Special values of the Riemann zeta function via several random variables

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Abstract: In this paper, using arcsine variables, we get a new elementary proof of $\zeta(2) = \frac{\pi^2}{6}$, known as the Basel problem and the Euler formula. Using exponential variables, we get the formula like Euler. We can also solve the Basel problem by using Wigner's semi-circle law and Legendre generating function.
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1 Introduction

First we review our previous paper([1]). Consider the Riemann zeta function

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s} \qquad \text{(for} \quad \mathrm{Re}\, s > 1\text{)}.$$

Using two independent Cauchy variables, we obtain the following Euler's formulae of the Riemann zeta function, which is very classical (see for example [9]):

Euler's Formulae

$$(1 - \frac{1}{2^{2n+2}})\zeta(2n+2) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+2}\frac{A_n}{\Gamma(2n+2)}.$$

Here, the coefficients A_n are featured in the series development

$$\frac{1}{\cos^2\theta} = \sum_{n=0}^{\infty}\frac{A_n}{(2n)!}\theta^{2n} \quad (|\theta| < \frac{\pi}{2}).$$

We remark that many authors ([3, 4, 7, 8, 10]) have written elementary proofs of $\zeta(2) = \frac{\pi^2}{6}$. The problem of finding this value is known as Basel problem ([2]).

We first review the density function of the ratio of two independent random variables from elementary probability theory.

Lemma1.1.

Consider two independent random variables X, Y such that $P(X > 0) = P(Y > 0) = 1$ and with density functions $f_X(x)$, $f_Y(x)$.

Then, $f_{\frac{Y}{X}}(x)$, the density function of $\frac{Y}{X}$ is given by:

$$f_{\frac{Y}{X}}(x) = \int_0^{\infty} f_X(u) f_Y(ux) u du$$

.

Proof.

For $x > 0$,

$P(\frac{Y}{X} < x) = \int\int_{\frac{u}{v} < x} f_X(u) f_Y(v) du dv = \int_0^{\infty} f_Y(v) dv \int_0^{vx} f_X(u) du$. Then differentiating both sides with respect to x, we get the result.

b) Applying Lemma 1.1. for

$f_{|\mathbb{C}_1|}(x) = f_{|\mathbb{C}_2|}(y) = \frac{2}{\pi}\frac{1}{1+x^2}1_{x>0}$ i.e.:

$|\mathbb{C}_1| \sim |\mathbb{C}_2| \sim |C|$ where C is a Cauchy variable with $f_C(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, we get , for $x > 0$:

$$f_{\frac{Y}{X}}(x) = \frac{4}{\pi^2}\int_0^{\infty} \frac{1}{(1+u^2)}\frac{1}{(1+(ux)^2)} u du$$

$$= \frac{2}{\pi^2}\int_0^{\infty} \frac{1}{(u+1)(1+ux^2)} du$$

$$= \frac{2}{\pi^2}\int_0^{\infty} (\frac{1}{1+u} - \frac{x^2}{1+ux^2})\frac{du}{1-x^2}$$

$$= \lim_{A\to\infty} \frac{2}{\pi^2}\int_0^{A} (\frac{1}{1+u} - \frac{x^2}{1+ux^2})\frac{du}{1-x^2}$$

$$= \frac{4}{\pi^2}\frac{\log x}{x^2-1}$$

c) Since $1 = \int_0^\infty f_{\frac{Y}{X}}(x)dx$, we have:

$$\frac{\pi^2}{4} = \int_0^\infty \frac{\log x}{x^2 - 1} dx$$

The righthand side R is equal to

$$
\begin{aligned}
R &= \int_0^1 \frac{\log x}{x^2-1}dx + \int_1^\infty \frac{\log x}{x^2-1}dx \\
&= 2\int_0^1 \frac{-\log x}{1-x^2}dx \\
&= 2\int_0^1 (-\log x)\sum_{k=0}^{\infty} x^{2k}dx \\
&= 2\sum_{k=0}^{\infty}\int_0^1 (-\log x)x^{2k}dx
\end{aligned}
$$

$$= 2\sum_{k=0}^{\infty}\int_0^{\infty} ue^{-2ku}e^{-u}du$$

$$= 2\sum_{k=0}^{\infty}\int_0^{\infty} \frac{y}{2k+1}e^{-y}\frac{dy}{2k+1}$$

$$= 2\Gamma(2)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$$

Thus, we obtained:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Noting that

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \frac{1}{2^2}\zeta(2),$$

we obtain the desired result, i.e.:

$$\zeta(2) = \frac{4}{3} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}\frac{4}{3} = \frac{\pi^2}{6}.$$

This is a probabilistic solution of Basel Problem.

