Stochastic optimal control with rough paths

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Introduction

Main motivation : stochastic/deterministic duality.
We are given a stochastic optimization problem (e.g. optimal stopping, optimal control of diffusions...).
The possible controls ν must be adapted (non-anticipating), denote the associated gain $J(\nu)$.
We want to compute the value

$$V = \sup_{\nu \text{ adapted}} \mathbb{E}[J(\nu)],$$

and (almost) optimal controls.
Lower bounds are given by any choice of policy ν, $V \geqslant \mathbb{E}\left[J(\nu)\right]$.
To know how good a policy is : need for upper bound.

Idea of the deterministic/stochastic duality

Information relaxation :

$$\sup_{\nu \text{ adapted}} \mathbb{E}[J(\nu)] \leqslant \sup_{\nu \text{ anticipating}} \mathbb{E}[J(\nu)] = \mathbb{E}\left[\sup_{\mu} J(\mu)(\omega)\right]$$

This inequality has no reason to be sharp ("value of information"), but the hope is that one can penalize anticipating controls

$$\sup_{\nu \text{ adapted}} \mathbb{E}[J(\nu)] = \inf_{P \in \mathcal{P}} \mathbb{E}\left[\sup_{\mu} J(\mu) - P(\mu)\right],$$

for some suitably chosen class of penalties $\mathcal{P}$.

Our context

Controlled diffusions

$$dX = b(X,\nu)dt + \sigma(X)dB_t.$$

Technical difficulty : need way to make sense of controlled SDEs with anticipating coefficients
$\longrightarrow$ rough path theory.

Outline

Outline

Rough path theory

ODE driven by a path $x_t = (x_t^1, \dots, x_t^d)$:

$$dy_t = V(y_t)dx_t := \sum_{i=1}^{d} V^i(y_t)dx_t^i. \tag{1}$$

Extension to non-smooth x ?

- Doss-Sussmann : When $d = 1$, solution $y_t = f(x_t)$, $\dot{f} = V(f)$.
  $\longrightarrow$ extension to any continous path x by continuity.
  Also for $d > 1$, when the vector fields V^i commute, $y_t = f(x_t^1, \dots, x_t^d)$.
- $x \mapsto y$ continuous in β-Hölder topology, $\beta > \frac{1}{2}$ (Young integration).

These do not cover multi-dimensional Brownian Motion !

Rough path theory

Key idea of Lyons (1998) : needs to consider extra data, the iterated integrals of x against itself :

$$
\begin{aligned}
I^2(x)_{s,t} &:= \int_s^t x_{s,r} \otimes dx_r = \left(\int_s^t x^i_{s,r} dx^j_r \right)_{1 \leqslant i,j \leqslant d} \\
&\dots \\
I^n(x)_{s,t} &:= \int_{s \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant t} dx_{t_1} \otimes \ldots \otimes dx_{t_n},
\end{aligned}
$$

- These iterated integrals are not well-defined a priori for nonsmooth x, but **taking them as given data**, one can solve any ODE driven by x.
- One only needs to consider a finite number of those, depending on the regularity of x, e.g. α when x is α-Hölder. For $\frac{1}{3} < \alpha \leqslant \frac{1}{2}$: level 2 is enough.

"Level 2" rough paths : definition

Fixed $\frac{1}{3} < \alpha \leqslant \frac{1}{2}$.

- Rough path will be

$$\mathbf{x} = (x_{s,t}, \underline{x}_{s,t})_{0 \leqslant s,t \leqslant T},$$

valued in $\mathbb{R}^d \times (\mathbb{R}^d)^{\otimes 2}$

- Rough path distance :

$$d_\alpha(\mathbf{x}, \widetilde{\mathbf{x}}) := \sup_{0 \leqslant s,t \leqslant T} \frac{|x_{s,t} - \widetilde{x}_{s,t}|}{|t-s|^\alpha} + \frac{|\underline{x}_{s,t} - \widetilde{\underline{x}}_{s,t}|}{|t-s|^{2\alpha}}.$$

- Geometric rough paths $\mathcal{D}^{0,\alpha}(\mathbb{R}^d)$: closure of (lift of) smooth paths under d_α.

