Some properties of real stable densities

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It is known that stable distributions have smooth densities which solve differential or integro-differential equations. Besides, there exist series representations. Zolotarev said in 1995 that these densities should be included in the family of special functions. The difficulty in analyzing stable densities comes from the absence of explicit formulæ, except in two particular cases (Lévy and Cauchy).

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$$\log[\mathbb{E}[e^{\mathrm{i}\lambda X(\alpha,\rho)}]] \;=\; \kappa_{\alpha,\rho}\,|\lambda|^{\alpha}(1 - \mathrm{i}\theta\tan(\pi\alpha/2)\,\mathrm{sgn}(\lambda)), \quad \lambda \in \mathbb{R},$$

with $\rho = 1/2 + (1/\pi\alpha)\tan^{-1}(\theta\tan(\pi\alpha/2))$ and $\kappa_{\alpha,\rho} = \cos(\pi\alpha(\rho - 1/2))$.

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with $\rho = 1/2 + (1/\pi\alpha)\tan^{-1}(\theta\tan(\pi\alpha/2))$ and $\kappa_{\alpha,\rho} = \cos(\pi\alpha(\rho - 1/2))$. Non-strictly stable laws are obtained from strict ones in adding a drift for $\alpha \neq 1$. If $\alpha = 1$ (the exotic class of skewed Cauchy distributions) they can be recovered by a limit in law. Recall that drifted Cauchy distributions are strictly stable.

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Theorem (Zolotarev) *One has*

$$X^+(\alpha, \rho) \;\stackrel{d}{=}\; \left(\frac{\mathrm{Z}_{\alpha\rho}}{\mathrm{Z}_\rho}\right)^\rho.$$

This result is actually equivalent to Zolotarev's duality, although this is not apparent from his book. In a sense, this reduces the analysis of the density of $X(\alpha, \rho)$ to a better understanding of the positive case.

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With our normalization, the density f_α of the positive stable random variable Z_α is such that

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One has the explicit formulæ

- $f_{1/2}(x) \;=\; \frac{1}{2\sqrt{\pi x^3}} e^{-1/4x}$
- $f_{1/3}(x) \;=\; \frac{1}{3\pi x^{3/2}} K_{1/3}(2/3\sqrt{3x})$ where $K_{1/3}$ is a MacDonald function.
- $f_{2/3}(x) \;=\; \sqrt{\frac{3}{\pi x}} e^{-2/27x^2} W_{1/2,1/6}(4/27x^2)$ where $W_{1/2,1/6}$ is a Whittaker function.

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For $\alpha = p/q$ rational, one can express f_α with the help of the solution on $\mathbb{R}^+$ to a certain ODE of order $q-1$, as explained in Zolotarev (1983). Those analytical expressions become however untractable for $q \geq 4$.

The Kanter factorization

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$$\mathrm{Z}_\alpha \stackrel{d}{=} \mathrm{L}^{-(1-\alpha)/\alpha} \times b_\alpha^{-1/\alpha}(\mathrm{U})$$

where $\mathrm{L} \sim \mathrm{Exp}(1)$, $\mathrm{U} \sim \mathrm{Unif}(0,\pi)$ independent of L, and

$$b_\alpha(u) = (\sin u / \sin(\alpha u))^\alpha (\sin u / \sin((1-\alpha)u))^{1-\alpha}, \quad u \in (0,\pi).$$

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We set $\mathrm{K}_\alpha = b_\alpha^{-1/\alpha}(\mathrm{U})$. The function b_α is bounded, decreasing and strictly concave. This shows that K_α is bounded with increasing density and that $\mathrm{K}_\alpha^{-1/\alpha}$ has an unbounded support not containing zero and a decreasing density.

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$$\Gamma_{1/2} \stackrel{d}{=} \Gamma_1 \times \mathrm{B}_{1/2,1/2},$$

which is a particular instance of the so-called Beta-Gamma algebra.

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Proposition (Yamazato) *One has the independent factorization*

$$\mathrm{Z}_\alpha \stackrel{d}{=} \mathrm{Y}_\alpha + \sum_{n\geq 1} \mathrm{Y}_{\alpha,n},$$

with $\mathrm{Y}_{\alpha,n} \sim \mathrm{Exp}((n\pi/\sin(\pi\alpha))^{1/\alpha})$ *for every* $n \geq 1$ *and* $\mathrm{Y}_\alpha \in \mathrm{ME}$.

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Such additive factorizations can actually be extended to all stable densities.

