When roll-overs do not qualify as numéraire: bond markets beyond short rate paradigms

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Most of the term structure models in the literature are based on the fundamental assumption that bond prices $P(t, T)$ together with a *numéraire bank account process* $B(t)$ form an arbitrage-free market. Formally speaking this means that we can find an equivalent local martingale measure for the collection of stochastic processes $(B(t)^{-1}P(t, T))_{0 \leq t \leq T}$ representing bond prices discounted by the bank account's current value. If we assume additionally that those local martingales are indeed martingales, then we arrive at the famous relationship

$$P(t, T) = E_Q\Big[\frac{B_t}{B_T}|\mathcal{F}_t\Big] \tag{1}$$

for $0 \leq t \leq T$. If we assume alternatively the existence of forward rates, we arrive at the Heath-Jarrow-Morton (HJM) drift condition for the stochastic forward rate process encoding the previous local martingale property.

Goal of this work

- Bond market theory without assumptions on the existence of a bank account numéraire.
- Existence of martingale measures as a consequence of trading arguments.
- Analysis whether bank account numéraires exist and what they could mean.
- Relationship of numéraires and bubbles.

Economic definition of a numéraire

A positive portfolio in a financial market is considered a numéraire if arbitrary positive and negative (sic!) quantities of this portfolio may be held at any time by the investor.

This means that we always find counterparties for going short in arbitrary quantities. There are two reasons for counterparties not to accept such a deal: first, the belief that the numéraire is overpriced and hence buying it is not a good idea, or the belief that the numéraire is underpriced but we default on the short position.

Let us demonstrate the problem with an example lent from Eckhard Platen's benchmark approach. Let S^* denote a growth optimal portfolio on a finite time horizon $T^* > 0$. We claim this portfolio to be a *numéraire portfolio* of our market, i.e. fair prices $(\pi_t(X))_{0 \leq t \leq T}$ of payoffs X at time $T > 0$ should satisfy the martingale equation

$$\frac{\pi_t(X)}{S_t^*} = E_Q\Big[\frac{X}{S_T^*}|\mathcal{F}_t\Big] \tag{2}$$

under appropriate integrability assumptions. In particular we obtain bond price processes from the equation

$$P(t,T) = E_Q\Big[\frac{S_t^*}{S_T^*}|\mathcal{F}_t\Big] \tag{3}$$

for $0 \leq t \leq T \leq T^*$, which defines our financial market consisting of the assets S^* and $P(.,T)$ for $T \geq 0$.

If bond prices are bounded by 1, i.e. in the case of non-negative interest rates, then the inverse growth optimal portfolio necessarily is a supermartingale, and vice versa. We recall that being a numéraire means in our view that arbitrary positive and negative multiples of S^* are available for trading.

Therefore in this setting *also* the terminal bond $P(., T^*)$ qualifies as numéraire (since negative multiples of it are still price processes of a claim), i.e. we can find an equivalent martingale measure Q^* such that the processes $\frac{P(.,T)}{P(.,T^*)}$ are Q^*-martingales for $0 \leq t \leq T^*$. Indeed, define a density process via the properly normalized positive martingale $\left(\frac{P(t,T^*)}{S_t^*}\right)_{0 \leq t \leq T^*}$, which is a martingale by construction, leading to the desired assertion. In other words: the benchmark approach leads to a bona fide, non-constant bond market without frictions and free lunches.

Roll-over portfolios

Consider now the (quite natural) case that $\frac{1}{S^*}$ is a strict local martingale.

Let us consider now a sequence of refining partitions $0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = T^*$ of $[0, T^*]$ define, for each n,

$$B_t^n = \begin{cases} \prod_{i=1}^{j} \frac{1}{P(t_{i-1}^n, t_i^n)} & \text{for } t = t_j^n,\, j = 1, \ldots, k_n, \\ B_{t_j^n}^n & \text{for } t_{j-1}^n < t \leq t_j^n,\, j = 1, \ldots, k_n. \end{cases}$$

We assume additionally that $\lim_{n \to \infty} B^n = B^\infty = B = 1$, which means that the bank account process has short rate 0.

Then by assumption the processes $\frac{B^\infty}{S^*}$ and $\frac{B^n}{S^*}$ are strict local local martingales, too, and do therefore not qualify as numéraires. Negative multiples of those portfolios are indeed not portfolio wealth processes anymore.

When pricing 1 at time T with respect to this deflator we obtain an alternative term structure $\tilde{P}^n(t,T)$

$$E_{Q^*}\Big[\frac{B^n_T}{P(T,T^*)}\frac{1}{E_{Q^*}[\frac{B^n_T}{P(T,T^*)}]}\frac{1}{B^n_T}\Big] = E_{Q^*}\Big[\frac{1}{P(T,T^*)}\frac{1}{E_{Q^*}[\frac{B^n_T}{P(T,T^*)}]}\Big] = \frac{1}{B^n_0}\tilde{P}^n(0,T)\,,$$

which yields

$$\tilde{P}^n(0,T) = \frac{B^n_0 P(0,T)}{P(0,T^*)E_{Q^*}[\frac{B^n_T}{P(T,T^*)}]} > P(0,T)\,, \tag{4}$$

for each n, i.e., the virtual term structures show lower interest rates (due to higher liquidity) than $P(t,T)$. In case of B^∞ we apparently obtain the virtual term structure $\tilde{P}^\infty(0,T) = 1$, which corresponds to the highest liquidity virtual term structure, with overnight borrowing at no cost available.