Rough differential equations

Rough differential equation (RDE)

$$dy_t = V(y_t) d\mathbf{x}_t$$

Existence of **continuous** solution map (for V regular enough)

$$\begin{array}{rcl} \mathcal{D}^{0,\alpha} \times \mathbb{R}^n & \rightarrow & \mathcal{C}([0,T], \mathbb{R}^n) \\ (\mathbf{x}, y_0) & \mapsto & y \end{array}$$

RDEs and SDEs

- Consistency with SDEs : define $\mathbf{B} = (B, \int B \otimes \circ dB)$ Stratonovich lift of Brownian Motion.
  Then $\mathbf{B} \in \mathcal{D}^{0,\alpha}$ a.s., and the solution to RDE

  $$dy_t = V(y_t) d\mathbf{B}_t(\omega)$$

  coincides a.s. with the solution to SDE

  $$dY_t = V(Y_t) \circ dB_t.$$

- One advantage (among others) : no difficulty to make sense of **anticipating** SDEs

  $$dY_t \quad = \quad V(Y_t, \omega) \circ dB,$$

  as long as $y \mapsto V(y, \omega)$ is a.s. regular enough.

Classical deterministic optimal control

Class of admissible controls $\mathcal{M} = \{\mu : [0, T] \to U \text{ measurable } \}$.
Controlled ODE :

$$dX_s^{t,x,\mu} = b\left(X_s^{t,x,\mu}, \mu_s\right) ds + \sigma\left(X_s^{t,x,\mu}\right) d\eta_s, \qquad X_t^{t,x,\mu} = x \quad \in \mathbb{R}^e$$

Here $\eta : [0, T] \to \mathbb{R}^d$ is a smooth path.
Optimization problem :

$$\begin{aligned} J(t, x; \mu, \eta) &:= \int_t^T f\left(s, X_s^{t,x,\mu}, \mu_s\right) ds + g\left(X_T^{t,x,\mu}\right), \\ v(t, x) &:= \sup_\mu J(t, x; \mu, \eta). \end{aligned}$$

Then to solve for the value function v and the optimal control :
HJB equation, Pontryagin maximum principle (PMP)...
We want to extend this to $\boldsymbol{\eta}$ rough path.

Controlled RDE

Now for $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}$,

$$dX_s^{t,x,\mu} = b\left(X_s^{t,x,\mu}, \mu_s\right) ds + \sigma\left(X_s^{t,x,\mu}\right) d\boldsymbol{\eta}_s \tag{2}$$

Regularity requirements :

- $b(\cdot, u) \in \mathrm{Lip}^1(\mathbb{R}^e)$ uniformly in $u \in U$
- $\sigma_1, \ldots, \sigma_d \in \mathrm{Lip}^\gamma(\mathbb{R}^e)$, for some $\gamma > \frac{1}{\alpha}$.

(For $\gamma = [\gamma] + \{\gamma\}$, where $[\gamma] \in \mathbb{N}$ and $\{\gamma\} \in (0, 1]$, f is in Lip^γ if it has derivatives up to order $[\gamma]$ which are $\{\gamma\}$-Hölder continuous.)

Theorem

For any μ in $\mathcal{M}$, (2) has a unique solution. Moreover the mapping

$$(\boldsymbol{\eta}, x_0) \mapsto X \in \mathcal{D}^{0,\alpha}$$

is locally Lipschitz continuous, uniformly in $\mu \in \mathcal{M}$.

The rough control problem

Payoff functions :

- $f : [0,T] \times \mathbb{R}^e \times U \to \mathbb{R}$ bounded, continuous and locally uniformly continuous in t, x, uniformly in u
- $g \in BUC(\mathbb{R}^e)$

Payoff

$$J(t,x;\mu;\boldsymbol{\eta}) \quad := \quad \int_t^T f\left(s, X_s^{t,x,\mu}, \mu_s\right) ds + g\left(X_T^{t,x,\mu}\right),$$

and value function

$$v(t,x) \quad := \quad \sup_{\mu} J(t,x;\mu,\boldsymbol{\eta}).$$

We will now show how in some sense, HJB equation and PMP hold for this rough control problem.