Considering even moments of $\log \frac{|\mathbb{C}_2|}{|\mathbb{C}_1|}$, we prove the Euler's formulae of the Riemann zeta function.

Lemma 1.2.

$$E(\left(\log \frac{|\mathbb{C}_2|}{|\mathbb{C}_1|}\right)^{2n}) = \frac{8}{\pi^2}\Gamma(2n+2)(1-\frac{1}{2^{2n+2}})\zeta(2n+2$$

Proof.

$$E\left(\left(\log \frac{|\mathbb{C}_2|}{|\mathbb{C}_1|}\right)^{2n}\right) = \int_0^\infty (\log x)^{2n} f_{\frac{|\mathbb{C}_2|}{|\mathbb{C}_1|}}(x)dx =$$

$$\frac{4}{\pi^2}\int_0^\infty \frac{(\log x)^{2n+1}}{x^2-1}dx = \frac{8}{\pi^2}\int_0^1 \frac{(\log x)^{2n+1}}{x^2-1}dx =$$

$$= \frac{8}{\pi^2}\int_0^1 (-\log x)^{2n+1} \sum_{k=0}^{\infty} x^{2k} dx$$

$$= \frac{8}{\pi^2}\sum_{k=0}^{\infty} u^{2n+1} e^{-2ku} e^{-u} du =$$

$$\frac{8}{\pi^2}\sum_{k=0}^{\infty}\int_0^\infty \left(\frac{u}{2k+1}\right)^{2n+1} e^{-u}\frac{du}{2k+1}$$

$$= \frac{8}{\pi^2}\Gamma(2n+2)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}}$$

$$= \frac{8}{\pi^2}\Gamma(2n+2)(1-\frac{1}{2^{2n+2}})\zeta(2n+2)$$

Using this Lemma, we obtain the following Euler Formula:

Theorem 1.1. (Euler's Formulae)

$$(1-\frac{1}{2^{2n+2}})\zeta(2n+2) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+2}\frac{A_n}{\Gamma(2n+2)}$$

where, the coefficients A_n are obtained in the series development

$$\frac{1}{\cos^2\theta} = \sum_{n=0}^{\infty} \frac{A_n}{(2n)!}\theta^{2n} \quad (|\theta| < \frac{\pi}{2}).$$

Proof.

We only need to prove that $E(|\mathbb{C}_1|^\alpha) = \frac{1}{\cos\frac{\pi}{2}\alpha} \quad (|\alpha| < 1)$, because by this, we can easily get that $E(e^{\alpha\log\frac{|\mathbb{C}_2|}{|\mathbb{C}_1|}}) = \frac{1}{(\cos\frac{\pi}{2}\alpha)^2}$,

which is equivalent to
$E((\log|\mathbb{C}_1/\mathbb{C}_2|)^{2n}) = (\frac{\pi}{2})^{2n} A_n$.

Noting that $\mathbb{C}_1 \sim \frac{N}{N'}$ where N and N' are two independent standard normal random variables, we get that $(\mathbb{C}_1)^2 \sim \frac{N^2}{(N')^2} \sim \frac{\gamma_{1/2}}{\gamma'_{1/2}}$ where $\gamma_{1/2}$ and $\gamma'_{1/2}$ are two independent gamma variables with parameter $1/2$, i.e. its density $f_{\gamma_{1/2}}(x) = \frac{x^{-1/2}}{\sqrt{\pi}} e^{-x} \quad (x > 0)$.

Then we get that

$E(|\mathbb{C}_1|^\alpha) = E((\gamma_{1/2})^{\frac{\alpha}{2}})E((\gamma_{1/2})^{-\frac{\alpha}{2}}) =$

$$\frac{\Gamma(\frac{1}{2}+\frac{\alpha}{2})}{\Gamma(1/2)}\frac{\Gamma(\frac{1}{2}-\frac{\alpha}{2})}{\Gamma(1/2)} = \frac{1}{\pi}\frac{\pi}{\sin\pi(\frac{1}{2}+\frac{\alpha}{2})}$$
$= \frac{1}{\cos\frac{\pi}{2}\alpha} \quad (|\alpha| < 1)$, where we used the fact : $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$ and $E((\gamma_a)^b) = \frac{\Gamma(a+b)}{\Gamma(a)}$, $f_{\gamma_a}(x) = \frac{x^{a-1}}{\Gamma(a)}e^{-x} \quad (x > 0)$.

