Multi-level Monte Carlo for Approximation of Distribution Functions and an Application to AF^4

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I. The Computational Problem

Given a random element X (random vector, stochastic process, ...) and a real-valued functional φ, such that $\varphi(X)$ has a Lebesgue density.

Approximate

- the distribution function F or
- the density ρ

of $\varphi(X)$ on a compact interval.

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Application

- X is the solution of an SDE (with reflection)

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dW(t) + \big(dn(t)\big),$$

- φ is a functional on the corresponding path space, e.g., an exit time.

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Goal Monte Carlo (randomized) algorithm $\mathcal{M}$ with error

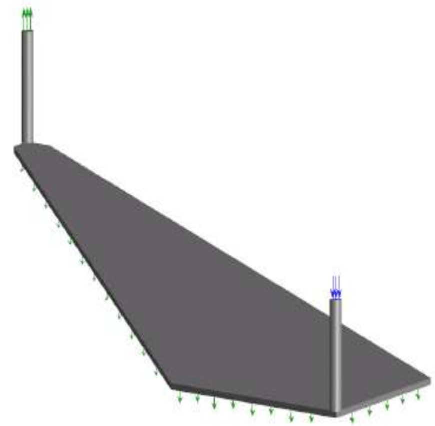
$$\mathrm{error}(\mathcal{M}) = \Big(\mathrm{E} \sup_{s \in [S_0, S_1]} |F(s) - \mathcal{M}|^2\Big)^{1/2}$$

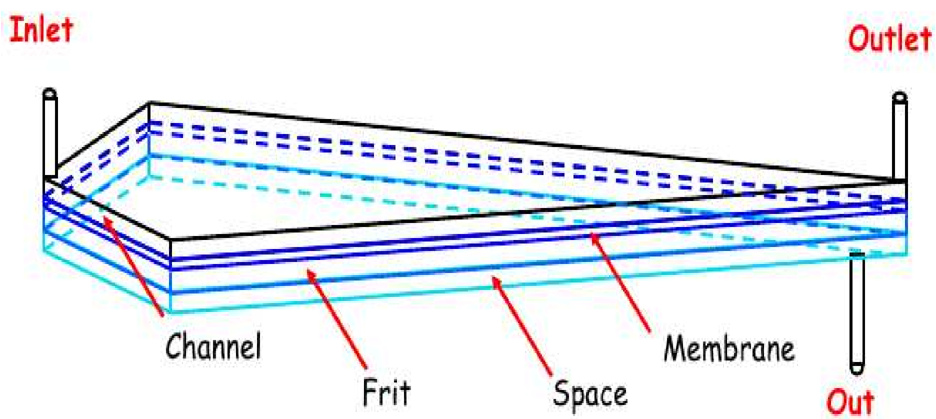
bounded by ε and with cost

$$\mathrm{cost}(\mathcal{M}) = \mathrm{E}(\text{\# operations and random number calls})$$

as small as possible. Likewise for ρ.

Asymmetric Flow Field Flow Fractionation Separation of nano-particles of different types.