The rough HJB equation

Formally,

$$-dv - H\left(x, Dv\right) dt - \left\langle \sigma\left(x\right), Dv \right\rangle d\boldsymbol{\eta} = 0, \qquad (3)$$
$$v(T, x) = g(x)$$

where the Hamiltonian is

$$H(x, p) := \sup_{u \in U} \left\{ \left\langle b\left(x, u\right), p \right\rangle + f\left(t, x, u\right) \right\}.$$

How to make sense of this equation ?

Definition

Caruana, Friz, Oberhauser (2011) :
v is a viscosity solution to a rough PDE

$$-dv - F(t,x,v,Dv,D^2v)dt - G(t,x,v,Dv)d\boldsymbol{\eta}_t = 0,$$
$$v(T,x) = \phi(x),$$

if for any smooth $\eta^n \to \boldsymbol{\eta}$, $v^{\eta^n} \to v$, where v^{η^n} is the unique solution to the PDE with $\boldsymbol{\eta}$ replaced by η^n.

In our case :

Proposition

v is the unique viscosity solution to the rough HJB (3).

Pontryagin maximum principle

Assume b, f, g be C^1 in x, such that the derivative is Lipschitz in x, u and bounded.
Assume $\sigma_1, \ldots, \sigma_d \in \mathrm{Lip}^{\gamma+2}(\mathbb{R}^e)$.

Given (X, μ), dual RDE for the costate:

$$-dp(t) = D_x b(X_t, \mu_t) p(t) dt + D_x \sigma(X_t) p(t) d\boldsymbol{\eta}_t + D_x f(X_t, \mu_t) dt,$$
$$p(T) = D_x g(X_T).$$

Theorem

Let $\bar{X}, \bar{\mu}$ be an optimal pair. Let $\bar{p}$ be the associated costate. Then

$$b(\bar{X}_t, \bar{\mu}_t) \cdot p(t) + f(\bar{X}_t, \bar{\mu}_t) = \sup_{u \in U} \left[b(\bar{X}_t, u) \cdot p(t) + f(\bar{X}_t, u) \right], \quad a.e. \; t.$$

Pontryagin maximum principle

Remarks :

- The proof is similar to the classical one :
  - $\mu^{\varepsilon}(t) = 1_I(t)\mu(t) + 1_{[0,T]\setminus I}(t)\bar{\mu}(t)$, where $|I| = \varepsilon$.
  - Expansion $J(\bar{\mu}) - J(\mu^{\varepsilon}) = \varepsilon F(\bar{\mu}, \mu) + o(\varepsilon)$,
  - First order condition : $F(\bar{\mu}, \mu) \leqslant 0 \Rightarrow$ PMP.
- Sufficient conditions ? Formally, would require convexity of

$$(x, u) \mapsto b(x, u) \cdot p + f(x, u) + \sigma(x) \cdot \dot{\eta},$$

can only happen if σ linear in x.

Why no μ-dependence in σ ?

The problem is NOT to make sense of controlled RDEs of the form

$$dX = b(t, X, \mu)dt + \sigma(X, \mu)d\boldsymbol{\eta}_t,$$

which could be done by restricting the class $\mathcal{M}$ of controls, e.g. piecewise constant, feedback controls,...

Real problem : degenerate control problem, i.e.

$$\sup_{\mu \in \mathcal{M}} \left\{ \int_t^T f\left(s, X_s^{t,x,\mu,\boldsymbol{\eta}}, \mu_s\right) ds + g\left(X_T^{t,x,\mu,\boldsymbol{\eta}}\right) \right\}$$
$$= \int_t^T (\sup_{\mu,x} f(s, \mu, x))ds + \sup_x g(x).$$

The reason is that if σ has enough u-dependence and η has unbounded variation on any interval (as is the case for typical Brownian paths), the system can essentially be driven to reach any point instantly.

