–U bridge ($t = 0.5$, 25000 bridges)



Diffusion bridge simulation

The density of the approximate diffusion bridge Z on the canonical space C_Δ (continuous functions defined on $[0, \Delta]$ with the usual σ-algebra):

$$f_a(x) = f_b(x)\pi_\Delta(x)/\pi_\Delta$$

f_b is the density of a $(0, a, \Delta, b)$-diffusion bridge

$$\pi_\Delta(x) = P(Y \in A_x) \qquad \pi_\Delta = P((X, Y) \in A)$$

where X and Y are independent
X is a $(0, a, \Delta, b)$-diffusion bridge
Y is a $p_\Delta(b)$-diffusion,

$$A_x = \{y \in C_\Delta \,|\, \mathrm{gr}(y) \cap \mathrm{gr}(x) \neq \emptyset\} \qquad A = \{(x, y) \in C_\Delta^2 \,|\, \mathrm{gr}(y) \cap \mathrm{gr}(x) \neq \emptyset\}$$

M-H algorithm: exact diffusion bridge

Simulate an initial approximate $(0, a, \Delta, b)$-diffusion bridge, $X^{(0)}$, set k = 1.

(1) Propose a new sample paths by simulating an approximate $(0, a, \Delta, b)$-diffusion bridge $X^{(k)}$

(2) Accept the proposed diffusion bridge with probability

$$\min\left(1, \frac{f_b(X^{(k)}) f_a(X^{(k-1)})}{f_b(X^{(k-1)}) f_a(X^{(k)})}\right) = \min\left(1, \frac{\pi_\Delta(X^{(k-1)})}{\pi_\Delta(X^{(k)})}\right)$$

Otherwise $X^{(k)} = X^{(k-1)}$

(3) Set $k = k + 1$ and GO TO (1)

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But we do not know $\pi_\Delta(x)$

Pseudo-marginal MH-algorithm

Andrieu and Robert (2009)

For a given $x \in C_\Delta$, define a random variable T as follows:

Simulate a sequence of independent $p_\Delta(b)$-diffusions $Y^{(1)}, Y^{(2)}, \ldots$ until a sample path is obtained that intersects x. Define T by

$$T = \min\{i : Y^{(i)} \in A_x\}.$$

Then

$$E(T) = 1/\pi_\Delta(x).$$

If $\mathbf{T} = (T_1, \ldots, T_N)$ is a vector of N independent draws of T, then an unbiased and consistent estimator of $1/\pi_\Delta(x)$ is

$$\hat{\rho}_\Delta(x; \mathbf{T}) = \frac{1}{N} \sum_{j=1}^{N} T_j.$$

M-H algorithm: exact diffusion bridge

Metropolis-Hastings Markov chain with state $(X^{(k)}, \mathbf{T}^{(k)})$.

Simulate an initial approximate $(0, a, \Delta, b)$-diffusion bridge, $X^{(0)}$, N independent T-values, $\mathbf{T}^{(0)} = (T_1^{(0)}, \ldots, T_N^{(0)})$ with $x = X^{(0)}$, and set k = 1.

(1) Propose a new sample paths by simulating an approximate $(0, a, \Delta, b)$-diffusion bridge $X^{(k)}$, and simulate N independent T-values, $\mathbf{T}^{(k)} = (T_1^{(k)}, \ldots, T_N^{(k)})$ with $x = X^{(k)}$

(2) Accept the proposed $(X^{(k)}, \mathbf{T}^{(k)})$ with probability

$$\min\left(1, \frac{\hat{\rho}_\Delta(X^{(k)}; \mathbf{T}^{(k)})}{\hat{\rho}_\Delta(X^{(k-1)}; \mathbf{T}^{(k-1)})}\right)$$