□

The above is a review of our previous papers([1], [5]).

In this paper, instead of Cauchy, we use arcsine random variables X with

$f_X(x) = \frac{2}{\pi\sqrt{1-x^2}} \quad (0 < x < 1)$ and can get the Euler formula.

2 Basel Problem via arcsine

Take $X_1 \sim X_2 \sim X(independent)$, then for $0 < x < 1$

$$f_{\frac{X_2}{X_1}}(x) = \int_0^1 \frac{1}{\sqrt{1-u^2}\sqrt{1-u^2x^2}} \frac{4}{\pi^2} u du$$

$$= \frac{2}{\pi^2} \int_0^1 \frac{1}{\sqrt{x^2((u - \frac{1+x^2}{2x^2})^2 - (\frac{x^2-1}{2x^2})^2)}} du$$

$$= \frac{2}{\pi^2 x} \int_{-\frac{1+x^2}{2x^2}}^{\frac{x^2-1}{2x^2}} \frac{du}{\sqrt{u^2 - (\frac{x^2-1}{2x^2})^2}}$$

$$= \frac{2}{\pi^2 x} [\log(u + \sqrt{u^2 - (\frac{x^2-1}{2x^2})^2})]_{-\frac{1+x^2}{2x^2}}^{\frac{x^2+1}{2x^2}}$$

$$= \frac{2}{\pi^2 x} \log \frac{1+x}{1-x} \quad (0 < x < 1)$$

For $x > 1$, $P(\frac{X_2}{X_1} \leqq x) = 1 - P(\frac{X_1}{X_2} \leqq \frac{1}{x})$, then $f_{\frac{X_2}{X_1}}(x) = -f_{\frac{X_2}{X_1}}(x)(\frac{1}{x})' =$ $\frac{2}{\pi^2 x} \log \frac{x+1}{x-1} \quad (x > 1)$.

Then we have the following proposition.

Proposition. 2.1

$$f_{\frac{X_2}{X_1}}(x) = \begin{cases} \frac{2}{\pi^2 x} \log \frac{1+x}{1-x} & (0 < x < 1) \\ \frac{2}{\pi^2 x} \log \frac{x+1}{x-1} & (1 < x < +\infty) \end{cases} .$$

Since $1 = \int_0^\infty f_{\frac{X_2}{X_1}}(x)dx = 2\int_0^1 f_{\frac{X_2}{X_1}}(x)dx = \frac{4}{\pi^2}\int_0^1 \frac{1}{x}\log\frac{1+x}{1-x}dx$, then

$\frac{\pi^2}{4} = \int_0^1 \frac{1}{x}(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots) - (-x -$

$\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots))dx =$

$2\int_0^1 (1 + \frac{x^2}{3} + \frac{x^4}{5} + \cdots)dx = 2\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2},$

therefore $\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} = \frac{\pi^2}{8}$ gives $\zeta(2) = \frac{\pi^2}{6}$

Remark.

We note that $X \sim \sqrt{\beta_{1/2,1/2}}$ where

$f_{\beta_{1/2,1/2}}(x) = \frac{1}{\pi} x^{-1/2}(1-x)^{-1/2} \quad (0 < x < 1).$

3 Euler formula via arcsine

In this section, we get the Euler formula for $\zeta(2n)$ by using arcsine variables.

Take X with $f_X(x) = \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}} \quad (0 < x < 1)$, then

For $\alpha > -1, E(X^\alpha) = \frac{2}{\pi}\int_0^1 \frac{x^\alpha}{\sqrt{1-x^2}}dx =$
$\frac{2}{\pi}\int_0^1 \frac{u^{\frac{\alpha}{2}}}{\sqrt{1-u}}\frac{1}{2}u^{-\frac{1}{2}}du$
$= \frac{1}{\pi}\int_0^1 u^{\frac{\alpha-1}{2}}(1-u)^{-\frac{1}{2}}du = \frac{1}{\pi}B(\frac{1+\alpha}{2}, \frac{1}{2})$.