These virtual term structures can be interpreted as high-liquidity term structures, which one would actually expect in the market if there was enough liquidity in the respective numéraire: this amounts to pricing with the corresponding supermartingale deflator

$$\frac{1}{E_{Q^*}[\frac{B_T^n}{P(T,T^*)}]} \frac{B_T^n}{P(T,T^*)} \frac{1}{B_T^n},$$

which is derived from changing measure by the local martingale density $\frac{B_T^n}{P(T,T^*)}$.

A strict local martingale deflator

In this section we consider the example touched upon in the introduction in more detail. Let S^* denote the growth optimal portfolio. In the benchmark approach fair prices of a payoff X at time $T > 0$ are given by

$$\pi_t(X) = S_t^* E_Q[\frac{X}{S_T^*}|\mathcal{F}_t],$$

and, as a consequence, one obtains bond prices of the form

$$P(t,T) = E_Q\Big[\frac{S_t^*}{S_T^*}|\mathcal{F}_t\Big]. \tag{5}$$

We give an example where the inverse of the growth optimal portfolio is related to a strict local martingale. Consider the case where

$$\frac{1}{S_t^*} = \frac{A(t)}{\| x + W_t \|^2} =: \xi_t$$

with a positive, deterministic, càdlàg function $A : [0, \infty) \mapsto (0, \infty)$, a four-dimensional standard Brownian motion W and $0 \neq x \in \mathbb{R}^4$. Then $(\| x + W_t \|^2)_{t \geq 0}$ is a squared Bessel process of dimension four and its inverse is a strict local martingale.

In this example, for each T^*, there exists an equivalent probability measure Q^* such that all bond prices with maturity $T \leq T^*$ discounted by the numéraire $P(., T^*)$ are martingales under Q^*. So we are in the situation of Theorem 12. Indeed, let $\alpha = \frac{1}{E_Q[\xi_{T^*}]}$ and define

$$\frac{dQ^*}{dQ} = \alpha \xi_{T^*}.$$

The density process Z_t^*, $0 \leq t \leq T^*$, satisfies

$$Z_t^* = \alpha E_Q[\xi_{T^*} | \mathcal{F}_t] = \alpha \xi_t P(t, T^*).$$

Therefore

$$Z_t^* \frac{P(t, T)}{P(t, T^*)} = \alpha E_Q[\xi_T | \mathcal{F}_t],$$

hence $\frac{P(t,T)}{P(t,T^*)}$, $0 \leq t \leq T$, is a martingale with respect to Q^*.

Using Markovianity of W and integrating over the transition density of squared Bessel processes one obtains the following explicit expression for $P(t,T)$.

$$P(t,T) = \frac{A(T)}{A(t)} E_Q\left[\frac{\| x + W_t \|^2}{\| x + W_T \|^2} \,\middle|\, \mathcal{F}_t \right] = \frac{A(T)}{A(t)} \left(1 - e^{-\frac{\| x + W_t \|^{-2}}{2(T-t)}} \right), \tag{6}$$

see work of Heath/Platen.

Assume that B_t^n is defined as in Definition 13, where we additionally assume that there exists a constant $K \geq 1$ such that, for all n, $\max_{1 \leq i \leq k_n} |t_i^n - t_{i-1}^n| \leq \frac{KT^*}{k_n}$. Then we get the following.

Lemma

$B_t^n \to B_t := \frac{A(0)}{A(t)}$ a.s., for $n \to \infty$.

By Theorem 15, $(B_t)_{0\leq t\leq T^*}$ discounted with respect to the numéraire $P(., T^*)$ is a Q^*–supermartingale. Lemma 21 and the definition of the measure Q^* moreover gives that $\left(\frac{B_t}{P(t,T^*)}\right)_{0\leq t\leq T^*}$ is a strict local martingale under Q^* as

$$Z_t^* \frac{B_t}{P(t, T^*)} = \alpha \xi_t B_t = \alpha \frac{A(0)}{\|x + W_t\|^2};$$

hence it does not qualify as strong nor as weak numéraire.

Besides the term structure given by $(P(t,T))_{0\le t\le T}$, we observe a second virtual term structure in correspondance with Definition: if there was enough liquidity such that the bank account qualifies as numéraire, one can price with the supermartingale deflator

$$\frac{1}{E_{Q^*}[\frac{B_T}{P(T,T^*)}]}\frac{B_T}{P(T,T^*)}\frac{1}{B_T},$$

which stems from changing measure by the supermartingale $\frac{B_T}{P(T,T^*)}$. Pricing 1 at time T with respect to this deflator, we obtain the virtual term structure

$$\begin{aligned}\tilde{P}(0,T) &= B_0 E_{Q^*}\Big[\frac{B_T}{P(T,T^*)}\frac{1}{E_{Q^*}[\frac{B_T}{P(T,T^*)}]}\frac{1}{B_T}\Big]\\ &= \frac{B_0}{B_T} = \frac{A(T)}{A(0)}.\end{aligned} \tag{7}$$

