Generalized Gamma Convolutions

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Outline

Generalized Gamma Convolutions (GGCs)

We first define the Thorin class of probability distributions on $\mathbb{R}_+$.

Originally this class was studied by Thorin(1977, two papers) when he wanted to prove the infinite divisibility of the Pareto distribution and of the log-normal distribution.

The Thorin class on $\mathbb{R}_+$, denoted by $T(\mathbb{R}_+)$, is the smallest class of distributions on $\mathbb{R}_+$ that contains all gamma distributions and is closed under convolution and weak convergence.

In other word, any element of $T(\mathbb{R}_+)$ is the weak limit of finite convolutions of gamma distributions.
A probability distribution in $T(\mathbb{R}_+)$ is called generalized gamma convolution (GGC).

This class was extended to $\mathbb{R}^d$ by Barndorff-Nielsen-M.-Sato(2006) as follows:

Call Γx an elementary gamma random variable in $\mathbb{R}^d$ if x is a non-random non-zero vector in $\mathbb{R}^d$ and Γ is a gamma random variable on $\mathbb{R}_+$.

Then the Thorin class on $\mathbb{R}^d$, denoted by $T(\mathbb{R}^d)$, is defined as the smallest class of distributions on $\mathbb{R}^d$ that contains all elementary gamma distributions on $\mathbb{R}^d$ and is closed under convolution and weak convergence.

(The Thorin class on $\mathbb{R}$ is already defined by Thorin(1978) as the name of the extended generalized gamma convolutions (EGGC).)

GGC is selfdecomposable and hence infinitely divisible

Definition

A probability distribution μ on $\mathbb{R}^d$ is *infinitely divisible* if, for any $n \in \mathbb{N}$, there exists a probability distribution μ_n on $\mathbb{R}^d$ such that $\widehat{\mu}(z) = \widehat{\mu}_n(z)^n$.
$I(\mathbb{R}^d)$ denotes the class of all infinitely divisible distributions on $\mathbb{R}^d$.

$\mu \in I(\mathbb{R}^d)$ is called *selfdecomposable* if for any $b \in (0,1)$, there exists some $\rho_b \in I(\mathbb{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_b(z)$.
($\exists X_b$ s.t. $X \overset{\mathrm{d}}{=} bX + X_b,\ X, X_b:$ indep.)

$L(\mathbb{R}^d)$ denotes the class of selfdecomposable distributions.

• Any selfdecomposable distribution can be obtained as the limiting distribution of suitably normalized partial sums of independent (not necessarily identically distributed) random variables with infinitesimal condition.

• $L(\mathbb{R}^d)$ is one of important subclasses of infinitely divisible distributions.

Fact

Any gamma distribution is selfdecomposable.

The support of gamma distribution is $[0, \infty)$. So, we can use the Laplace transform (LT) $\pi(s) = \int_0^\infty e^{-sx}\mu(dx), s \geq 0$.

Definition

A nonnegative function $g(x)$ on $(0, \infty)$ is *completely monotone* (c.m.) if it is of class C^∞ and for any $n \geq 1$ and $x > 0$, $(-1)^n g^{(n)}(x) \geq 0$.

• Examples of c.m. functions are
$f(x) = e^{-x}$; $f(x) = (a + x)^{-\gamma}, a \in \mathbb{R}, \gamma > 0$.

Fact

$\mu \in I(\mathbb{R}_+)$ *is selfdecomposable* ($\Leftrightarrow \pi_\mu(s) = \pi_\mu(bs)\pi_{\mu_b}(s)$) *iff for any* $b \in (0, 1), \pi_{\mu_b}(s)$ *is c.m.*

The density function of gamma distribution (with two parameters $(r, \lambda), r > 0, \lambda > 0$) is

$$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}.$$

and its LT is $\left(\frac{\lambda}{\lambda+s}\right)^r$.
For $b \in (0,1)$, $\pi_{\mu_b}(s)$ can be written as

$$\pi_{\mu_b}(s) = \left(\frac{\lambda + bs}{\lambda + s}\right)^r = \left(b + (1-b)\frac{\lambda}{\lambda + s}\right)^r,$$

which is completely monotone.