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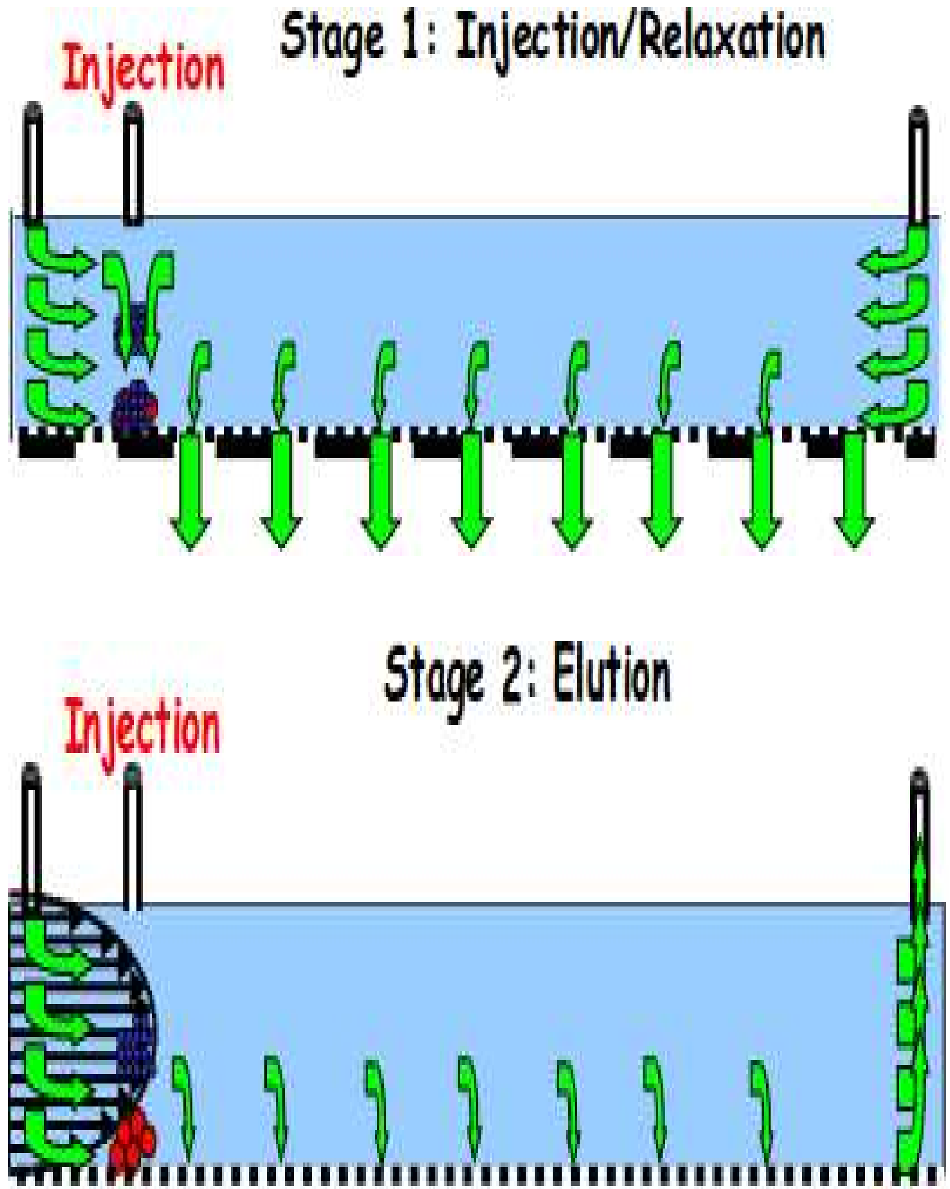

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We are interested in the stopped exit time φ, i.e.,

$$\varphi(x) = \inf\{t \geq 0 : x(t) \in D_0\} \wedge T,$$

where $D \subset \mathbb{R}^d$ is a bounded domain and $D_0 \subset \partial D$.

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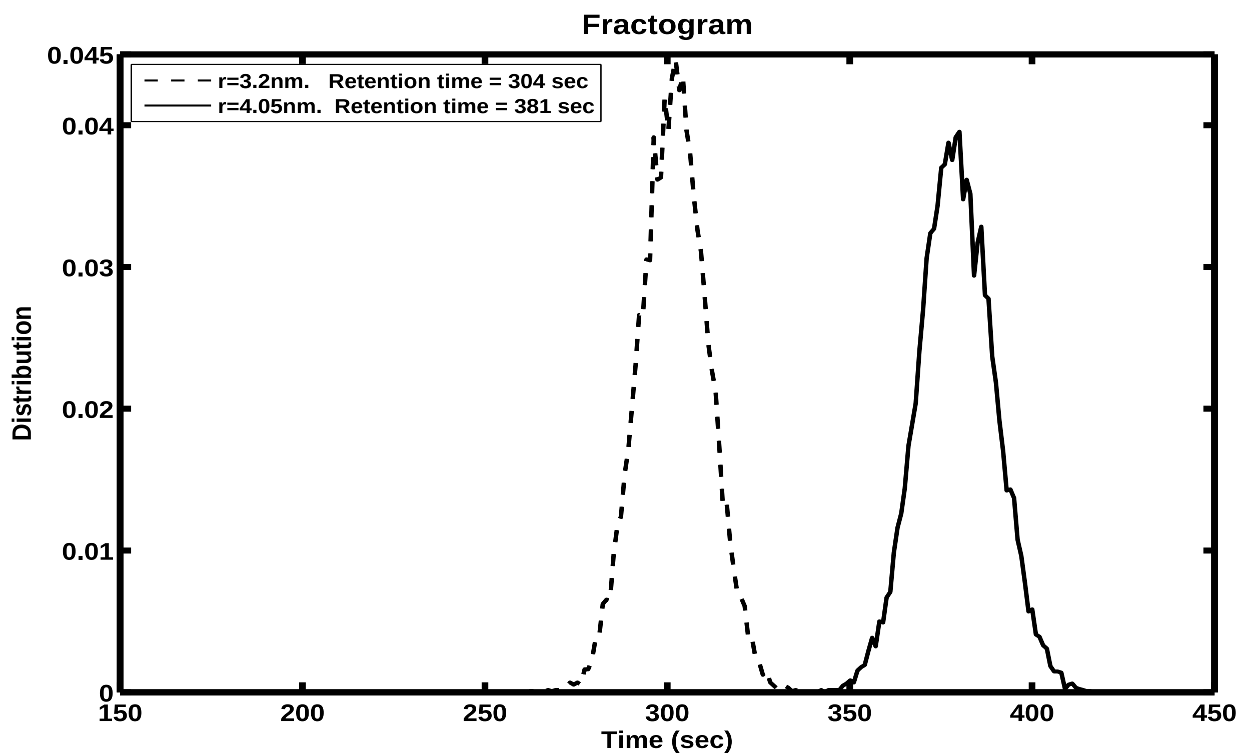
A measurement result (cf. emprical densities of two exit times)