The illiquidity premium for the claim $Y \equiv 1$ at time 0 can be computed from (9) and we obtain

$$\begin{aligned}1-\Big(E_{Q^*}\big[\frac{B_T}{P(T,T^*)}\big]\Big)^{-1} &= 1-\frac{1}{A(0)\,E_{Q^*}\big[(A(T)P(T,T^*))^{-1}\big]}\\ &= 1-\frac{A(T^*)}{A(0)\,E_{Q^*}\Big[\Big(1-e^{-\frac{\|x+W_T\|^{-2}}{2(T^*-T)}}\Big)^{-1}\Big]},\end{aligned}$$

by (16). The expectation is given in terms of the transition density of Bessel processes.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where the filtration satisfies the usual conditions. The price process of traded assets $(\mathbf{X}_t)_{t\in[0,T]} = (X_t^0, \dots, X_t^d)_{t\in[0,T]}$ is a $d+1$-dimensional adapted process with càdlàg trajectories, where at least one process, say X^0, is positive, i.e. $X^0 > 0$. We introduce the process of discounted assets,

$$\mathbf{S} := (1, \frac{X^1}{X^0}, \dots, \frac{X^d}{X^0})$$

and assume without loss of generality that we are dealing from now on with a semimartingale S.

Let $\mathbf{H}$ be a predictable $\mathbf{S}$-integrable process and denote by $(\mathbf{H} \cdot \mathbf{S})$ the stochastic integral process of $\mathbf{H}$ with respect to $\mathbf{S}$, the *(portfolio) wealth process*. The process $\mathbf{H}$ is called an *a-admissible trading strategy* if there is $a \geq 0$ such that $(\mathbf{H} \cdot \mathbf{S})_t \geq -a$ for all $t \in [0, T]$. A strategy is called admissible if it is a-admissible for some $a \geq 0$. Define

$$\mathbf{K} = \{(\mathbf{H} \cdot \mathbf{S})_T : H \text{ admissible}\} \text{ and}$$
$$\mathbf{C} = \{g \in L^\infty(P) : g \leq f \text{ for some } f \in K\}.$$

Then $\mathbf{K}$ and $\mathbf{C}$ form convex cones in $L^0(\Omega, \mathcal{F}, P)$.

The condition *no free lunch with vanishing risk* (NFLVR) is the right concept of no arbitrage, see work of Delbaen/Schachermayer.

Definition

The market $\mathbf{S}$ satisfies (NFLVR) if

$$\bar{\mathbf{C}} \cap L^{\infty}_{+}(P) = \{0\},$$

where $\bar{\mathbf{C}}$ denotes the closure of C with respect to the norm topology of $L^{\infty}(P)$.

This means that a free lunch with vanishing risk exists, if there exists a free lunch $f \in L_+^\infty(P)$, which can be approximated by a sequence of portfolio wealth processes $(f_n) = ((\mathbf{H_n} \cdot \mathbf{S})) \in K$ with $\frac{1}{n}$-admissible integrands $\mathbf{H}_n$, such that

$$\lim_{n\to\infty} \| f - f_n \|_\infty = 0$$

with respect to the norm topology of $L^\infty(P)$. Define the set $\mathbf{M}_e$ of equivalent separating measures as

$$\mathbf{M}_e = \{Q \sim P|_{\mathcal{F}_T} : E_Q[f] \leq 0 \text{ for all } f \in \mathbf{K}\}\,.$$

If $\mathbf{S}$ is (locally) bounded then $\mathbf{M}_e$ consists of all equivalent probability measures such that $\mathbf{S}$ is a (local) martingale.

Definition

Fix $Q \in \mathbf{M}_e$. Consider an extension of the original market $\mathbf{S}' := (\mathbf{S}, \mathbf{Y})$ by finitely many assets $\mathbf{Y}$ such that the process $\mathbf{S}'$ is a Q-local martingale. Consider furthermore a predictable, $\mathbf{S}'$-integrable process φ and the sequence of hitting times

$$\sigma_n := \inf\{t \geq 0 : (\varphi \cdot \mathbf{S}')_t \leq -n\}, \quad n \geq 1.$$

The trading strategy φ is called *Q-admissible* (such as the corresponding stochastic integral, the wealth process), if

$$\liminf_{n\to\infty} E_Q[(\varphi \cdot \mathbf{S}')^-_{\sigma_n} \mathbb{1}_{\{\sigma_n < \infty\}}] = 0\,.$$

Define

$$\mathbf{L}^Q = \{x + (\varphi \cdot \mathbf{S}') : x \in \mathbb{R}, \quad \varphi \text{ is } Q\text{-admissible}\}.$$

and

$$\mathbf{L} = \cup_{Q \in \mathbf{M}_e} \mathbf{L}^Q \,.$$

We extend the set of admissible portfolios but we do not introduce arbitrages, since every wealth process $(\varphi \cdot \mathbf{S}')$ for a Q-admissible strategy is a supermartingale. We also do not introduce free lunches, since this notion *only* depends on a-admissible strategies.

Now we are in the position to make our intuitive definition of numéraire portfolios precise: a numéraire portfolio is a strictly positive portfolio which allows for short-selling, i.e. the negative of its wealth process is still given by a Q-admissible trading strategy for some $Q \in \mathbf{M}_e$, and hence is an element of $\mathbf{L}$.

Definition

A strictly positive process $N \in \mathbf{L}$ with $N_0 = 1$ is called a *strong numéraire* (in discounted terms with respect to S^0), if

$$N \in \mathbf{L}^Q \text{ and } -N \in \mathbf{L}^Q \tag{8}$$

for all $Q \in \mathbf{M}_e$. It is called *weak numéraire* (in discounted terms with respect to S^0), if (8) holds for at least one $Q \in \mathbf{M}_e$, i.e. N and $-N$ are elements of $\mathbf{L}$.