It is easily seen that $L(\mathbb{R}_+)$ is closed under convolution and weak convergence. Thus, any GGC is also selfdecomposable.

Theorem

$$I(\mathbb{R}_+) \supset L(\mathbb{R}_+) \supset T(\mathbb{R}_+)$$

It will be seen how rich the class $T(\mathbb{R})$ is.

1. L. Bondesson, *Generalized Gamma Convolutions and Related Classes of Distributions and Densities*. Lecture Notes in Statistics **76**, Springer, 1992.
2. L.F. James, B. Roynette and M. Yor, Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples, *Probability Surveys* **5** (2008), 341-415.
3. F.W. Steutel and K. Van Haan, *Infinite Divisibility of Probability Distributions on the Real Line*, Dekker, 2004, Chapter VI, Section 6.

The infinite divisibility of the log-normal distribution and the Pareto distribution

The log-normal distribution is the distribution of the random variable e^Z with Z standard normal, and its density function f is

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left\{ -\frac{1}{2} (\log x)^2 \right\}, \ x > 0.$$

The density of the Pareto distribution is

$$f(x) = \left(\frac{1}{a+x} \right)^a, \ a > 1.$$

Theorem

The log-normal distribution is a GGC, and thus is selfdecomposable and infinitely divisible.

It was very hard to prove the infinite divisibility of the log-normal distribution and the Pareto distribution before the theory on *hyperbolic complete monotonicity* which was developed by Thorin.

Definition

A nonnegative function $\psi(u)$ on $(0, \infty)$ is *hyperbolically complete monotone* (h.c.m.) if for every $u > 0$ the function

$$v \mapsto \psi(uv)\psi\left(\frac{u}{v}\right), \ v > 0$$

is c.m. (on $(2, \infty)$) as a function of $w := v + \frac{1}{v}$.

• Examples of h.c.m. functions are
$\psi(s) = s^{\alpha}; \alpha \in \mathbb{R};$
$\psi(s) = e^{-s};\ \ \psi(s) = e^{-1/s};$
$\psi(s) = \exp\{-s^{\alpha}\}, |\alpha| \le 1$:
$\psi(s) = (1+s)^{-\gamma}, \gamma > 0.$

• The set of h.c.m. functions is closed under each of the following operations:

(i) scale transformation; (ii) point wise multiplication;
(iii) point wise limit; (iv) composition with the function $s \mapsto s^{\alpha}, |\alpha| \le 1.$

The density function of gamma distribution (with two parameters $(r, \lambda), r > 0, \lambda > 0$) is

$$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}.$$

Note that the gamma density is h.c.m.

Proof of the infinite divisibility of log-normal distribution

The density function of log-normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left\{ -\frac{1}{2} (\log x)^2 \right\}, x > 0.$$

Theorem 5.18 of [Steutel and van Harn] says

Theorem

If $\mu \in \mathcal{P}(\mathbb{R}_+)$ has a h.c.m. density, then μ is a GGC and is hence selfdecomposable and infinitely divisible.

It is not difficult to see that $g(x) := \exp\left\{ -\frac{1}{2} (\log x)^2 \right\}$ is h.c.m.

For the Pareto distribution, see [Steutel and van Harn]. (Note that again the concept of hyperbolic complete monotonicity has to be used.)

A characterization of the class $T(\mathbb{R}_+)$ in terms of LT $\pi(s)$

$\mu \in T(\mathbb{R}_+)$ has the Laplace transform:

$$\begin{aligned}\pi(s) :=& \int_0^\infty e^{-sx}\mu(dx), \quad s > 0,\\ =& \exp\left\{-\gamma s - \int_0^\infty (1 - e^{-sx})\nu(dx)\right\},\end{aligned}$$

where $\gamma \geq 0$ and $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$.