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OUTLINE

II. Single-level MC for Real-valued Functionals

III. Multi-level MC for Real-valued Functionals

IV. Multi-level MC for Distribution Functions and Densities

V. Complexity of Infinite-Dimensional Integration Problems

Joint work with

M. Giles (Oxford), *O. Iliev* (ITWM), *T. Nagapetyan* (ITWM),

J. Creutzig (Darmstadt), *S. Dereich* (Münster), *T. Müller-Gronbach* (Passau)

II. Single-level MC for Real-valued Functionals

Compute $a = \mathrm{E}(\varphi(X)) \in \mathbb{R}$ for a real-valued functional φ.

Single-level Monte Carlo is defined by

$$\mathcal{M}_N^L = \frac{1}{N} \sum_{i=1}^{N} \varphi(X_i^{(L)})$$

with $L \in \mathbb{N}_0$, $N \in \mathbb{N}$, and independent copies $X_1^{(L)}, \ldots, X_N^{(L)}$ of $X^{(L)}$.

We have

$$\begin{aligned} \mathrm{error}^2(\mathcal{M}_N^L) &= \mathrm{E}(a - \mathcal{M}_N^L)^2 \\ &= \big(a - \mathrm{E}(\varphi(X^{(L)}))\big)^2 + \frac{1}{N}\,\mathrm{Var}(\varphi(X^{(L)})), \end{aligned}$$

$$\mathrm{cost}(\mathcal{M}_N^L) \le c \cdot N \cdot \mathrm{cost}(\varphi(X^{(L)})).$$

ASSUMPTIONS

Cost for simulating the distribution of $\varphi(X^{(\ell)})$

There exists $M > 1$, $c > 0$ such that, for $\ell \in \mathbb{N}_0$,

$$\mathrm{cost}(\varphi(X^{(\ell)})) \leq c \cdot M^{\ell}.$$

Weak error estimate

There exist $\alpha > 0$, $c > 0$ such that, for $\ell \in \mathbb{N}_0$,

$$\left| a - \mathrm{E}(\varphi(X^{(\ell)})) \right| \leq c \cdot M^{-\ell \cdot \alpha}.$$

Bounded variances $\sup_{\ell \in \mathbb{N}_0} \mathrm{Var}(\varphi(X^{(\ell)})) < \infty$.

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Example For the solution X of an SDE, the Euler approximation $X^{(\ell)}$ with step-size $2^{-\ell}$, and $\varphi : C([0,T], \mathbb{R}^d) \to \mathbb{R}$ being Lipschitz continuous

$$M = 2, \qquad \alpha = 1/2 - \varepsilon.$$

Terminology A sequence of Monte Carlo algorithms $\mathcal{M}_n$ with $\lim_{n\to\infty} \mathrm{error}(\mathcal{M}_n) = 0$ achieves the order of convergence $\gamma > 0$ if

$$\exists\, c > 0 \,\exists\, \varepsilon > 0 \,\forall\, n \in \mathbb{N} :$$

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Theorem

With suitably chosen parameters, the single-level Monte Carlo algorithms $\mathcal{M}_N^L$ achieve the order of convergence

$$\gamma = 2 + \frac{1}{\alpha}.$$

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Example For SDEs, the Euler approximation, and Lipschitz continuous functionals φ on the path space

$$\gamma = 4.$$

III. Multi-level MC for Real-valued Functionals

Compute $a = \mathrm{E}(\varphi(X)) \in \mathbb{R}$ for a real-valued functional φ.

Choose $L \in \mathbb{N}_0$ as previously. Clearly

$$\mathrm{E}(\varphi(X^{(L)})) = \mathrm{E}(\underbrace{\varphi(X^{(0)})}_{=\Delta^{(0)}}) + \sum_{\ell=1}^{L} \mathrm{E}\big(\underbrace{\varphi(X^{(\ell)}) - \varphi(X^{(\ell-1)})}_{=\Delta^{(\ell)}}\big).$$

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Idea Variance reduction, compared to single-level MC, by approximating $\mathrm{E}\Delta^{(0)}, \ldots, \mathrm{E}\Delta^{(L)}$ separately with independent MC algorithms.