This definition has a clear economic meaning and easy consequences: as it should be, a weak numéraire qualifies as an accounting unit, where the classical change of numéraire technique is possible: there exist an equivalent measure $Q \in \mathbf{M}_e$ under which $N = (1 + (\varphi \cdot \mathbf{S}'))$ is a true Q-martingale.

Theorem

The following statements are equivalent:

1. *A strictly positive process N with $N_0 = 1$ is a weak numéraire.*
2. *There exists $Q \in \mathbf{M}_e$ such that N is a Q-martingale.*

Definition

A strictly positive process $B \in \mathbf{L}$ is (modeled) in a *strong bubble state* if $-B \notin \mathbf{L}$, i.e. for all $Q \in \mathbf{M}_e$ the wealth process B is a strict local martingale. It is (modeled) in a *weak bubble state* if $-B \notin \mathbf{L}^Q$ for some $Q \in \mathbf{M}_e$, i.e. for this $Q \in \mathbf{M}_e$ the wealth process B is a strict local martingale.

Theorem

A strictly positive portfolio $B \in \mathbf{L}$ with $B_0 = 1$ is in a strong bubble state if and only if B does not qualify as weak numéraire portfolio.
A strictly positive portfolio $B \in \mathbf{L}$ with $B_0 = 1$ is in a weak bubble state if and only if B does not qualify as strong numéraire.

Virtual Market

Let $V \in \mathbf{L}$ with $V_0 = 1$ be a weak bubble, i.e. there is $Q \in \mathbf{M}_e$ such that V is a strict Q-local martingale. Consider $T \in [0, T^*]$ and define the probability measure Q^{V_T} by

$$E_{Q^{V_T}}[Y] := \frac{E_Q[YV_T]}{E_Q[V_T]}$$

for bounded measurable Y. We call the market discounted by V a *virtual market* and its prices *virtual prices* with respect to V_T, i.e. the price of a discounted (with respect to S^0) $\mathcal{F}_T$-measurable claim Y in this virtual market with respect to the pricing measure Q^{V_T} is given by

$$E_{Q^{V_T}}[\frac{Y}{V_T}] = \frac{E_Q[Y]}{E_Q[V_T]}\,.$$

We call the difference of prices

$$E_{Q^{V_T}}\Big[\frac{Y}{V_T}\Big] - E_Q[Y] = E_Q[Y]\Big(1 - \frac{1}{E_Q[V_T]}\Big), \quad 0 \leq T \leq T^* \quad (9)$$

the *term structure of (il-)liquidity premia* of the numéraire V with respect to the pricing measure Q.

We consider the following model for a bond market. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space where the filtration satisfies the usual conditions. For each $T \in [0, \infty)$ we denote by $(P(t, T))_{0\leq t\leq T}$ the price process of a bond with maturity T. For all T, $(P(t, T))_{0\leq t\leq T}$ is a strictly positive càdlàg stochastic process adapted to $(\mathcal{F}_t)_{0\leq t\leq T}$ with $P(T, T) = 1$. We assume that the price process is almost surely right continuous in the second variable, where the nullset does not depend on t, indeed we make

Assumption

There is $N \in \mathcal{F}$ with $P(N) = 0$ such that

$$N \supseteq \bigcup_{t\in[0,\infty)} \{\omega : T \to P(t, T)(\omega) \text{ is not right continuous}\}.$$

Assumption

We make the following assumption on uniform local boundedness for $P(., T)$ and local boundedness for $P(., T)^{-1}$:

1) *For any T there is $\varepsilon > 0$, an increasing sequence of stopping times $\tau_n \to \infty$ and $\kappa_n \in [0, \infty)$ such that*

$$P(t, U)^{\tau_n} \leq \kappa_n,$$

for all $U \in [T, T + \varepsilon)$ and all $t \leq T$.

2) *There exists a nonempty set $\mathcal{T} \subset [0, \infty)$ such that $\left(\frac{1}{P(t,T^*)}\right)_{0 \leq t \leq T^*}$ is locally bounded for all $T^* \in \mathcal{T}$.*

In the following assumption we consider a numéraire related to a *terminal maturity* $T^* \in \mathcal{T}$.

Assumption

For all finite collections of maturities $T_1 < T_2 < \cdots < T_n \leq T^*$ *with* $T^* \in \mathcal{T}$ *there exists a measure* $Q \sim P|_{\mathcal{F}_{T^*}}$ *such that* $\left(\frac{P(t,T_i)}{P(t,T^*)}\right)_{0 \leq t \leq T_i}$ *is a local* Q*-martingale,* $i = 1, \dots, n$.

The measure Q from Assumption 3 is called the T^*-forward-measure for the finite market consisting of bonds $P(., T_i)$, $i = 1, \dots, n$ and the numéraire $P(., T^*)$.

Note that we do not assume the existence of a short rate or even a bank account. Moreover, we do not assume that $P(., T)$ is a semimartingale. However, Assumption (3) implies that, for a finite collection of maturities, only bonds in terms of the numéraire $P(., T_n)$ are semimartingales under the objective measure P, because they are local martingales under the equivalent measure Q. Moreover, they are locally bounded because we assumed that $P(., T)$ is locally bounded, for any T, and $P(., T^*)^{-1}$ is locally bounded for $T^* \in \mathcal{T}$.