Note that

$$\mu \in T(\mathbb{R}_+) \Leftrightarrow \nu(dx) = \frac{k(x)}{x}dx \text{ and } k(x) \text{ is c.m. on } (0, \infty).$$

Since the c.m. function is the LT of some σ-finite and positive measure (let's say, σ) by Bernstein's theorem, we have

$$k(x) = \int_0^\infty e^{-xy}\sigma(dy),$$

where σ is called the Thorin measure.

A characterization of the class $T(\mathbb{R}^d)$ in terms of characteristic function $\widehat{\mu}(z)$

For any infinitely divisible distribution μ on $\mathbb{R}^d$, we have the following Lévy-Khintchine representation of the characteristic function $\widehat{\mu}(z), z \in \mathbb{R}^d$, of μ:

$$\widehat{\mu}(z) = \exp\left\{-\tfrac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right\},$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ (called Lévy measure) satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d}(1 \wedge |x|^2)\nu(dx) < \infty,$$

and $\gamma \in \mathbb{R}^d$.

- Polar decomposition of Lévy measure ν

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

where λ is a measure on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$,
$\{\nu_\xi : \xi \in S\}$ is a family of positive measures on $(0, \infty)$,
We call ν_ξ the radial component of ν.

$\mu \in T(\mathbb{R}^d)$ if and only if

$$\nu_\xi(dr) = \frac{k_\xi(r)}{r} dr, r > 0,$$

and $k_\xi(r)$ is completely monotone.

Note: $\mu \in L(\mathbb{R}^d)$ if and only if

$$\nu_\xi(dr) = \frac{k_\xi(r)}{r} dr, r > 0,$$

and $k_\xi(r)$ is decreasing.

The Rosenblatt process (Joint work with C.A. Tudor)

Let $0 < D < \frac{1}{2}$. The Rosenblatt process is defined, for $t \geq 0$, as

$$Z_D(t) = C(D) \int_{\mathbb{R}^2}' \left(\int_0^t (u - s_1)_+^{-(1+D)/2} (u - s_2)_+^{-(1+D)/2} du \right) dB(s_1) dB(s_2),$$

where $\{B(s), s \in \mathbb{R}\}$ is a standard Brownian motion, $\int_{\mathbb{R}^2}'$ is the integral over $\mathbb{R}^2$ except the hyperplane $s_1 = s_2$ and $C(D)$ is a normalizing constant. The distribution of $Z_D(1)$ is called the Rosenblatt distribution.

The Rosenblatt process is H-selfsimilar with $H = 1 - D$ and has stationary increments.
The Rosenblatt process lives in the so-called second Wiener chaos.
Consequently, it is not a Gaussian process.
In the last few years, this stochastic process has been the object of several papers. (See Pipiras-Taqqu(2010), Tudor(2008), Tudor-Viens(2009), Veillette-Taqqu(2012) among others.)

Our first theorem is as follows.

Theorem (1)

For every $t_1, ..., t_d \geq 0$*,*

$$(Z_D(t_1), ..., Z_D(t_d)) \stackrel{\mathrm{d}}{=} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1), ..., \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1) \right),$$

where $\{\varepsilon_n\}$ *are i.i.d.* $N(0,1)$ *random variables.*

The case $d = 1$ was shown by Taqqu (see Proposition 2 of Dobrushin-Major(1979)).

The proof is enough to extend the idea of Taqqu from one dimension to multi-dimension.

Theorem (2)

For every $t_1, ..., t_d \geq 0$, the law of $(Z_D(t_1), ..., Z_D(t_d))$ belongs to $T(\mathbb{R}^d)$.