For $|\alpha| < 1, E((\frac{X_2}{X_1})^\alpha) = E(X_2^\alpha)E(X_1^{-\alpha}) =$
$\frac{1}{\pi}B(\frac{1+\alpha}{2}, \frac{1}{2})\frac{1}{\pi}B(\frac{1-\alpha}{2}, \frac{1}{2})$

$$= \frac{1}{\pi^2} \frac{\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\alpha}{2}+1)} \frac{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1-\alpha}{2})} =$$

$$\frac{1}{\pi} \frac{\pi}{\sin \pi (\frac{1+\alpha}{2})} \frac{1}{\frac{\alpha}{2}\Gamma(\frac{\alpha}{2})\Gamma(\frac{1-\alpha}{2})}$$

$$= \frac{1}{\cos \frac{\pi}{2}\alpha} \frac{1}{\frac{\alpha}{2}} \frac{\sin \frac{\pi}{2}\alpha}{\pi} = \frac{\tan \frac{\pi\alpha}{2}}{\frac{\pi\alpha}{2}} \quad |\alpha| < 1,$$ where

we used the fact

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \quad (|\alpha| < 1).$$

Here we note that

$$\frac{\tan x}{x} = \sum_{n=0}^{\infty} \frac{A_n}{(2n+1)!} x^{2n} \quad (*)$$ by integrating

from 0 to x $\sum_{n=0}^{\infty} \frac{A_n}{(2n)!} u^{2n} = \frac{1}{\cos^2 u}$.

Lemma 3.1.

$$E((\log X_2 - \log X_1)^{2n})$$
$$= \frac{8}{\pi^2} \Gamma(2n+1)(1 - \frac{1}{2^{2n+2}})\zeta(2n+2).$$

Proof.

$E((\log X_2 - \log X_1)^{2n}) = E((\log \frac{X_2}{X_1})^{2n})$

$$= \int_0^\infty (\log x)^{2n} f_{\frac{X_2}{X_1}}(x)dx$$

$$= 2\int_0^1 (\log x)^{2n} \frac{2}{\pi^2 x} \log \frac{1+x}{1-x} dx$$

$$= \frac{4}{\pi^2} \int_0^1 2 \sum_{k=0}^{\infty} \frac{x^{2k}}{2k+1} (\log x)^{2n} dx$$

$$= \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{2k+1} \int_0^1 x^{2k} (\log x)^{2n} dx$$

$$= \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{2k+1} \int_0^\infty e^{-2ku} u^{2n} e^{-u} du$$

$$= \tfrac{8}{\pi^2} \sum_{k=0}^{\infty} \int_0^{\infty} e^{-u} (\frac{u}{2k+1})^{2n} \frac{du}{2k+1}$$

$$= \tfrac{8}{\pi^2} \Gamma(2n+1) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+2}}$$

$$= \tfrac{8}{\pi^2} \Gamma(2n+1)(1 - \tfrac{1}{2^{2n+2}})\zeta(2n+2).$$

Comparing (*) and

$E((\frac{X_1}{X_2})^{\alpha}) = E(e^{\alpha(\log X_1 - \log X_2)}) =$

$\sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} E((\log X_1 - \log X_2)^{2n}) \quad (\because$

$$E((\log X_1 - \log X_2)^{odd})) = 0),$$

we get that

$$\frac{8}{\pi^2}\Gamma(2n+1)(1-\frac{1}{2^{2n+2}})\zeta(2n+2) = \frac{A_n}{2n+1}(\frac{\pi}{2})^{2n},$$

Using arcsine varialbles, we also can get the following Euler formula.

Theorem. 3.1.

$$\zeta(2n+2) = \frac{1}{2}(\frac{\pi}{2})^{2n+2}\frac{1}{1-\frac{1}{2^{2n+2}}}\frac{A_n}{\Gamma(2n+2)}.$$

4 Euler like formula via exponential

In this section, using exponential random variables, we can get the Euler like formula for special values of the Riemann zeta function.

Take $\mathfrak{e}_1 \sim \mathfrak{e}_2 \sim \mathfrak{e}$ with $f_{\mathfrak{e}}(x) = e^{-x} \quad (x > 0)$, then

For $\alpha > 0, E(\mathfrak{e}^{\alpha}) = \int_0^{\infty} x^{\alpha} e^{-x} dx = \Gamma(1+\alpha)$. For $|\alpha| < 1, E((\frac{\mathfrak{e}_2}{\mathfrak{e}_1})^{\alpha}) = \Gamma(1+\alpha)\Gamma(1-\alpha) =$

$$\alpha\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi\alpha}{\sin\pi\alpha} \quad (|\alpha| < 1).$$

Here we put that $\frac{x}{\sin x} = \sum_{n=0}^{\infty} \frac{C_n}{(2n)!} x^{2n} \quad (**).$

Lemma 4.1.

$$E((\log \mathfrak{e}_2 - \log \mathfrak{e}_1)^{2n}) = 2\Gamma(2n+1)(1 - \frac{1}{2^{2n-1}})\zeta(2n).$$

Proof.