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Cost for simulating the joint distribution of $\varphi(X^{(\ell)})$ and $\varphi(X^{(\ell-1)})$

There exists $M > 1$, $c > 0$ such that, for $\ell \in \mathbb{N}$,

$$\mathrm{cost}\big(\varphi(X^{(\ell)}), \varphi(X^{(\ell-1)})\big) \leq c \cdot M^{\ell}.$$

Weak error estimate

There exist $\alpha > 0$, $c > 0$ such that, for $\ell \in \mathbb{N}_0$,

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Strong error estimate

There exist $\beta > 0$, $c > 0$ such that, for $\ell \in \mathbb{N}_0$,

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$$M = 2, \qquad \alpha = 1/2 - \varepsilon, \qquad \beta = 1 - \varepsilon.$$

DEFINITION OF THE MULTI-LEVEL ALGORITHM

Consider an

- independent family of random elements $\big(X_i^{(\ell)}, Y_i^{(\ell)}\big)$ such that $\big(X_i^{(\ell)}, Y_i^{(\ell)}\big) \stackrel{\mathrm{d}}{=} \big(X^{(\ell)}, X^{(\ell-1)}\big)$.

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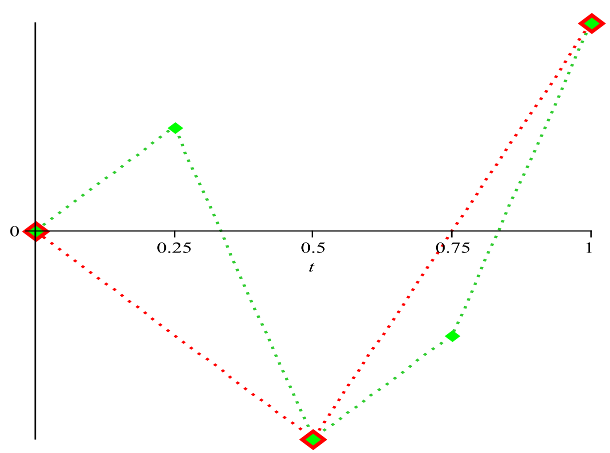
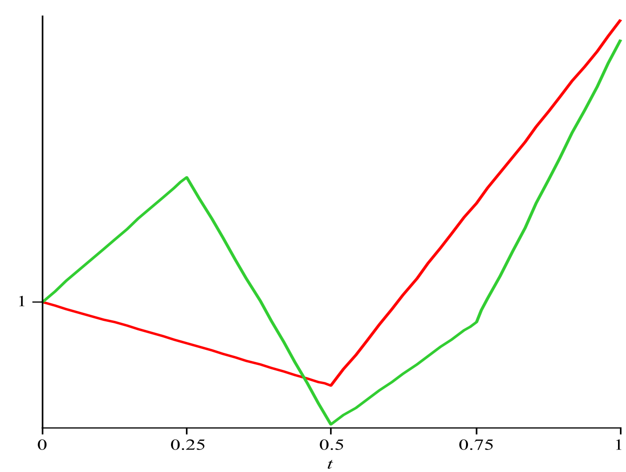
Example Coupled Euler scheme

W



$\left(X^{(2)}, X^{(1)}\right)$



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Put

$$\mathcal{M}^{L_0,L_1}_{N_{L_0},\dots,N_{L_1}} = \underbrace{\frac{1}{N_{L_0}} \cdot \sum_{i=1}^{N_{L_0}} \varphi(X_i^{(L_0)})}_{\to \mathrm{E}(\varphi(X^{(L_0)}))} + \sum_{\ell=L_0+1}^{L_1} \underbrace{\frac{1}{N_\ell} \cdot \sum_{i=1}^{N_\ell} \Big(\varphi(\color{blue}{X_i^{(\ell)}}) - \varphi(\color{blue}{Y_i^{(\ell)}}) \Big)}_{\to \mathrm{E}\left(\varphi(X^{(\ell)}) - \varphi(X^{(\ell-1)})\right)}.$$

Theorem *Giles (2008)*

With suitably chosen parameters, the multi-level Monte Carlo algorithms $\mathcal{M}^{L_0,L_1}_{N_{L_0},\ldots,N_{L_1}}$ achieve the order of convergence

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Remark Recall that single-level MC achieves

$$\gamma = 2 + \frac{1}{\alpha}.$$

Example For SDEs, the Euler approximation, and Lipschitz continuous functionals φ on the path space

$$\gamma = 2 \qquad \text{vs.} \qquad \gamma = 4.$$

Multi-level Monte Carlo

- Abstract analysis, integral equations, parametric integration

  Heinrich (1998), Heinrich, Sindambiwe (1999), ...