Anticipating stochastic control

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with brownian motion B.
Take $\boldsymbol{\eta} = \mathbf{B}(\omega)$, Stratonovich lift of Brownian motion, which is in $\mathcal{D}^{0,\alpha}$ $\mathbb{P}$-a.e. ω $(\frac{1}{2} > \alpha \geqslant \frac{1}{3})$.
If $\nu \in \mathcal{A}$, set of progressively measurable maps : $\Omega \times [0, T] \to U$, then

$$X|_{\mu=\nu(\omega), \boldsymbol{\eta}=\mathbf{B}(\omega)} = \widetilde{X}, \qquad \mathbb{P} - a.s., \tag{4}$$

where $\widetilde{X}$ is the (usual) solution to the controlled SDE

$$\widetilde{X}_t = x_0 + \int_0^t b(\widetilde{X}_r, \nu_r) dr + \int_0^t \sigma(\widetilde{X}) \circ dB_r.$$

Outline

Deterministic/stochastic duality

General idea :

$$\begin{aligned}\sup_{\nu} \mathbb{E}[J(\nu)] &= \mathbb{E}\left[\sup_{\mu} \{J(\mu) - P^*(\mu)\}\right], \\ &= \inf_{P \in \mathcal{P}} \mathbb{E}\left[\sup_{\mu} \{J(\mu) - P(\mu)\}\right].\end{aligned}$$

Long History :

- Rockafellar, Wets (70s),
- Davis and coauthors (late 80s–early 90s),
- Rogers, Haugh–Kogan, Brown–Smith–Sun (2001–present)

Example : duality for optimal stopping

Optimal stopping problem :

$$Y_0^* := \sup_{\tau \text{ stopping time}} \mathbb{E}\left[Z_\tau\right].$$

- Davis–Karatzas (1994),
- Rogers (2001), (also Haugh–Kogan (2002)):

$$Y_0^* = \inf_{M \in H_0^1} \mathbb{E}\left[\sup_t (Z_t - M_t)\right],$$

where H_0^1 is the space of martingales M s.t. $\sup_t M \in L^1$, and $M_0 = 0$.
Application to Monte-Carlo pricing of american options.

Duality for optimal control

Some previous works :

- Optimal control of diffusions : Davis–Burstein (1992). Use anticipative stochastic calculus based on flow decomposition.
- Discrete-time controlled Markov processes :
  Rogers (2006), Brown–Smith–Sun (2010),....
  Numerically useful to compute upper-bounds !

We extend these approaches in our framework.

Setting : optimal control of diffusions

For $\nu \in \mathcal{A}$,

$$\begin{aligned} X_t^{t,x,\nu} &= x, \\ dX_s^{t,x,\nu} &= b\left(X_s^{t,x,\nu}, \nu_s\right) ds + \sum_{i=1}^{d} \sigma_i\left(X_s^{t,x,\nu}\right) \circ dB_s^i, \\ &= \widetilde{b}\left(X_s^{t,x,\nu}, \nu_s\right) ds + \sum_{i=1}^{d} \sigma_i\left(X_s^{t,x,\nu}\right) dB_s^i. \end{aligned}$$

Value function :

$$V(t,x) := \sup_{\nu \in \mathcal{A}} \mathbb{E}\left[\int_t^T f\left(s, X_s^{t,x,\nu}, \nu_s\right) ds + g\left(X_T^{t,x,\nu}\right)\right].$$

Anticipating stochastic control

$$
\begin{aligned}
& \sup_{\nu \in \mathcal{A}} \mathbb{E}\left[\int_t^T f\left(s, X_s^{t,x,\nu}, \nu_s\right) ds + g\left(X_T^{t,x,\nu}\right)\right] \\
= \quad & \sup_{\nu \in \mathcal{A}} \mathbb{E}\left[J(t,x;\mu,\boldsymbol{\eta})\big|_{\mu=\nu(\omega),\boldsymbol{\eta}=\mathbf{B}(\omega)}\right]. \\
\leqslant \quad & \mathbb{E}\left[\sup_{\mu \in \mathcal{M}} \left\{J(t,x;\mu,\boldsymbol{\eta})\right\}\Big|_{\mu=\nu(\omega),\boldsymbol{\eta}=\mathbf{B}(\omega)}\right].
\end{aligned}
$$

We don't expect equality !
Difference between the two sides of this equation : "value of information".