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If $n > p \geq 1$ are two integers, define the following indices: $q_0 = 0, q_p = n$ and if $p \geq 2$,

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for all $j = 1, \ldots p-1$. Using the explicit fractional moments of Z_α :

$$\mathbb{E}[\mathrm{Z}_\alpha^s] \;=\; \int_0^\infty f_\alpha(x) x^s \, dx \;=\; \frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}$$

and Legendre-Gauss multiplication formula, it is possible to show (S. 2013) that

$$\mathrm{Z}_{\frac{p}{n}}^{-p} \;\stackrel{d}{=}\; \frac{n^n}{p^p(n-p)^{n-p}} \, \mathrm{L}^{n-p} \;\times\; \prod_{j=0}^{p-1} \left(\prod_{i=q_j+1}^{q_{j+1}-1} \mathrm{B}_{\frac{i}{n}, \frac{i-j}{n-p} - \frac{i}{n}} \right) \;\times\; \prod_{j=1}^{p-1} \mathrm{B}_{\frac{q_j}{n}, \frac{j}{p} - \frac{q_j}{n}} .$$

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By Zolotarev's duality and since $\mathrm{L} \stackrel{d}{=} -\log[\mathrm{B}_{1,1}]$, all one-sided strictly stable branches with rational parameters can be factorized with the sole Beta distribution.

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When $p = 1$, this boils down to

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a formula originally due to Williams (1977). When $p > 1$ there are $(p-1)$ Beta factors and $(n-p)$ Gamma factors in the product. For example

$$4 \, Z_{2/3}^{-2} \stackrel{d}{=} 27 \, \mathrm{B}_{1/3,1/6} \times \Gamma_{2/3} \quad \text{and} \quad 4 \, Z_{2/5}^{-2} \stackrel{d}{=} 3125 \, \mathrm{B}_{2/5,1/10} \times \Gamma_{1/5} \times \Gamma_{3/5} \times \Gamma_{4/5}.$$

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Beta-Gamma factorizations are not canonical, contrary to the Beta factorization and its interpretation in terms of the Meijer G-function, but they are more useful.

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$$X^+(\alpha,\rho) \stackrel{d}{=} \left(\frac{\mathrm{Z}_{\alpha\rho}}{\mathrm{Z}_\rho}\right)^\rho \stackrel{d}{=} \mathrm{L}^{(\alpha\rho-1)/\alpha} \times \mathrm{L}^{1-\rho} \times \mathrm{K}_{\alpha\rho}^{-1/\alpha} \times \mathrm{K}_\rho,$$

shows the unimodality of $X^+(\alpha,\rho)$. Indeed, $\mathrm{K}_{\alpha\rho}^{-1/\alpha} \times \mathrm{K}_\rho$ is unimodal and the property is stable by independent multiplication with exponential powers (Cuculescu and Theodorescu, 1998).

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This property is false for positive self-decomposable laws in general, but it is conjectured to hold true as soon as the spectral function is infinite at zero.

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for all $\alpha \in (0,1)$. This is a comparison of Z_α with the two extremal Fréchet distributions corresponding to the behaviour of its density at zero and at infinity.

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$$M_\alpha \;<\; m_\alpha \;<\; \mathbb{E}[\mathrm{Z}_\alpha]\;(=+\infty)$$

as soon as $\alpha < 1/(1+\log(2)) \sim 0.5906$ or α is close enough to 1.

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for all $0 < \beta < \alpha < 1$, and that

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for every $1/2 \leq \beta < \alpha < 1$.

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- The family $\{\Gamma(1+\alpha)\mathrm{Z}_\alpha^{-\alpha},\, 1/2 \geq \alpha > 0\}$ (further Mittag-Leffler distributions).
- The family $\{\Gamma(1+\alpha)^2\Gamma(2-\alpha)^{-1}\mathrm{Z}_\alpha^{-\alpha} \times \mathrm{Z}^{\alpha}_{\frac{\alpha}{1-\alpha}},\, 1/2 \geq \alpha > 0\}$ (negative stable branches).
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Observe that the normalization is the same for the first and the third family.

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- The median-mode or mean-median-mode inequality (in one or another direction). As observed by Dharmadikari and Joag-dev (1988), this inequality would be a consequence of the perfect skewness. But it should be less difficult to obtain. By the way, I do not know any infinitely divisible counterexample to the median-mode or mean-median-mode inequality and this is another open question.

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Proposition *The map* $\rho \mapsto \Gamma(1+\rho)\Gamma(1-\rho)X^+(1, \rho)$ *is non-increasing for the convex order on* $(0, 1)$.

A conjecture is that for every $\alpha \in (1, 2)$, there exists an Asian subroof connecting the parameters $(1, 0), (\alpha, 1/2)$ and $(1, 1)$ such that moving along the top of this subroof from left to right, we get a family of one-sided stable branches which is non-increasing for the convex order after suitable normalization.