- SDEs, two-level construction, density estimation

  Kebaier (2005), *Kebaier, Kohatsu-Higa (2008)*

- S(P)DEs, computational finance, . . .

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  Giles (2008, . . .), **. . .**

- Non-standard SDEs, non-standard functionals

  Avikainen (2009), *Giles, Higham, Mao (2009)*, *Altmayer, Neuenkirch (2013)*,
  Dereich, Neuenkirch, Szpruch (2012), Dereich (2011),
  Hutzenthaler, Jentzen (2011,. . .)

- **. . .**

See **http://people.maths.ox.ac.uk/gilesm/mlmc_community.html**

IV. Multi-level MC for Distribution Functions

ASSUMPTIONS

Smoothness of the density ρ of $\varphi(X)$

There exists $r \in \mathbb{N}_0$ and $\delta > 0$ such that $\rho \in C^r([S_0 - \delta, S_1 + \delta])$.

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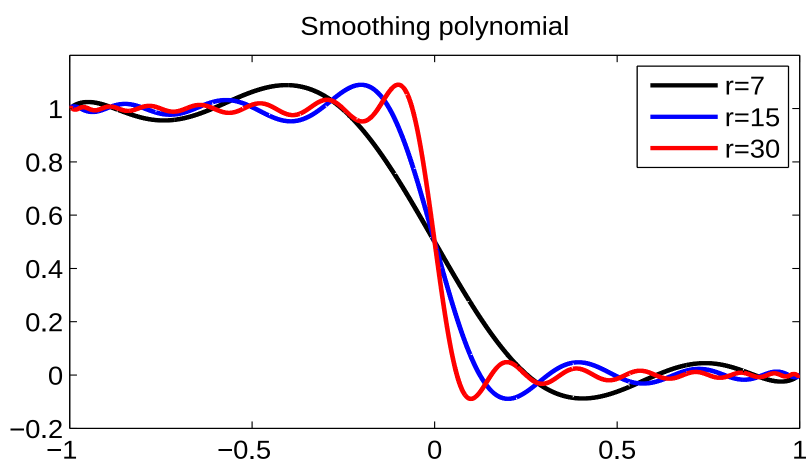

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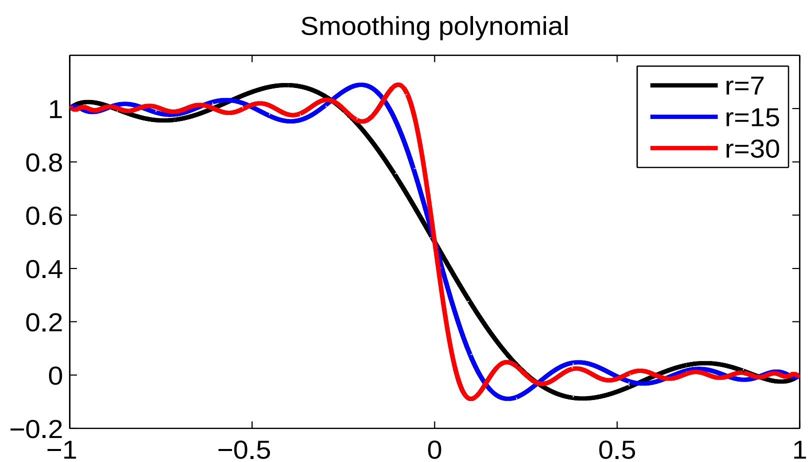

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There exist $\alpha_1 \geq 0$, $\alpha_2 > \alpha_3 \geq 0$, $c > 0$ such that, for $\ell \in \mathbb{N}_0$ and $\delta \in \,]0, 1]$,

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Strong error estimate

There exist $\beta_1 \geq 0$, $\beta_2 > 0$, $c > 0$ such that, for $\ell \in \mathbb{N}_0$ and $\delta \in]0, 1]$,

$$\mathrm{E} \min((\tau - \tau^{(\ell)})^2/\delta^2, 1) \leq c \cdot \delta^{-\beta_1} \cdot M^{-\ell \cdot \beta_2}.$$

Example 1 (SDE, smooth path-independent functional)

Let $\tau = \varphi(X_T)$, where

- X is the solution of a d-dimensional system of SDEs,
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Then $M = 2$ and, due to *Bally, Talay (1996)*,

$$\alpha_1 = 0, \qquad \alpha_2 = 1, \qquad \alpha_3 = 0,$$

Recall

$$\text{weak error} \leq c \cdot \min\left(\delta^{-\alpha_1} \cdot M^{-\ell\cdot\alpha_2}, M^{-\ell\cdot\alpha_3}\right).$$

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Recall

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Example 2 (SDE, smooth path-dependent functional)

Let $\tau = \varphi(X)$, where

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- φ : $C([0,T],\mathbb{R}^d) \to \mathbb{R}$ is Lipschitz continuous.