Noting

$f_{\frac{\mathfrak{e}_2}{\mathfrak{e}_1}}(x) = \frac{1}{(1+x)^2} \quad (x > 0)$ (Parato distribution), for $n \geqq 1$,

$$E(\log \mathfrak{e}_2 - \log \mathfrak{e}_1)^{2n}) = E((\log \frac{\mathfrak{e}_2}{\mathfrak{e}_1})^{2n})$$

$$= \int_0^\infty (\log x)^{2n} f_{\frac{\mathfrak{e}_2}{\mathfrak{e}_1}}(x) dx$$

$$= \int_0^\infty (\log x)^{2n} \frac{1}{(1+x)^2} dx$$

$$= 2 \int_0^1 (\log x)^{2n} \frac{1}{(1+x)^2} dx$$

$$= 2 \int_0^1 \sum_{k=1}^{\infty} k(-x)^{k-1} (\log x)^{2n} dx$$

$$= 2 \sum_{k=1}^{\infty} k \int_0^1 (-x)^{k-1} (\log x)^{2n} dx =$$

$$2\sum_{k=1}^{\infty} k \int_0^{\infty} e^{-(k-1)u} u^{2n} e^{-u} du$$

$$= 2\sum_{k=1}^{\infty} k \int_0^{\infty} (\frac{u}{k})^{2n} e^{-u} \frac{du}{k} =$$

$$2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2n}} \Gamma(2n+1)$$

$$= 2\Gamma(2n+1)(\sum_{k=1}^{\infty} \frac{-1}{(2k)^{2n}} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n}})$$

$$= 2\Gamma(2n+1)(\tfrac{-1}{2^{2n}}\zeta(2n) + (1-\tfrac{1}{2^{2n}})\zeta(2n))$$

$$= 2\Gamma(2n+1)(1-\tfrac{1}{2^{2n-1}})\zeta(2n).$$

Comparing (**) and

$$E((\tfrac{\mathfrak{e}_2}{\mathfrak{e}_1})^{\alpha}) = E(e^{\alpha(\log\mathfrak{e}_2 - \log\mathfrak{e}_1)})$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} E((\log\mathfrak{e}_2 - \log\mathfrak{e}_1)^{2n}) \quad (\because$$

$$E((\log\mathfrak{e}_2 - \log\mathfrak{e}_1)^{odd})) = 0,$$

we get that

$$2\Gamma(2n+1)(1-\frac{1}{2^{2n-1}})\zeta(2n) = C_n\pi^{2n},$$

Theorem 4.1.

$$\zeta(2n) = \frac{C_n \pi^{2n}}{2\Gamma(2n+1)(1 - \frac{1}{2^{2n-1}})}.$$

Remark.

A decomposition

$$\frac{1}{\cos\theta} = \frac{\tan\theta}{\theta}\frac{\theta}{\sin\theta}$$

is interesting for our discussion.

From $\frac{\tan\theta}{\theta}$, we have the Euler formula. From $\frac{\theta}{\sin\theta}$, we have the Euler fomula, too. From $\frac{1}{\cos\theta}$, we can get $L_{\chi_4}(2n+1)$ (:special values of $L_{\chi_4}(s) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^s}$) which was already discussed in [1]).

This decomposition has following Brownian interpretation.

Let B_t be a Brownian motion. We define $g_t = \sup\{s | B_s = 0, s \leqq t\}$ and

$T_a^* = \inf\{t||B_t| = a\}$ for $a > 0$. Then we can see $g_{T_a^*}$ and $T_a^* - g_{T_a^*}$ are independent and for $\lambda > 0$, $E(e^{-\lambda T_a^*}) = \frac{1}{\cosh\sqrt{2\lambda}a}$,
$E(e^{-\lambda(T_a^* - g_{T_a^*})}) = \frac{\sqrt{2\lambda}a}{\sinh\sqrt{2\lambda}a}$ and
$E(e^{-\lambda g_{T_a^*}}) = \frac{\tanh\sqrt{2\lambda}a}{\sqrt{2\lambda}a}$.