Assumption (3) means that for a finite selection of bonds considered with respect to a certain numéraire (the bond with the largest maturity) there exists an equivalent local martingale measure. Our aim will be the following: for a fixed maturity $T^* \in \mathcal{T}$, we aim at finding a measure Q^* such that all bonds with maturity $T \leq T^*$ are local martingales under Q^* in terms of the numéraire $P(t, T^*)$.

We choose a finite time horizon $T > 0$ as this will be sufficient for our purpose. Let $(\mathbf{S}^n_t)_{t\in[0,T]}$, $n = 1, 2, \dots,$ be a sequence of semimartingales, where $\mathbf{S}^n$ takes values in $\mathbb{R}^{d(n)}$, based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ where the filtration satisfies the usual assumptions. For each $n \geq 1$ we define a classical market model (referred to as *finite market n*) given by the $\mathbb{R}^{d(n)}$–valued semimartingale $\mathbf{S}^n$ (which describes the price processes of $d(n)$ tradable assets). We assume that the assets are already discounted with respect to one numéraire, so we have that one of the $d(n)$ assets equals 1. For our purposes it is sufficient to assume that there is a sequence S^i, $i = 0, 1, \dots$ of semimartingales, such that $S^0_t \equiv 1$ and such that $(\mathbf{S}^n_t) = (S^0_t, S^1_t, \dots, S^n_t)$. In this case $d(n) = n + 1$.

Let $\mathbf{H}$ be a predictable $\mathbf{S}^n$-integrable process and, as previously, $(\mathbf{H} \cdot \mathbf{S}^n)_t$ the stochastic integral of $\mathbf{H}$ with respect to $\mathbf{S}$. The process $\mathbf{H}$ is an admissible trading strategy if $\mathbf{H}_0 = 0$ and there is $a > 0$ such that $(\mathbf{H} \cdot \mathbf{S}^n)_t \geq -a$, $0 \leq t \leq T$. Define

$$\mathbf{K}^n = \{(\mathbf{H} \cdot \mathbf{S}^n)_T : H \text{ admissible}\} \text{ and } \mathbf{C}^n = (\mathbf{K}^n - L^0_+) \cap L^\infty. \quad (10)$$

$\mathbf{K}^n$ can be interpreted as the cone of all replicable claims in the finite market n, and $\mathbf{C}^n$ is the cone of all claims in L^∞ that can be superreplicated in this market.

Define the set $\mathbf{M}_e^n$ of equivalent separating measures for the finite market n as

$$\begin{aligned}\mathbf{M}_e^n &= \{Q \sim P : E_Q[f] \leq 0 \text{ for all } f \in \mathbf{C}^n\} \\ &= \{Q \sim P : E_Q[f] \leq 0 \text{ for all } f \in \mathbf{K}^n\}.\end{aligned} \tag{11}$$

If $\mathbf{S}^n$ is (locally) bounded then $\mathbf{M}_e^n$ consists of all equivalent probability measures such that $\mathbf{S}^n$ is a (local) martingale.

A *large financial market* is the sequence of the finite market models n, i.e. the sequence of the market models induced by the $d(n)$-dimensional semimartingales $\mathbf{S}^n$. As a consequence, we cannot trade with an actually infinite number of securities (so that we avoid artificially introduced infinite-dimensional trading strategies), but we can trade in more and more assets and in this way approximate something infinite-dimensional.

We impose the following assumption, which is standard in the theory of large financial markets:

$$\mathbf{M}_e^n \neq \emptyset, \qquad \text{for all } n \in \mathbb{N}. \tag{12}$$

This implies that any no arbitrage condition (such as *no arbitrage*, *no free lunch with vanishing risk*, *no free lunch*) holds for each finite market n.

No asymptotic free lunch

(NAFL) is the large financial markets analogue of the classical no free lunch condition (NFL) of Kreps. We will first recall the classical NFL condition here for a finite market n. Let $\mathbf{C}^n$ be defined as in (10).

Definition

The condition NFL holds on the finite market n if

$$\overline{\mathbf{C}^n}^* \cap L^\infty_+ = \{0\}, \tag{13}$$

where $\overline{\mathbf{C}^n}^*$ denotes the weak-star-closure of $\mathbf{C}^n$.

This means by superreplicating claims in an admissible way with a finite number of assets we cannot approximate in a weak-star sense a strictly positive gain.

Now NAFL can be defined in analogous way as the condition NFL but for the whole sequence of sets $\mathbf{C}^n$:

Definition

A given large financial market satisfies NAFL if

$$\overline{\bigcup_{n=1}^{\infty} \mathbf{C}^n}^{*} \cap L_+^\infty = \{0\}.$$

If NAFL holds then it is not possible to approximate a strictly positive profit in a weak-star sense by trading in any finite number of the given assets (although we can use more and more of them).

Note that in the literature the term large financial market is used for a more general concept where each market n is based on a different filtered probability space. So, in our setting, we will not have to deal with technicalities which are common in large financial markets.