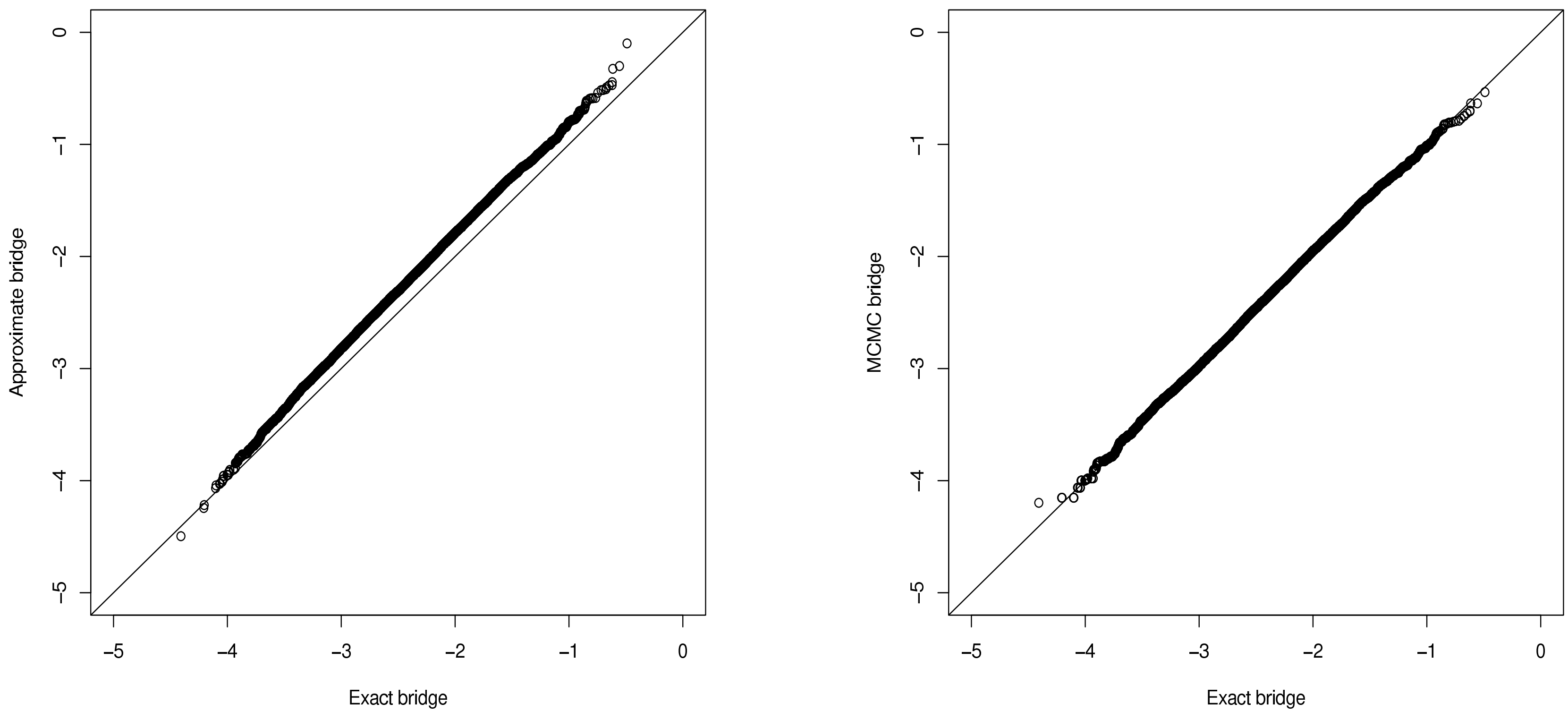
Otherwise $X^{(k)} = X^{(k-1)}$ and $\mathbf{T}^{(k)} = \mathbf{T}^{(k-1)}$

(3) Set $k = k + 1$ and GO TO (1)