However, we can hope to penalize the r.h.s. to obtain equality.

Penalization

$\mathcal{Z}_{\mathcal{F}}$ class of admissible penalties $z : \mathcal{D}^{0,\alpha} \times \mathcal{M} \to \mathbb{R}$ such that

- z is bounded, measurable, and continuous in $\boldsymbol{\eta} \in \mathcal{D}^{0,\alpha}$ uniformly over $\mu \in \mathcal{M}$
- $\mathbb{E}[z(\mathbf{B}, \nu)] \geq 0$, if ν is adapted

Theorem

$$V(t,x) = \inf_{z \in \mathcal{Z}_{\mathcal{F}}} \mathbb{E}\left[\sup_{\mu \in \mathcal{M}} \{J(t,x;\mu,\boldsymbol{\eta}) + z(\boldsymbol{\eta},\mu)\} \Big|_{\mu=\nu(\omega),\boldsymbol{\eta}=\mathbf{B}(\omega)} \right].$$

Proof :
$' \leqslant '$ is obvious.
$' \geqslant '$: $z^*(\boldsymbol{\eta}, \mu) := V(t, x) - J(t, x; \mu, \boldsymbol{\eta})$. □

Of course this is not so interesting, since we already need to know V to get z^*.

We now describe two concrete (parametrized) subsets of $\mathcal{Z}_{\mathcal{F}}$ for which duality still holds.

The Rogers penalty (value-function based)

Define $L^u \phi = \widetilde{b}(x,u) \cdot \nabla \phi + \frac{1}{2} \mathrm{Tr}[\sigma\sigma^T(x,u) D^2 \phi]$.

Theorem

$$V(t,x) = \inf_{h \in C_b^{1,2}} \mathbb{E}\left[\sup_{\mu \in \mathcal{M}} \left\{ J(t,x;\mu,\boldsymbol{\eta}) - M_{t,T}^{t,x,\mu,\boldsymbol{\eta},h} \right\} \Big|_{\boldsymbol{\eta} = \mathbf{B}(\omega)} \right],$$

where

$$M_{t,T}^{t,x,\mu,\boldsymbol{\eta},h} := h\left(T, X_T^{t,x,\mu,\boldsymbol{\eta}}\right) - h\left(t, X_t^{t,x,\mu,\boldsymbol{\eta}}\right) - \int_t^T \left(\partial_s + L^{\mu_s}\right) h\left(s, X_s^{t,x,\mu,\boldsymbol{\eta}}\right) ds.$$

Moreover, if $V \in C_b^{1,2}$ the infimum is achieved at $h^* = V$.

Proof (1/3)

First remark : for $\nu \in \mathcal{A}$, $\mathbb{P}$-a.s, $M^{t,x,\nu(\omega),h}$ is the martingale increment $\int_t^T Dh(s, X_s)dB_s$, so has zero expectation.

The proof is more or less a verification argument for the HJB equation for V :

$$\begin{aligned} -\partial_t V - \sup_{u \in U} \left[L^u V + f(x,u) \right] &= 0, \\ V(T,\cdot) &= g. \end{aligned}$$

Denote by S_s^+ the class of $\mathcal{C}_b^{1,2}$ supersolutions to this equation.