5 Basel problem via Wigner's semi-circle law

Let W be a random variable with $f_W(x) = \frac{4}{\pi}\sqrt{1-x^2} \quad (0 < x < 1)$, called Wigner's semi-circle random variables.

Take $W_1 \sim W_2 \sim W$(independent), then for $0 \leqq x \leqq 1$,

$f_{\frac{W_2}{W_1}}(x) = \int_0^1 \frac{16}{\pi^2}\sqrt{1-u^2}\sqrt{1-x^2u^2}udu = \frac{8}{\pi^2}\int_0^1 \sqrt{(1-u)(1-x^2u)}du.$

Then $\frac{1}{2} = P(W_2 \leqq W_1) = P(\frac{W_2}{W_1} \leqq 1) = \frac{8}{\pi^2} \iint_{0 \leqq u, x \leqq 1} \sqrt{(1-u)(1-x^2u)} du dx$, and $\frac{\pi^2}{16}$

$$= \iint_{0 \leqq u, x \leqq 1} \sqrt{1-u}(\sqrt{1-ux^2} - 1 + 1) du dx$$

$$= \frac{2}{3} - \iint_{0 \leqq u, x \leqq 1} \sqrt{1-u}(1 - \sqrt{1-ux^2}) du dx$$

$$= \frac{2}{3} - \iint_{0 \leqq u, x \leqq 1} \sqrt{1-u} \sum_{k=1}^{\infty} \frac{1}{2^{2k}(2k-1)} \binom{2k}{k} x^{2k} u^k d$$

$$= \frac{2}{3} - \sum_{k=1}^{\infty} \frac{1}{2^{2k}(2k-1)} \binom{2k}{k} \frac{1}{2k+1} B(k+1, \frac{3}{2})$$

$$= \frac{2}{3} - \sum_{k=1}^{\infty} \frac{1}{2^{2k}(2k-1)} \binom{2k}{k} \frac{1}{2k+1} \frac{\Gamma(k+1)\Gamma(\frac{3}{2})}{\Gamma(k+\frac{5}{2})}$$

$$= \frac{2}{3} - \sum_{k=1}^{\infty} \frac{1}{2^{2k}(2k-1)} \frac{(2k)!}{k!k!} \frac{1}{2k+1} \frac{2^{2k+2} k!(k+1)!}{(2k+3)!}$$

$$= \frac{2}{3} - 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)^2(2k+3)}$$

, where we used that

$$1-\sqrt{1-x}=\sum_{k=1}^{\infty}\frac{1}{2^{2k}(2k-1)}\binom{2k}{k}x^k.$$

This solves the Basel problem.

Remark.

We remark that

$$E(|W|^\alpha)=\frac{2}{\pi}B(\tfrac{\alpha+1}{2},\tfrac{3}{2}),$$

$$E(|\tfrac{W_2}{W_1}|^\alpha)=\frac{1}{1-\frac{\alpha^2}{4}}\frac{\tan\frac{\pi\alpha}{2}}{\frac{\pi}{2}},$$

and $W\sim\sqrt{\beta(\frac{1}{2},\frac{3}{2})}$ and $W\sim U^{1/2}X$.

6 Two simple derivations of Basel Problem

In this section, using some double integral related to the Legendre Polynomials for the first one, we get another ways to Basel Problem.

Theorem 6.1.
$\int_{-1}^{1} dt \int_{-1}^{1} \frac{1}{1-2xt+t^2} dx = \int_{-1}^{1} dx \int_{-1}^{1} \frac{1}{1-2xt+t^2} dt$
makes

$$\zeta(2) = \frac{\pi^2}{6}.$$

Proof.

For $t(-1<t<1)$, $\int_{-1}^{1} \frac{1}{1-2xt+t^2}dx = [\frac{1}{-2t}\log|1-2xt+t^2|]_{-1}^{1} = \frac{1}{-2t}(\log(1-t)^2 - \log(1+t)^2) = \frac{1}{t}(\log(1+t)-\log(1-t)) = \frac{1}{t}((t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\frac{t^5}{5}+\cdots)-(-t-\frac{t^2}{2}-\frac{t^3}{3}-\frac{t^4}{4}-\frac{t^5}{5}+\cdots)) = 2(1+\frac{t^2}{3}+\frac{t^4}{5}+\frac{t^6}{7}+\cdots) \qquad (*)$.