Ornstein-Uhlenbeck bridge

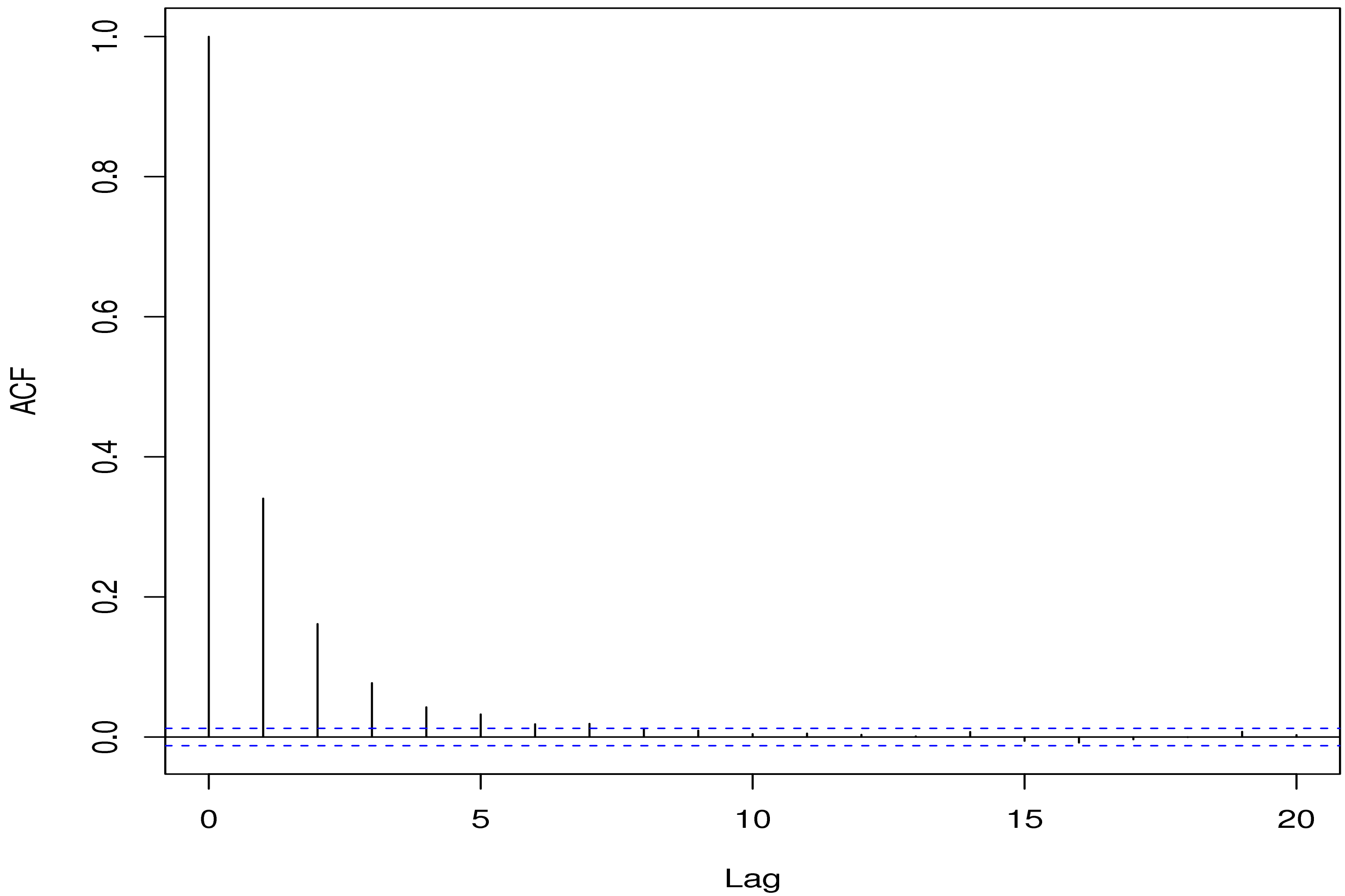
$(0, -3, 1, -2)$ O–U bridges ($\theta = 0.5$, $\sigma = 1.0$, $t = 0.5$, 25000 bridges)

Approximate simulation and exact MH-simulation (N=10, burn-in 5000)



Autocorrelations for the exact MH-algorithm

Autocorrelations for the exact MH-algorithm of the state at time 0.5 for simulated $(0, -3, 1, -2)$ O–U bridges ($\theta = 0.5$, $\sigma = 1.0$)