Proof (2/3)

Then :

$$
\begin{aligned}
& V(t,x) \\
&= \inf_{h \in C_b^{1,2}} \sup_{\nu \in \mathcal{A}} \mathbb{E}\left[\left\{ J(t,x;\mu,\boldsymbol{\eta}) - M_{t,T}^{t,x,\mu,\boldsymbol{\eta},h} \right\}\Big|_{\mu=\nu(\omega),\boldsymbol{\eta}=\mathbf{B}(\omega)} \right] \\
&\le \inf_{h \in C_b^{1,2}} \mathbb{E}\left[\sup_{\mu \in \mathcal{M}} \left\{ J(t,x;\mu,\boldsymbol{\eta}) - M_{t,T}^{t,x,\mu,\boldsymbol{\eta},h} \right\}\Big|_{\boldsymbol{\eta}=\mathbf{B}(\omega)} \right] \\
&= \inf_{h \in C_b^{1,2}} \Big(h(t,x) + \mathbb{E}\Big[\sup_{\mu \in \mathcal{M}} \big\{ g(X_T^{t,x,\mu,\boldsymbol{\eta}}) - h(T, X_T^{t,x,\mu,\boldsymbol{\eta}}) \\
&\qquad + \int_t^T f(s, X_s^{t,x,\mu,\boldsymbol{\eta}}, \mu_s) + (\partial_s + L^{\mu_s}) h(s, X_s^{t,x,\mu,\boldsymbol{\eta}}) ds \Big\}\Big|_{\boldsymbol{\eta}=\mathbf{B}(\omega)} \Big] \Big) \\
&\le \inf_{h \in S_s^+} h(t,x).
\end{aligned}
$$

Proof (3/3)

Now :

- If V is $\mathcal{C}_b^{1,2}$, then $V \in S_s^+$.
- In general : by a result of Krylov (2000), it is actually true that $V = \inf_{h \in S_s^+} h$.

Hence all inequalities are equalities, and we get the result.

The Davis–Burstein penalty

Additional assumptions :

- $b \in C_b^5$, $\sigma \in C_b^5$, $\sigma\sigma^T > 0$,
- U compact convex subset of $\mathbb{R}^n$,
- existence of an optimal feedback control $u^*(t,x)$ continuous, C^1 in t and C_b^4 in x, taking values in the interior of U,
- an additional convexity assumption.

The Davis–Burstein penalty

Let A be the class of all $\lambda : [0, T] \times \mathbb{R}^e \times \mathcal{D}^{0,\alpha} \to \mathbb{R}^d$ such that

- λ is bounded and uniformly continuous on bounded sets
- λ is future adapted, i.e. for any fixed t, x, $\lambda(t, x, \mathbf{B}) \in \mathcal{F}_{t,T}$
- $\mathbb{E}[\lambda(t, x, \mathbf{B})] = 0$ for all t, x.

Theorem

Under these assumptions,

$$V(t,x) = \inf_{\lambda \in A} \mathbb{E}\Big[\sup_{\mu \in \mathcal{M}} \left\{ J(t,x;\mu,\boldsymbol{\eta}) + \int_t^T \langle \lambda(r, X_r^{t,x,\mu,\boldsymbol{\eta}}, \boldsymbol{\eta}), \mu_r \rangle dr \right\}\Big|_{\boldsymbol{\eta}=\mathbf{B}(\omega)} \Big].$$

Moreover the infimum is achieved at some λ^*.

Rough idea of the Davis–Burstein proof

- Find a λ^* such that for (almost) any $\boldsymbol{\eta}$, the feedback control $u^*(t,x)$ is still optimal for the deterministic rough problem :

$$\begin{aligned} &\sup_{\mu \in \mathcal{M}} \left[g(X_T^{t,x,\mu,\boldsymbol{\eta}}) + \int_t^T f(s, X_s^{t,x,\mu,\boldsymbol{\eta}}, \mu_s) - \langle \lambda^*(s, X_s^{t,x,\mu,\boldsymbol{\eta}}; \boldsymbol{\eta}), \mu_s \rangle \, ds \right] \\ = \quad & g(X_T^{u^*}) + \int_t^T f(s, X_s^{u^*}, u^*(s, X_s^{u^*})) - \left\langle \lambda^*(s, X_s^{u^*}; \boldsymbol{\eta}), u^*(s, X_s^{u^*}) \right\rangle ds \end{aligned}$$