Definition

Let $T^* \in \mathcal{T}$ where $\mathcal{T}$ is the set from Assumption 2. Fix a sequence $(T_i)_{i\in\mathbb{N}}$ in $[0, T^*]$. Define the $n+1$-dimensional stochastic process $(\mathbf{S}^n) = (S^0, \ldots, S^n)$ on $[0, T^*]$ as follows:

$$S_t^i = \begin{cases} \frac{P(t,T_i)}{P(t,T^*)} & \text{for } 0 \le t \le T_i \\ \frac{1}{P(T_i,T^*)} & \text{for } T_i < t \le T^* \end{cases}, \tag{14}$$

for $i = 1, \ldots, n$ and $S_t^0 = \frac{P(t,T^*)}{P(t,T^*)} \equiv 1$. The large financial market consists of the sequence of classical market models given by the $(n+1)$-dimensional stochastic processes $(\mathbf{S}^n)_{t\in[0,T^*]}$ based on the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T^*]}, P|_{\mathcal{F}_{T^*}}\right)$.

Definition

The bond market $(P(t,T))_{0\leq t\leq T}$ for $0 \leq T \leq T^*$ is said to satisfy NAFL if there exists a dense sequence $(T_i)_{i\in\mathbb{N}}$ in $[0, T^*]$, such that the large financial market of Definition 10 satisfies the condition NAFL.

Since all involved semimartingales $\mathbf{S}^n$ are locally bounded due to Assumption 2, it is sufficient to deal with equivalent local martingale measures. Hence, the set $\mathbf{M}_e^n$ from (11) is given as follows:

$$\mathbf{M}_e^n = \{Q^n \sim P|_{\mathcal{F}_T} : \mathbf{S}^n \text{ local } Q^n\text{-martingale}\}.$$

By Assumption 3 we have that $\mathbf{M}_e^n \neq \emptyset$ for all $n \in \mathbb{N}$, so the standard assumption (12) for large financial markets holds. Note that this also implies that each $\mathbf{S}^n$ is a semimartingale, so this is not a problem in Definition 10.

Theorem

Fix any $T^* \in \mathcal{T}$ and let Assumption 1, 2 and Assumption 3 hold. Then, the bond market satisfies NAFL, if and only if there exists a measure $Q^* \sim P|_{\mathcal{F}_{T^*}}$ such that $\left(\frac{P(t,T)}{P(t,T^*)}\right)_{0 \leq t \leq T}$ is a local Q^*-martingale for all $T \in [0, T^*]$.

In Theorem 12 we consider the NAFL condition for the large financial market as in Definition 10 with respect to one fixed, dense sequence of T_i in $[0, T^*]$. However, as there is a local martingale measure for all bond prices discounted by the numéraire, the general theorem about NAFL in large financial markets implies that for any large financial market (induced by the bond market via any sequence of maturities) NAFL holds.

It is possible to obtain a candidate process for the bank account by a limit of rolled over bonds as we show now. Throughout this section we assume that all the assumptions of Theorem 12 hold.

Definition

Let $0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = T^*$ be a sequence of refining partitions of $[0, T^*]$. Define, for each n, the *roll-over* B^n as follows: $B_0^n = 1$ and

$$B_t^n = \begin{cases} \prod_{i=1}^{j} \frac{1}{P(t_{i-1}^n, t_i^n)} & \text{for } t = t_j^n,\ j = 1, \dots, k_n \\ B_{t_j^n}^n P(t, t_j^n) & \text{for } t_{j-1}^n < t \leq t_j^n,\ j = 1, \dots, k_n \end{cases}$$

Lemma

There exists a self-financing strategy $\hat{\mathbf{H}}_t^n = (\hat{H}_t^1, \ldots, \hat{H}_t^{k_n})$ on the market containing the k_n-dimensional asset $\hat{\mathbf{S}}^n(\cdot) = (P(., t_1^n), \ldots, P(., t_{k_n}^n))$ such that $B_t^n = \langle \hat{\mathbf{H}}_t, \hat{\mathbf{S}}_t \rangle$. Discounted by the numéraire $P(t, t_{k_n}^n) = P(t, T^*)$ this gives an admissible strategy $\mathbf{H}^n$ such that $\frac{B_t^n}{P(t,T^*)} = \frac{1}{P(0,T^*)} + (\mathbf{H}^n \cdot \mathbf{S}^n)_t > 0$, where $\mathbf{S}^n$ is the process $\hat{\mathbf{S}}^n$ discounted by the numéraire $P(t, T^*)$. In particular, $\left(\frac{B_t^n}{P(t,T^*)}\right)_{0 \le t \le T^*}$ is a positive local martingale and hence a supermartingale with respect to the measure Q^* of Theorem 12.

Theorem

Let $((B^n_t)_{0\leq t\leq T^*})$ be the sequence of roll-overs given as in Definition 13. There exists a sequence of convex combinations $\tilde{B}^n \in conv(B^n, B^{n+1}, \dots)$ and a càdlàg stochastic process $(B_t)_{0\leq t\leq T^*}$, henceforward called generalized bank account, such that

$$B_t = \lim_{q\downarrow t} \lim_{n\to\infty} \tilde{B}^n_q,$$

with $B_0 \leq 1$ and $0 \leq B_t < \infty$, for all $t \leq T^*$. The generalized bank account has the following properties.

1. *The process $(V_t)_{0\leq t\leq T^*}$, where $V_t = \frac{B_t}{P(t,T^*)}$, is a supermartingale with respect to the measure Q^* of Theorem 12.*
2. *If $0 < P(t,T) \leq 1$, for all $T \leq T^*$, then $P(B_t \geq 1) = 1$, for all $t \leq T^*$.*

With a view what it means to be a numéraire we can ask why just terminal bonds qualify as numéraires by default in our setting: the answer is that we could take any other reasonably behaved stochastic process (the inverse has to be locally bounded) and plug it into Assumption 3 instead of $P(., T^*)$. Conclusions would remain the same, of course with a different meaning on what we would call numéraire in this setting. For instance we could think of taking discrete roll-over bonds as numéraires if we want to claim that this portfolio can be shortened.