Take $\tau^{(\ell)} = \varphi(X^{(\ell)})$, where

- $X^{(\ell)}$ is the Euler scheme with 2^ℓ equidistant time-steps.

Then $M = 2$ and

$$\alpha_1 = \varepsilon, \qquad \alpha_2 = 1/2 - \varepsilon, \qquad \alpha_3 = 1/2 - \varepsilon,$$
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Cf. *Avikainen (2009)*.

Example 3 (SDE, stopped exit time)

Let $\tau = \varphi(X)$, where

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According to *Bouchard, Geiss, Gobet (2013)*, for $1 \leq p < \infty$,

$$\left(\mathrm{E}|\tau - \tau^{(\ell)}|^p\right)^{1/p} \leq c_p \cdot M^{-\ell/(2\,p)}.$$

DEFINITION OF THE MULTI-LEVEL ALGORITHM

Step 1 Approximation of the distribution function F of τ at discrete points.

Choose

- a number $k \in \mathbb{N}$ of point,
- a smoothing parameter $\delta > 0$,
- minimal and maximal levels $L_0, L_1 \in \mathbb{N}$, and
- replication numbers $N_\ell \in \mathbb{N}$ at the levels $\ell = L_0, \ldots, L_1$.

Put

$$s_i = S_0 + i \cdot (S_1 - S_0)/k, \qquad i = 0, \ldots, k,$$
$$g^{k,\delta}(t) = \left(g(\tfrac{t-s_1}{\delta}), \ldots, g(\tfrac{t-s_k}{\delta})\right), \qquad t \in \mathbb{R}.$$

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Consider an independent family of random vectors $(\tau_i^{(\ell)}, \sigma_i^{(\ell)})$ such that $(\tau_i^{(\ell)}, \sigma_i^{(\ell)}) \stackrel{\mathrm{d}}{=} (\tau^{(\ell)}, \tau^{(\ell-1)})$.

For approximation of $(F(s_1), \dots F(s_k))$ define the multi-level algorithm

$$\mathcal{M}^{k,\delta,L_0,L_1}_{N_{L_0},\dots,N_{L_1}} = \frac{1}{N_{L_0}} \cdot \sum_{i=1}^{N_{L_0}} g^{k,\delta}(\tau_i^{(L_0)}) + \sum_{\ell=L_0+1}^{L_1} \frac{1}{N_\ell} \cdot \sum_{i=1}^{N_\ell} \Big(g^{k,\delta}(\tau_i^{(\ell)}) - g^{k,\delta}(\sigma_i^{(\ell)}) \Big).$$

Step 2 Extension to functions on $[S_0, S_1]$.

Put $\|f\|_\infty = \sup_{s\in[S_0,S_1]} |f(s)|$ and $|x|_\infty = \sup_{i=1,\dots,k} |x_i|$.

Take linear mappings $Q_k : \mathbb{R}^k \to C([S_0, S_1])$ such that $\exists\, c > 0 \,\forall\, k \in \mathbb{N}$

$$\forall x \in \mathbb{R}^k : \quad \mathrm{cost}(Q_k(x)) \le c \cdot k,$$

$$\forall x \in \mathbb{R}^k : \quad \|Q_k(x)\|_\infty \le c \cdot |x|_\infty,$$

$$\|F - Q_k(F(s_1), \dots, F(s_k))\|_\infty \le c \cdot k^{-(r+1)}.$$

E.g., Q_k piecewise polynomial interpolation of degree $\max(r, 1)$.

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Steps 1 and 2 yield the multi-level algorithm

$$\mathcal{A}^{k,\delta,L_0,L_1}_{N_{L_0},\ldots,N_{L_1}} = Q_k(\mathcal{M}^{k,\delta,L_0,L_1}_{N_{L_0},\ldots,N_{L_1}}).$$

Put

$$q = \min\left(\frac{r+1+\alpha_1}{\alpha_2}, \frac{r+1}{\alpha_3}\right).$$

Theorem *Giles, Nagapetyan, R (2013)*

With suitably chosen parameters, the algorithms $\mathcal{A}^{k,\delta,L_0,L_1}_{N_{L_0},\dots,N_{L_1}}$ achieve

$$\gamma = 2 + \frac{\vartheta}{r+1}$$

for approximation of F on $[S_0, S_1]$, where

$$\vartheta = \begin{cases} \max(1,q), & \text{if } q \le \max(1, \beta_1/\beta_2), \\ \max(1, \beta_1/\beta_2, \beta_1 + (1-\beta_2)\cdot q), & \text{otherwise.} \end{cases}$$

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Remark

- Similar results holds for approximation of $F(s)$ and of ρ on $[S_0, S_1]$.
- Single-level MC ‘suffices’, i.e., $L_0 = L_1$, if $q \leq \max(1, \beta_1/\beta_2)$.