Integrated diffusions

$$dX_t = b(X_t;\alpha)dt + \sigma(X_t;\beta)dW_t, \qquad X_0 \sim \mu_{\alpha,\beta}$$

Data:

$$Y_i = \int_{t_{i-1}}^{t_i} X_s\,ds + Z_i, \qquad i = 1, \ldots, n$$

$Z_i \sim N(0, \tau^2)$, independent

Bollersev and Wooldridge (1992)

Gloter (2000, 2006)

Ditlevsen and Sørensen (2004)

Comte, Genon-Catalot and Rozenholc (2008)

Forman and Sørensen (2008)

Baltazar-Larios and Sørensen (2010)

Sørensen (2011)

Likelihood inference

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Likelihood conditional on the diffusion process:

$$\prod_{i=1}^{n} \varphi\left(Y_i; \int_{t_{i-1}}^{t_i} X_s\,ds,\ \tau^2\right)$$

φ Gaussian density function

Likelihood with full diffusion observation

$$dX_t = b(X_t;\alpha)dt + \sigma(X_t;\beta)dW_t, \qquad X_0 \sim \mu_{\alpha,\beta}$$

Likelihood if we had observed X continuously in $[t_0, t_n]$:

$$U_t = h(X_t;\beta) \qquad\qquad h(x;\beta) = \int^x \frac{1}{\sigma(y;\beta)}dy$$

$$dU_t = c(U_t;\alpha,\beta)dt + dB_t,$$

$$c(x;\alpha,\beta) = \frac{b\left(h^{-1}(x;\beta);\alpha\right)}{\sigma\left(h^{-1}(x;\beta);\beta\right)} - \frac{1}{2}\sigma'\left(h^{-1}(x;\beta);\beta\right)$$

Likelihood with full diffusion observation

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$$dU_t = c(U_t;\alpha,\beta)dt + dB_t,$$

$$c(x;\alpha,\beta) = \frac{b\left(h^{-1}(x;\beta);\alpha\right)}{\sigma\left(h^{-1}(x;\beta);\beta\right)} - \frac{1}{2}\sigma'\left(h^{-1}(x;\beta);\beta\right)$$

$$Y_i = \int_{t_{i-1}}^{t_i} h^{-1}(U_s;\beta)\,ds + Z_i, \qquad i = 1,\dots,n$$

Likelihood with full diffusion observation

$$\log L(\alpha, \beta, \tau^2) = \sum_{i=1}^{n} \log \varphi \left(Y_i; \int_{t_{i-1}}^{t_i} h^{-1}(U_s; \beta)\, ds,\ \tau^2 \right) + a(U_{t_n}; \alpha, \beta) - a(U_{t_0}; \alpha, \beta) - \frac{1}{2} \int_{t_0}^{t_n} \big(c(U_t; \alpha, \beta)^2 + c'(U_t; \alpha, \beta) \big)\, dt$$

$$a(x; \alpha, \beta) = \int^x c(u; \alpha, \beta) du.$$

EM algorithm

$\theta = (\alpha, \beta, \tau^2)$

$\tilde{\theta}$ any value of the parameter vector

(1) (E–step) Generate sample paths of $X^{(k)}$, $k = 1, \ldots, M$ conditional on $Y_1, \ldots, Y_n$ using the parameter value $\tilde{\theta}$, and calculate

$$g(\theta) = \frac{1}{M - M_0} \sum_{k=M_0+1}^{M} \log L\left(\theta; Y_1, \ldots, Y_n, h(X_t^{(k)}; \tilde{\beta}), t \in [t_0, t_n]\right)$$

(for a suitable burn-in period M_0)

(2) (M–step) $\tilde{\theta} =$ argmax $g(\theta)$

(3) GO TO (1)

Conditional diffusion simulation

Chib, Pitt & Shephard (2006)