In this section, we relax the assumptions on the bond market and investigate under which conditions there is a supermartingale deflator. This is motivated by the fact that we are lead to supermartingale deflators by the very structure of bond market models. Indeed if we have a non-vanishing generalized bank account and decide to choose it as market numéraire, see Theorem 15, then our theory only provides us with a supermartingale deflator structure. The results about supermartingale deflators in this section are related to and inspired by results of Kostas Kardaras.

Consider a large financial market induced by a sequence of semimartingales S^i, $i = 0, 1, \ldots$ on a fixed filtered probability space, where the filtration satisfies the usual conditions, such that $(\mathbf{S}^n_t)_{t\in[0,T^*]} = (S^0_t, S^1_t, \ldots, S^n_t)_{t\in[0,T^*]}$. Recall that $S^0_t \equiv 1$, i.e. the numéraire has been fixed and prices are discounted by the chosen numéraire. The sets $\mathbf{K}^n$, $\mathbf{C}^n$, $\mathbf{M}^n_e$ are defined as previously. We assume that each finite market satisfies (NFLVR), i.e. (12) holds for all n. In contrast to the previous sections, we do not assume here that the semimartingales are locally bounded. In this case, the set $\mathbf{M}^n_e$ as in (11) consists of all equivalent probability measures Q such that stochastic integrals $(\mathbf{H}^n \cdot \mathbf{S}^n)_t$, $0 \le t \le T^*$, with admissible integrands $\mathbf{H}^n$ (i.e. $(\mathbf{H}^n \cdot \mathbf{S}^n)_{T^*} \in \mathbf{K}^n$) are Q-supermartingales. It was shown by Delbaen/Schachermayer under the condition no free lunch with vanishing risk the set of equivalent sigma-martingale measures for $\mathbf{S}^n$ is dense in the set $\mathbf{M}^n_e$.

The notion *no asymptotic arbitrage of first kind* (NAA1) was introduced by Kabanov/Kramkov:

Definition

A large financial market admits an asymptotic arbitrage opportunity of first kind if there exists a subsequence, again denoted by n, and trading strategies $\mathbf{H}^n$ with

1. $(\mathbf{H}^n \cdot \mathbf{S}^n)_t \geq -\varepsilon_n$ for all $t \in [0, T^*]$,
2. $P((\mathbf{H}^n \cdot \mathbf{S}^n)_{T^*} \geq C_n) \geq \alpha$,

for all n, where $\alpha > 0$, $\varepsilon_n \to 0$ and $C_n \to \infty$.

We say that the large financial market satisfies the condition NAA1 if there are no asymptotic arbitrage opportunities of first kind.

The following result for large financial markets provides us with supermartingale deflators for bond markets.

Theorem

Consider the large financial market induced by the sequence of semimartingales $(\mathbf{S}_t^n)_{0\le t\le T^*} = (S_t^0, S_t^1, \dots, S_t^n)_{t\in[0,T^*]}$, $n = 1, 2, \dots$ *and assume that (12) holds for all* n. *Then NAA1 holds, if and only if there exists a strictly positive supermartingale* $(Z_t)_{0\le t\le T^*}$ *with* $Z_0 \le 1$, *such that* $(Z_t X_t)_{0\le t\le T^*}$ *is a supermartingale for all processes* X *with* $X_{T^*} \in \bigcup_{n=1}^{\infty} \mathbf{K}^n$.

Moreover, if NAA1 holds, then:

1. *if* $S_t^i \ge -a$, $0 < t \le T^*$, *for some* $i \in \mathbb{N}$ *and some* $a > 0$ *and* $S_0^i \ge 0$, *then* $(Z_t S_t^i)_{0\le t\le T^*}$ *is a supermartingale.*
2. *If* $S_0^i = 0$ *and* $(S_t^i)_{0\le t\le T^*}$ *is locally bounded for some* $i \in \mathbb{N}$, *then* $(Z_t S_t^i)_{0\le t\le T^*}$ *is a local martingale.*

The supermartingale Z is called *supermartingale deflator for the large financial market*.

In the sequel we apply the results to bond markets: again, for each $T \in [0, T^*]$, $(P(t, T))_{0 \leq t \leq T}$ is a strictly positive càdlàg stochastic process adapted to $(\mathcal{F}_t)_{0 \leq t \leq T}$ with $P(T, T) = 1$. We assume that, for fixed t, the function $T \mapsto P(t, T)$ is almost surely right-continuous. Note, that in this section, we do not have any local boundedness assumptions on $P(t, T)$ or $\frac{1}{P(t,T)}$. We will again have to assume that in the case of a finite number of assets discounted with the numéraire $P(t, T^*)$ we will not have any arbitrage opportunities. This is again a consequence of the following assumption on existence of the T^*-forward measure. Consider any $0 < T_1 < \cdots < T_n \leq T^*$ and define the cone $\mathbf{C}(T_1, \ldots, T_n, T^*)$ as in (10) where S^i is defined as in (14). Note that we do not assume here that $T^* \in \mathcal{T}$.