Proof. By Theorem (1),

$$\begin{aligned}
(Z_D(t_1),&..., Z_D(t_d)) \\
&\stackrel{\mathrm{d}}{=} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1), ..., \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1) \right) \\
&= \sum_{n=1}^{\infty} \varepsilon_n^2(\lambda_n(t_1), ..., \lambda_n(t_d)) - \left(\sum_{n=1}^{\infty} \lambda_n(t_1), ..., \sum_{n=1}^{\infty} \lambda_n(t_d) \right),
\end{aligned}$$

where $\varepsilon_n^2(\lambda_n(t_1), ..., \lambda_n(t_d)), n = 1, 2, ...,$ are the elementary gamma random variables in $\mathbb{R}^d$. Since they are independent, by the properties of the class $T(\mathbb{R}^d)$ that the class is closed under convolution and weak convergence, we see that $\sum_{n=1}^{\infty} \varepsilon_n^2(\lambda_n(t_1), ..., \lambda_n(t_d))$ belongs to $T(\mathbb{R}^d)$, and so does $(Z_D(t_1), ..., Z_D(t_d))$. This completes the proof. $\square$

More about second Weiner chaos

Let $I_2^B(f)$ be a double Wiener-Itô integral with respect to standard Brownian motion B, where $f \in L^2_{\mathrm{sym}}(\mathbb{R}^2_+)$.

Proposition

$$I_2^B(f) \stackrel{\mathrm{d}}{=} \sum_{n=1}^{\infty} \lambda_n(f)(\varepsilon_n^2 - 1),$$

where the series converges in $L^2(\Omega)$ and almost surely.
Also

$$\widehat{\mu}_{I_2^B(f)}(z) = \exp\left\{ \frac{1}{2} \int_{\mathbb{R}_+} (e^{izx} - 1 - izx) \frac{1}{x} \left(\sum_{n=1}^{\infty} e^{-x/\lambda_n} \right) dx \right\}.$$

Thus $\mathcal{L}\left(I_2^B(f)\right) \in T(\mathbb{R})$.

(For the proof, see, e.g. I. Nourdin and G. Prccati, *Normal approximations with Malliavin Calculus*, 2012.)

Stochastic integral representations with respect to Lévy process of the Rosenblatt distribution

The Rosenblatt distribution is represented by double Wiener-Itô integral.
However, the distributions in $T(\mathbb{R})$ have several stochastic integral representations with respect to Lévy processes.
We regard them as members of the class of selfdecomposable distributions, which is a larger class than the Thorin class.
This allows us to obtain a new result related to the Rosenblatt distribution.

We know that any selfdecomposable random variable X has the stochastic integral representation with respect to some Lévy process $\{X_t\}$ in law.
Namely, $X \overset{\mathrm{d}}{=} \int_0^\infty e^{-t} dZ_t$. However, for the Rosenblatt distribution, we can give an explicit form of $\{X_t\}$.

Theorem

$$Z_D(1) \overset{\mathrm{d}}{=} \int_0^\infty e^{-t} dX_t,$$

where $\{X_t\}$ is a Lévy process.

Other recent examples of GGC.

(1) Bertoin-Fujita-Roynette-Yor(2006) (Random excursion of Bessel processes)

Let $\{R_t, t \geq 0\}$ be a Bessel process with $R_0 = 0$, with dimension $d = 2(1-\alpha)$. $(0 < \alpha < 1$, equivalently $0 < d < 2.)$
When $\alpha = \frac{1}{2}$, $\{R_t\}$ is a Brownian motion. Let

$$g_t^{(\alpha)} := \sup\{s \leq t : R_s = 0\},$$

$$d_t^{(\alpha)} := \inf\{s \geq t : R_s = 0\}$$

and

$$\Delta_t^{(\alpha)} := d_t^{(\alpha)} - g_t^{(\alpha)},$$

which is the length of the excursion above 0, straddling t, for the process $\{R_u, u \geq 0\}$, and let ε be a standard exponential random variable independent of $\{R_u, u \geq 0\}$. Let $\Delta_\alpha := \Delta_\varepsilon^{(\alpha)}$. Then