Simulate an unrestricted stationary sample path of X in $[t_0, t_n]$

Repeat the following

1) Randomly draw $\nu_2 < \cdots < \nu_{K+1}$ from the set $\{1, 2, \ldots, n-1\}$ and set $\nu_1 = 0$, $\nu_{K+2} = n$, $\tau_j = t_{\nu_{j+1}}$, $j = 0, \ldots, K+1$

2) In each interval $[\tau_{j-1}, \tau_j]$ update by simulating a diffusion bridge conditional on the values of the diffusion process at the times τ_{j-1} and τ_j obtained in the previous iteration and on $Y_{t_{\nu_j}}, Y_{t_{\nu_j}+1}, \ldots, Y_{t_{\nu_{j+1}}}$

Conditional bridge simulation

Simulate an initial $(\tau_{j-1}, v_{i0}, \tau_j, v_{i1})$-diffusion bridge, $X^{(0)}$, and set k = 1.

(1) Propose a new sample paths by sampling a $(\tau_{j-1}, v_{i0}, \tau_j, v_{i1})$-diffusion bridge $X^{(k)}$

(2) Accept the proposed diffusion bridge with probability

$$\min\left(1, \prod_{i=1}^{n_j} \frac{\varphi\left(Y_{\nu_j+i};\ \int_{t_{i-1}}^{t_i} X_s^{(k)}\, ds,\ \tau^2\right)}{\varphi\left(Y_{\nu_j+i};\ \int_{t_{i-1}}^{t_i} X_s^{(k-1)}\, ds,\ \tau^2\right)}\right).$$

Otherwise $X^{(k)} = X^{(k-1)}$

(3) Set $k = k + 1$ and GO TO (1)

Integrated O-U process: simulation study

$$dX_t = -\alpha X_t dt + \sigma dW_t$$

$\alpha = 0.1 \quad \sigma = 0.5 \quad \tau^2 = 1.25$

$Y_i, \quad t_i = i, \quad i = 1.\ldots, 1500$

$M = 10000, \quad M_0 = 1000$

1000 simulated datasets

Average of parameter estimates:

λ	α	σ	τ^2
10	0.106	0.523	1.229
20	0.101	0.507	1.235
30	0.084	0.458	1.252

Integrated CIR process: simulation study

$$dX_t = (\alpha - \beta X_t)dt + \sigma\sqrt{X_t}dW_t$$

$\alpha = 0.5 \quad \beta = 0.2 \quad \sigma = 0.5 \quad \tau^2 = 1.25$

$Y_i, \quad t_i = i, \quad i = 1 \ldots, 1500$

$M = 10000, \quad M_0 = 1000$

1000 simulated datasets

Average of parameter estimates:

λ	α	β	σ	τ^2
30	0.4802	0.2056	0.4787	1.2432
20	0.4727	0.2043	0.4698	1.2406
10	0.4587	0.1965	0.4609	1.2287

Multivariate diffusions

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t, \quad X'_\Delta = b$$

$$dX'_t = \alpha(X'_t)dt + \sigma(X'_t)dW'_t, \quad X'_0 = a$$

Multivariate diffusions

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t, \quad X'_\Delta = b$$

$$dX'_t = \alpha(X'_t)dt + \sigma(X'_t)dW'_t, \quad X'_0 = a$$

Lindvall & Rogers (1986), Chen &Li (1989)

$$dW'_t = \left[I - (1-\gamma)\Pi(X_t, X'_t)\right] dW_t + \sqrt{1-\gamma^2}\,u(X_t, X'_t)dU_t.$$

$\gamma \in [-1, 1)$, $\quad U$ is a univariate standard Wiener process independent of W

$$\Pi(x, x') = u(x, x')u(x, x')^T \qquad u(x, x') = \frac{\sigma(x')^{-1}(x - x')}{|\sigma(x')^{-1}(x - x')|}.$$