Application SDE, Euler, $r \in \mathbb{N}$

	F	ρ	$F(s)$
smooth functional	$2 + \frac{2}{r+1}$	$2 + \frac{4}{r}$	$2 + \frac{1}{r+1}$
stopped exit time	$3 + \frac{2}{r+1}$	$3 + \frac{5}{r}$	$3 + \frac{2}{r+1}$

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Remark

- Multi-level 'superior' to single level in all these cases.
- The same orders for path-independent and path-dependent functionals.
- For $d = 1$, path-independent functionals, and the Milstein scheme

	F	ρ	$F(s)$
smooth, path-indep.	$2 + \frac{1}{r+1}$	$2 + \frac{3}{r}$	2

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Remark Corresponding results available for the approximation of $\mathrm{E}(\varphi(X))$ by means of multi-level Euler algorithms. For smooth functionals

$$\gamma = 2,$$

and for stopped exit times, see Higham *et al.* (2013),

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Question Optimality?

A Numerical Experiment Let $\tau = X_1$, where $X_0 = 1$ and

$$dX_t = 0.05\, X_t\, dt + 0.2\, X_t\, dW_t.$$

Let $S_0 = 0$ and $S_1 = 2$.

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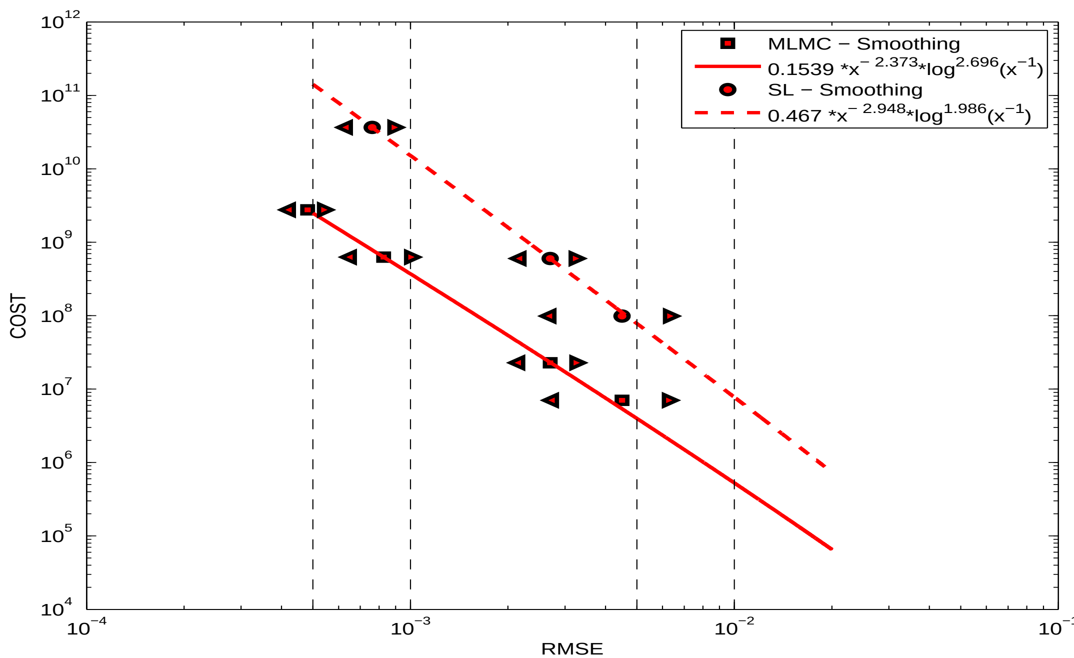