$$\mathcal{L}(\Delta_\alpha) \in T(\mathbb{R}_+)(\subset L(\mathbb{R}_+)).$$

They showed that

$$E\left[e^{-s\Delta_\alpha}\right] = \exp\left\{-(1-\alpha)\int_0^\infty \left(1-e^{-sx}\right)\frac{E[e^{-xG_\alpha}]}{x}dx\right\},\ \ s>0,$$

with a random variable G_α. (The density function of G_α is explicitly given.) Since $k(x) := E[e^{-xG_\alpha}]$ is completely monotone by Bernstein's theorem, $\mathcal{L}(\Delta_\alpha)$ belongs to $T(\mathbb{R}_+)$.

(2) Handa(2012)
(Continuous state branching processes with immigration)

CBCI-process with quadruplet (a, b, n, δ) :

Continuous state **B**ranching process with **C**ontinuous **I**mmigration with the generator

$$L_\delta f(x) = axf''(x) - bxf'(x) + x\int_0^\infty [f(x+y) - f(x) - yf'(x)]n(dy) + \delta f'(x),$$

where n is a measure on $(0, \infty)$ satisfying $\int_0^\infty (y \wedge y^2) n(dy) < \infty$.

Let $\mu \in T(\mathbb{R}_+)$. Then

$$
\begin{aligned}
\pi(s) &= \int_0^\infty e^{-sx}\mu(dx), \quad s > 0, \\
&= \exp\left\{-\gamma s - \int_0^\infty (1 - e^{-sx})\nu(dx)\right\} \\
&= \exp\left\{-\gamma s - \int_0^\infty (1 - e^{-sx})\frac{1}{x}\left(\int_0^\infty e^{-xy}\sigma(dy)\right)dx\right\}.
\end{aligned}
$$

Since GGC on $\mathbb{R}_+$ is determined by γ and σ, we call it the GGC with pair (γ, σ).

Theorem

Let $\gamma \geq 0$ and suppose that σ is a non-zero Thorn measure.

(1) There exist (a, b, M) such that

$$\gamma + \int \frac{1}{s+u}\sigma(du) = \frac{1}{as + b + \int \frac{s}{s+u} M(du)}, s > 0.$$

(2) Any GGC with pair (γ, σ) is a unique stationary solution of CBCI-process with quadruplet $(a, b, n, 1)$, where n is a measure on $(0, \infty)$ defined by

$$n(dy) = \left(\int_0^\infty u^2 e^{-yu} M(du) \right) dy.$$

(3) Takemura-Tomisaki(2012) (Lévy density of inverse local time of some diffusion processes)

Example (Also, Shilling-Song-Vondraček(2010),p.201)

Let $I = (0, \infty)$ and $-1 < \nu < 0$.
Let $\mathcal{G}^{(\nu)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx}$.
Assume 0 is reflecting.
$\mathbb{D}^{(\nu)}$: the diffusion process on I with the generator $\mathcal{G}^{(\nu)}$
$n^{(\nu)}$: the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(\nu)}$

$$\Longrightarrow \qquad n^{(\nu)}(x) = C\frac{1}{x}x^{-|\nu|} \quad \text{(GGC)}$$

Example

Let $I = (0, \infty)$ and $-1 < \nu < 0$.

$\mathcal{G}^{(\nu)} = 2x\frac{d^2}{dx^2} + (2\nu + 2)\frac{d}{dx}$

$\mathbb{D}^{(\nu)}$: the diffusion process with the generator $\mathcal{G}^{(\nu)}$
and the end point 0 being reflecting.

$n^{(\nu)}$: the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(\nu)}$

$$\Longrightarrow \qquad n^{(\nu)}(x) = C\frac{1}{x}x^{-|\nu|} \quad \text{(GGC)}$$

Example

Let $-1 < \nu < 1$ and $\beta > 0$. Let

$$\mathcal{G}^{(\nu,\beta)} = \frac{1}{2}\frac{d^2}{dx^2} + \left\{ \frac{1}{2x} + \sqrt{2\beta}\frac{K_\nu'(\sqrt{2\beta}x)}{K_\nu(\sqrt{2\beta}x)} \right\} \frac{d}{dx},$$

where $K_\nu(x)$ is the modified Bessel function.