Multivariate diffusions

Assume that X is ergodic with invariant probability density function ν

Time-reversed diffusion:

$$dX_t^* = \alpha^*(X_t^*)dt + \sigma(X_t^*)dW_t$$

$$\alpha_i^*(x) = -\alpha_i(x) + \nu(x)^{-1} \sum_{j=1}^{p} \partial_{x_j} \left(\nu(x) V(x)_{ij} \right), \quad i = 1, \ldots, d,$$

$V(x) = \sigma(x)\sigma(x)^T$

Provided that

$$\int_D \left| \sum_{j=1}^{d} \partial_{x_j} \left(\nu(x) V_{ij}(x) \right) \right| dx < \infty, \quad i = 1, \ldots, d.$$

Multivariate Ornstein-Uhlenbeck process

$$dX_t = -BX_t dt + dW_t,$$

$$B = \left\{ \begin{array}{ll} 1.5 & 1 \\ 1 & 1.5 \end{array} \right\}$$

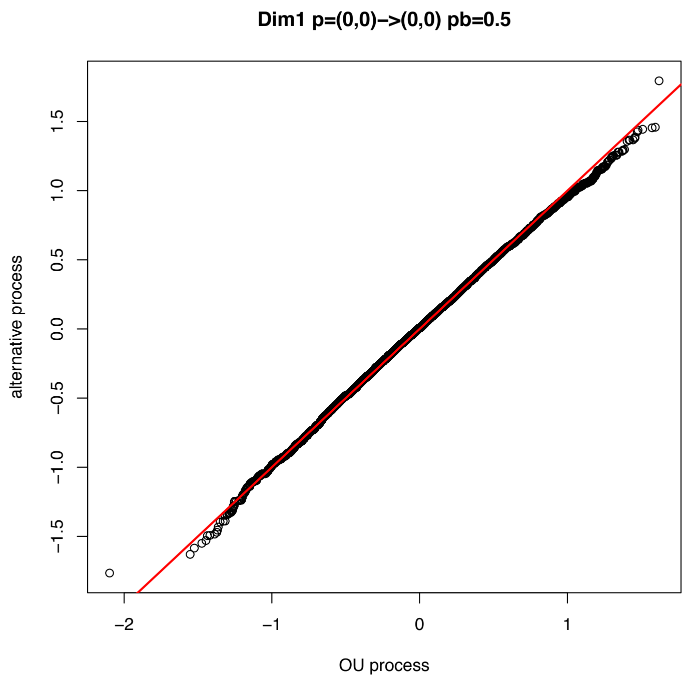
Ergodic and time-reversible with stationary distribution $N_2(0, \Gamma)$, where

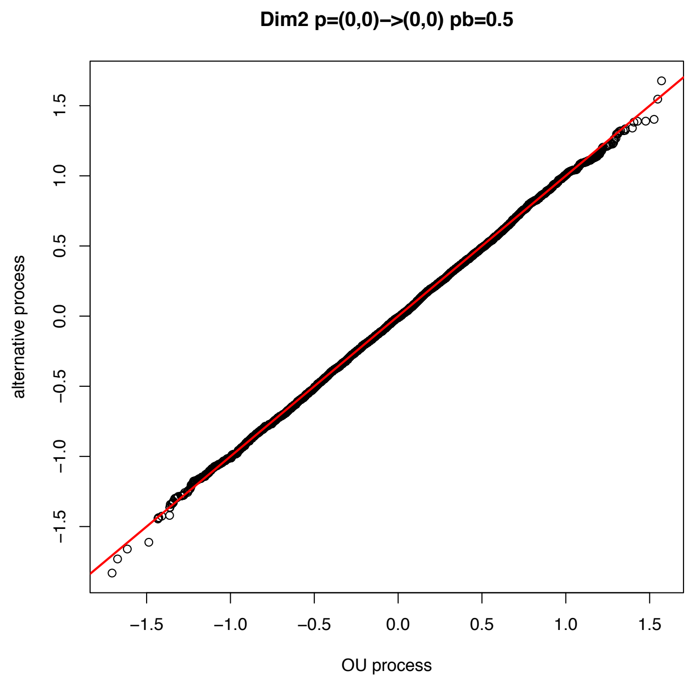
$$\Gamma = \left\{ \begin{array}{rr} 0.6 & -0.4 \\ & \\ -0.4 & 0.6 \end{array} \right\}$$