- Based on HJB verification argument for this deterministic control problem, i.e. find W s.t. (formally)

$$\begin{aligned} 0 \quad = \quad & -\partial_t W - \sup_{u \in U} \{ \langle b(x,u), DW \rangle + f(t,x,u) - \langle u, \lambda^*(t,x;\boldsymbol{\eta}) \rangle \} \\ & - \langle \sigma(x), DW \rangle \dot{\boldsymbol{\eta}} \\ = \quad & -\partial_t W - \langle b(x, u^*(t,x)), DW \rangle + f(t,x,u^*(t,x)) - \langle u^*(t,x), \lambda^*(t,x;\boldsymbol{\eta}) \rangle \\ & - \langle \sigma(x), DW \rangle \dot{\boldsymbol{\eta}} \end{aligned}$$

Explicit computations in LQ problems : additive noise

Additive noise :
Dynamics

$$dX = (MX + N\nu)dt + dB_t,$$

and control problem

$$V(t,x) = \inf_{\nu \in \mathcal{A}} \mathbb{E}\left[\frac{1}{2}\int_t^T (\langle QX_s, X_s\rangle + \langle R\nu_s, \nu_s\rangle)ds + \frac{1}{2}\langle GX_T, X_T\rangle\right].$$

Of course in that case the solution is well-known. Still interesting to compute and compare the optimal penalties.
Let P be the solution to the matrix Riccati equation

$$\begin{aligned} P(T) &= G, \\ P'(t) &= -P(t)M - {}^tMP(t) + PNR^{-1\,t}NP(t) - Q, \end{aligned}$$

Explicit computations in LQ problems : additive noise

Proposition

For this LQ control problem the optimal penalty corresponding to Theorem 7 (Davis–Burstein) is given by

$$z^1(\boldsymbol{\eta}, \mu) = -\int_t^T \langle \lambda^1(s; \boldsymbol{\eta}), \mu_s \rangle ds,$$

where

$$\lambda^1(t; \boldsymbol{\eta}) = -{}^t N \int_t^T e^{{}^t M(s-t)} P(s) d\boldsymbol{\eta}_s.$$

The optimal penalty corresponding to Theorem 6 (Rogers) is given by

$$z^2(\boldsymbol{\eta}, \mu) = z^1(\boldsymbol{\eta}, \mu) + \gamma^R(\boldsymbol{\eta}),$$

where $\gamma^R(\boldsymbol{\eta})$ is a random (explicit) constant (not depending on the control) with zero expectation.

Explicit computations in LQ problems : multiplicative noise

Dynamics

$$dX \quad = \quad (MX + N\nu)dt + \sum_{i=1}^{n} C_i X \circ dB_t^i,$$

and same cost criterion.
For notational simplicity we take $d = n = 1$.
Let $\Gamma_{t,s}$ be the solution to the RDE

$$d_s\Gamma_{t,s} = M\Gamma_{t,s}ds + C\Gamma_{t,s}d\boldsymbol{\eta}_s, \quad \Gamma_{t,t} = 1$$

For $t \leqslant r \leqslant T$, define

$$\Theta_r = \int_r^T P_s \Gamma_{r,s}^2 (d\boldsymbol{\eta}_s - Cds).$$

Explicit computations in LQ problems : multiplicative noise

Proposition

Then the optimal penalty corresponding to Theorem 7 (D–B) is given by

$$z^1(\boldsymbol{\eta}, \mu) = CNx \int_t^T \Theta_s \mu_s ds,$$

while the optimal penalty corresponding to Theorem 6 (R) is given by

$$z^2(\boldsymbol{\eta}, \mu) = C\Theta_t x^2 + CNx \int_t^T \Gamma_{t,s} \Theta_s \mu_s ds + CN^2 \int_t^T \int_t^T \Gamma_{r\wedge s, r\vee s} \Theta_{r\vee s} \mu_r \mu_s drds.$$

Conclusion

Summary :

- We define and study optimal control of RDEs.
- Special case : SDEs with anticipative control. This allows us to formulate a deterministic/stochastic duality in continuous time.

Some remaining questions :

- Theoretical results, but are there useful applications ?
- Extension to diffusions with control in the volatility $\sigma(X, \mu)dB$, non-Brownian noise...