$\mathbb{D}^{(\nu,\beta)}$: the diffusion process on I with the generator $\mathcal{G}^{(\nu,\beta)}$ and the end point 0 being reflecting.

$n^{(\nu,\beta)}$: the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(\nu,\beta)}$

$$\Longrightarrow \qquad n^{(\nu,\beta)}(x) = C\tfrac{1}{x}x^{-|\nu|}e^{-\beta x}. \quad \text{(GGC)}$$

(When $\nu = 0$, Shilling-Song-Vondraček(2010),p.202.)

Example (Shilling-Song-Vondraček(2010),p.201)

Let $0 < \nu < 1$ and $\beta > 0$. Let

$$\mathcal{G}^{(\nu,\beta)} = \frac{1}{2}\frac{d^2}{dx^2} + \left\{ \frac{\beta - 1}{2x} + \sqrt{2\beta}\frac{K_\nu'(\sqrt{2\beta}x)}{K_\nu(\sqrt{2\beta}x)} \right\} \frac{d}{dx},$$

where $K_\nu(x)$ is the modified Bessel function.

$\mathbb{D}^{(\nu,\beta)}$: the diffusion process on I with the generator $\mathcal{G}^{(\nu,\beta)}$ and the end point 0 being reflecting.

$n^{(\nu,\beta)}$: the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(\nu,\beta)}$

$$\Longrightarrow \qquad n^{(\nu,\beta)}(x) = C\tfrac{1}{x}x^{-\nu}e^{-\beta x}. \quad \text{(GGC)}$$

Example

Let $-1 < \nu < 1$ and $\beta > 0$. Let

$$\mathcal{G}^{(\nu,\beta)} = 2x\frac{d^2}{dx^2} + 2\left\{1 + \sqrt{2\beta x}\frac{K'_\nu(\sqrt{2\beta x})}{K_\nu(\sqrt{2\beta x})}\right\}\frac{d}{dx}.$$

$\mathbb{D}^{(\nu,\beta)}$: the diffusion process with the generator $\mathcal{G}^{(\nu,\beta)}$ and the end point 0 being reflecting.

$n^{(\nu,\beta)}$: the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(\nu,\beta)}$

$$\Longrightarrow \qquad n^{(\nu,\beta)}(x) = C\tfrac{1}{x}x^{-|\nu|}e^{-\beta x}. \quad \text{(GGC)}$$

GGC in finance

Lévy process plays an important role in asset modelling, and among others a typical pure jump Lévy process is a subordination of Brownian motion.

One of them is Variance-gamma process $\{Y_t\}$ by Madan and Seneta (1990), which is a time-changed Brownian motion $B = \{B(t)\}$ subordinated by Gamma process $G = \{G(t)\}$; namely

$$Y_t = B(G(t)).$$

(*Note*: Gamma process $\{G(t)\}$ is a Lévy process such that $\mathcal{L}(G(1))$ is a gamma distribution, and GGC process $\widetilde{G} = \{\widetilde{G}(t)\}$ is a Lévy process such that $\mathcal{L}(\widetilde{G}(1))$ is a GGC.)

• Variance-gamma process $\Rightarrow$ Variance-GGC process, where G is replaced by $\widetilde{G}$.

(1) Case that B is a standard Brownian motion is treated by Geman, Madan and Yor (1999).
(2) Case that B is a Brownian motion with drift is treated recently by Privault and Yang (2013).

Proposition

Y_t is decomposed as $Y_t = U_t - W_t$, where $\{U_t\}$ and $\{W_t\}$ are two independent GGC process, and thus $\mathcal{L}(Y_t) \in T(\mathbb{R})$ (EGGC).

Thank you very much!