Simulation of Ornstein-Uhlenbeck

Bridge from $(0,0)$ to $(0,0)$, $t = 0.5$, 20000 bridges

QQ-plots of approximate bridge against exact bridge (1st and 2nd coordinate)

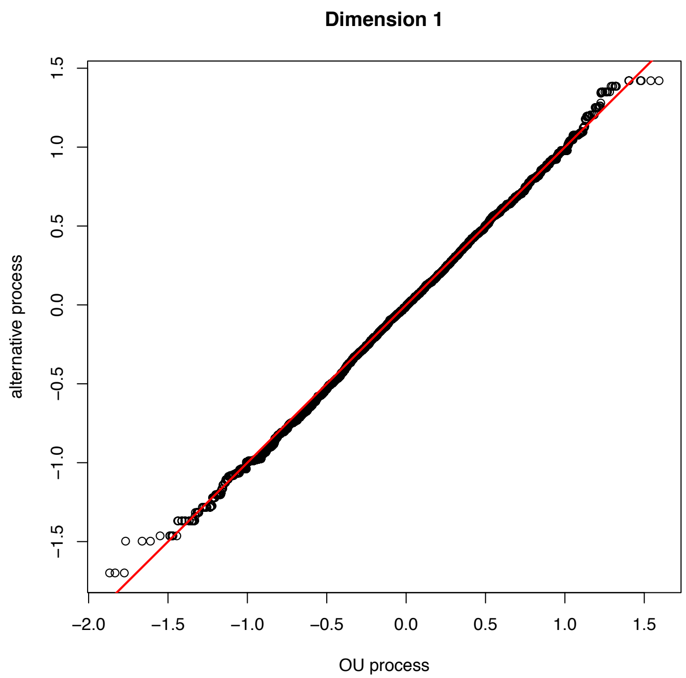


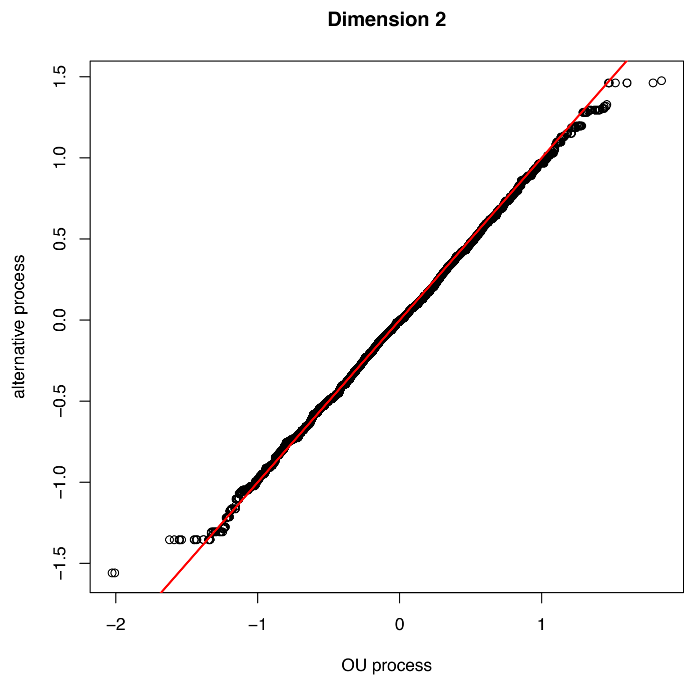


Simulation of Ornstein-Uhlenbeck

Bridge from $(0, 0)$ to $(0, 0)$, $t = 0.5$, 20000 bridges

QQ-plots of mcmc bridge against exact bridge (1st and 2nd coordinate)





Simulation of Ornstein-Uhlenbeck

Bridge from $(0,0)$ to $(0,0)$, $t = 0.5$, 20000 bridges

Level curves for empirical copula of approximate bridge and exact bridge