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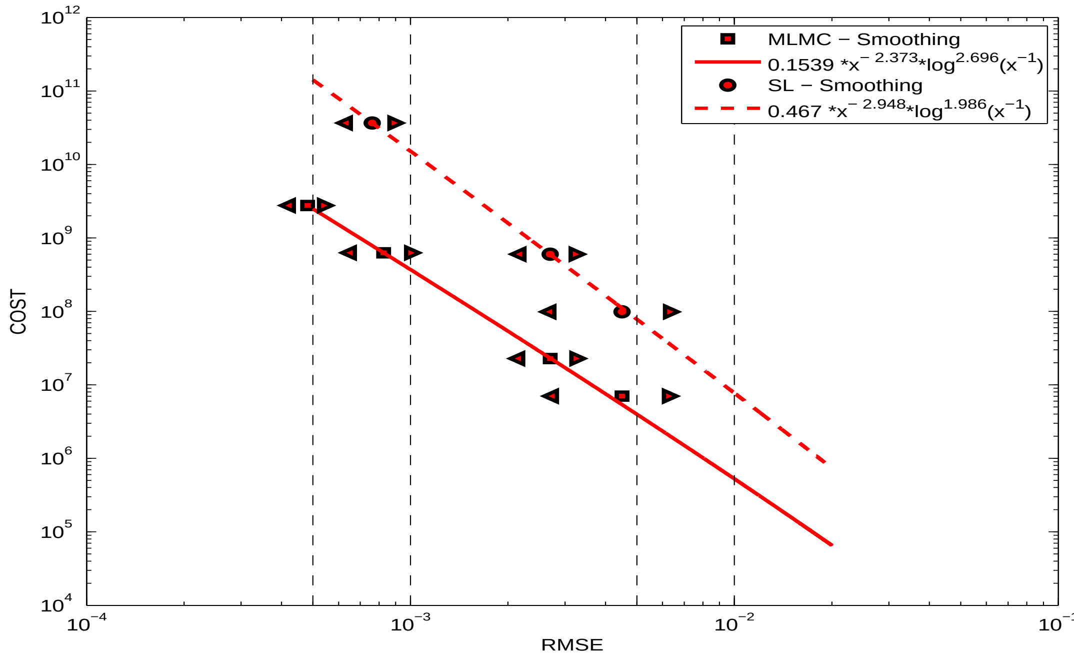

Order of convergence (theoretically, empirically)

single-level: 3.00, 2.94,

multi-level: 2.25, 2.37.

V. Complexity of Infinite-Dimensional Integration

Given

- a Borel probability measure μ on a separable Banach space $\mathfrak{X}$,
- a class F of integrable functions $\varphi : \mathfrak{X} \to \mathbb{R}$.

Compute

$$I(\varphi) = \int_{\mathfrak{X}} \varphi \, d\mu, \qquad \varphi \in F.$$

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The classical case

- $\mathfrak{X} = \mathbb{R}^n$ and μ uniform distribution on $[0,1]^n$ or standard normal distribution.

In the sequel

- μ distribution of the solution of an SDE on $\mathfrak{X} = C([0,T], \mathbb{R}^d)$.

Randomized (Monte Carlo) algorithms

- exact computation with real numbers; cost one per operation
- perfect random number generator for every distribution; cost one per call,
- oracle for $\varphi(x)$ for every $x \in \bigcup_{\ell \in \mathbb{N}_0} \mathfrak{X}_\ell$ with any scale of finite-dimensional subspaces

$$\mathfrak{X}_0 \subset \mathfrak{X}_1 \subset \ldots \subset C([0,T], \mathbb{R}^d);$$

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Worst case analysis: error and cost of a randomized algorithm $\mathcal{M}$, complexity of the integration problem

$$\operatorname{error}(\mathcal{M}, F) = \sup_{\varphi \in F} \Big(\mathrm{E} \big| I(\varphi) - \mathcal{M}(\varphi)) \big|^2 \Big)^{1/2},$$

$$\operatorname{cost}(\mathcal{M}, F) = \sup_{\varphi \in F} \mathrm{E} \big(\text{\# op's} + \text{\# random number calls} + \text{oracle cost} \big),$$

$$\operatorname{comp}(\varepsilon, F) = \inf \{ \operatorname{cost}(\mathcal{M}) : \operatorname{error}(\mathcal{M}) \leq \varepsilon \}.$$

Let $F = \mathrm{Lip}(1)$, i.e.,

$$|f(x) - f(y)| \leq \|x - y\|_\infty, \qquad x, y \in C([0,T], \mathbb{R}^d).$$

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Theorem *Creutzig, Dereich, Müller-Gronbach, R (2009)*

There exist $c_1, c_2 > 0$, which only depend on the coefficients and the initial value of the SDE, such that for every $\varepsilon \in \,]0, 1/2[$

$$c_1 \cdot \varepsilon^{-2} \leq \mathrm{comp}(\varepsilon, \mathrm{Lip}(1)) \leq c_2 \cdot \varepsilon^{-2} \cdot (\log \varepsilon^{-1})^2.$$

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Remark

- The upper bound is achieved by the multi-level Euler algorithm.
- Deterministic algorithms merely yield $\exp(\varepsilon^{-2})$.

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Remark General result for

- every probability measure μ on any separable Banach space $\mathfrak{X}$ and
- $F = \mathrm{Lip}(1)$.

Upper and lower bounds for $\mathrm{comp}(\varepsilon, \mathrm{Lip}(1))$ in terms of

- quantization numbers and Kolmogorov widths of μ.