Then

$$I = \int_{-1}^{1} dt \int_{-1}^{1} \frac{1}{1-2xt+t^2}dx = 4\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

Because each term is positive, the interchange

of sum and integral is clear. By Fubini Theorem,
$I = \int_{-1}^{1} dx \int_{-1}^{1} \frac{1}{1-2xt+t^2} dt =$
$\int_{-1}^{1} dx \int_{-1}^{1} \frac{1}{(t-x)^2+1-x^2} dt =$
$\int_{-1}^{1} dx [\frac{1}{\sqrt{1-x^2}} \arctan \frac{t-x}{\sqrt{1-x^2}}]_{-1}^{1} =$
$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} (\arctan \sqrt{\frac{1-x}{1+x}} + \arctan \sqrt{\frac{1+x}{1-x}}) dx =$
$\frac{\pi}{2} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2} [\arcsin x]_{-1}^{1} = \frac{\pi^2}{2}$, where we used the fact that $0 < \alpha < \frac{\pi}{2}, 0 < \beta < \frac{\pi}{2}$ and $\tan \alpha \tan \beta = 1$ gives $\alpha + \beta = \frac{\pi}{2}$ because $0 < \alpha + \beta < \pi \quad$ and

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=$
$\cos\alpha\cos\beta(1-\tan\alpha\tan\beta)=0$. This makes
$\alpha+\beta=\frac{\pi}{2}$.

Then $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi^2}{8}$ and

$$\zeta(2)=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}+\sum_{k=1}^{\infty}\frac{1}{(2k)^2}=$$

$\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}+\frac{1}{4}\zeta(2)$. These make

$$\zeta(2) = \tfrac{4}{3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

Remark.

(*) is interpretated as the Parseval formula for the Legendre generating function

$$g(t,x) = \tfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \text{ where}$$

$P_n(x)$ are the Legendre Polynomials which is defined by $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$, because $\sqrt{2n+1}P_n(x)(n = 0, 1, 2, \dots)$ are C.O.N.S. in

$\mathbb{L}_2[-1,1]$.

Theorem 6.2.

$\int_{-1}^{1} dt \int_0^{+\infty} \frac{1}{(1+x^2)+t(1-x^2)} dx = \int_0^{\infty} dx \int_{-1}^{1} \frac{1}{(1+x^2)+t(1-x^2)} dt$

makes

$$\zeta(2) = \frac{\pi^2}{6}.$$

Proof.

For $-1 < t < 1$, $\int_0^{+\infty} \frac{1}{(1-t)+x^2(1+t)} dx = \frac{1}{1+t} \frac{1}{\sqrt{\frac{1-t}{1+t}}} [\arctan \frac{x}{\sqrt{\frac{1-t}{1+t}}}]_0^{\infty} = \frac{\pi}{2} \frac{1}{\sqrt{1-t^2}}$. Then

L.H.S. $= \int_{-1}^{1} \frac{\pi}{2} \frac{1}{\sqrt{1-t^2}} .dt = \frac{\pi^2}{2}$.

For $\int_{-1}^{1} \frac{1}{(1-t)+x^2(1+t))} dt =$

$[\frac{1}{x^2-1} \log |(x^2 - 1)t + (x^2 + 1)|]_{-1}^{1} = \frac{\log x^2}{x^2-1}$.

So, $= \int_{0}^{\infty} \frac{\log x^2}{x^2-1} dx = 2 \int_{0}^{1} \frac{\log x^2}{x^2-1} dx$. because

$\int_{1}^{\infty} \frac{\log x^2}{x^2-1} dx = \int_{0}^{1} \frac{\log x^{-2}}{x^{-2}-1} \frac{1}{x^2} dx = \int_{0}^{1} \frac{\log x^2}{x^2-1} dx$.

Then, R.H.S.$= 2 \int_{0}^{1} \frac{-\log x^2}{1-x^2} dx =$

$$2 \int_{0}^{1} (-\log x^2) \sum_{k=0}^{\infty} x^{2k} dx =$$

$$4\sum_{k=0}^{\infty}\int_0^1(-\log x)x^{2k}dx =$$

$$4\sum_{k=0}^{\infty}\int_0^{\infty} ue^{-2kx}e^{-u}du = 4\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.$$

Then L.H.S.=R.H.S. gives

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$ and $\zeta(2) = \frac{\pi^2}{6}$.

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