Assumption

For all finite collections of maturities $T_1 < T_2 < \cdots < T_n \le T^*$ there exists a measure $Q \sim P|_{\mathcal{F}_{T^*}}$ such that for all $f \in \mathbf{C}(T_1, \ldots, T_n, T^*)$ we have that $E_Q[f] \le 0$.

The condition $E_Q[f] \le 0$ means that the measure $Q \in \mathbf{M}_e(T_1, \ldots, T_n, T^*)$, where the definition of the set of separating measures is analogous as in (11). Note that Assumption 4 implies the existence of an equivalent sigma-martingale measure for $S^1, \ldots, S^n$ given as as in (14), and therefore these processes are semimartingales. As in Definition 10, any sequence $(T_i)_{i \in \mathbb{N}}$ in $[0, T^*]$ induces a large financial market.

Definition

The bond market satisfies NAA1 w.r.t. a sequence $(T_i)_{i\in\mathbb{N}}$ in $[0, T^*]$ if for the large financial market induced by $(T_i)_{i\in\mathbb{N}}$ there does not exist an asymptotic arbitrage of first kind.

Theorem

Fix a sequence $(T_i)_{i\in\mathbb{N}}$ *in* $[0, T^*]$. *The bond market satisfies NAA1 w.r.t.* $(T_i)_{i\in\mathbb{N}}$ *if and only if there exists a strictly positive supermartingale deflator* $(Z_t)_{0\le t\le T^*}$ *for the large financial market induced by* $(T_i)_{i\in\mathbb{N}}$. *If* $(T_i)_{i\in\mathbb{N}}$ *is dense in* $[0, T^*]$, *then* $\left(Z_t \frac{P(t,T)}{P(t,T^*)}\right)_{0\le t\le T}$ *is a supermartingale for all* $T \le T^*$.

Finally we will show that under the weaker assumptions of this section we will still be able to define a generalized bank account.

Theorem

Let (T_i) be a dense sequence in $[0, T^*]$ such that NAA1 holds. Let $(B^n_t)_{t\in[0,T^*]}$ be the sequence of roll-overs as in Definition 13, where the refining partition $\{t^n_1, \dots, t^n_{k_n}\}$ is chosen such $\bigcup_{n\in\mathbb{N}}\{t^n_1, \dots, t^n_{k_n}\} \subseteq (T_i)$. Then there exists a sequence of convex combinations $\tilde{B}^n \in conv(B^n, B^{n+1}, \dots)$ and a càdlàg stochastic process $(B_t)_{0\le t\le T^*}$ (the generalized bank account) such that

$$B_t = \lim_{q\downarrow t} \lim_{n\to\infty} \tilde{B}^n_q,$$

with $B_0 \le 1$ and $0 \le B_t < \infty$, for all $t \le T^*$. The generalized bank account has the following properties.

1. *The process $(V_t)_{0\le t\le T^*}$, where $V_t = Z_t\left(\frac{B_t}{P(t,T^*)} - \frac{1}{P(0,T^*)}\right)$, is a supermartingale, where Z is the supermartingale deflator as in Theorem 19. This implies that the bank account B discounted with respect to $P(., T^*)$ multiplied by Z is a supermartingale as*

$$\frac{Z_t B_t}{P(t,T^*)} = V_t + \frac{Z_t}{P(0,T^*)}.$$

2. *If $0 < P(t,T) \le 1$, for all $T \le T^*$, then $P(B_t \ge 1) = 1$, for all $t \le T^*$.*

The interpretation of this Theorem is, that for any refining sequence of partitions, which does not produce an asymptotic arbitrage opportunity of first kind in the induced large financial market, there does exist a generalized bank account. In particular, if the bond market does not allow an asymptotic arbitrage opportunity of first kind for any sequence of maturities in $[0, T^*]$ (for the respective induced large financial market as in Definition 10), then any refining sequence of partitions gives a generalized bank account in the sense of Theorem 20. If, moreover, the bond $P(t, T) \leq 1$, $0 \leq t \leq T \leq T^*$, then we can say that B_t is bounded from below by 1 a.s. This corresponds to the case, where a non-negative short rate exists.

A strict local martingale deflator

In this section we consider the example touched upon in the introduction in more detail. Let S^* denote the growth optimal portfolio. In the benchmark approach fair prices of a payoff X at time $T > 0$ are given by

$$\pi_t(X) = S_t^* E_Q[\frac{X}{S_T^*}|\mathcal{F}_t],$$

and, as a consequence, one obtains bond prices of the form

$$P(t,T) = E_Q\big[\frac{S_t^*}{S_T^*}|\mathcal{F}_t\big]. \tag{15}$$

We give an example where the inverse of the growth optimal portfolio is related to a strict local martingale. Consider the case where

$$\frac{1}{S_t^*} = \frac{A(t)}{\| x + W_t \|^2} =: \xi_t$$

with a positive, deterministic, càdlàg function $A : [0, \infty) \mapsto (0, \infty)$, a four-dimensional standard Brownian motion W and $0 \neq x \in \mathbb{R}^4$. Then $(\| x + W_t \|^2)_{t \geq 0}$ is a squared Bessel process of dimension four and its inverse is a strict local martingale.

In this example, for each T^*, there exists an equivalent probability measure Q^* such that all bond prices with maturity $T \leq T^*$ discounted by the numéraire $P(., T^*)$ are martingales under Q^*. So we are in the situation of Theorem 12. Indeed, let $\alpha = \frac{1}{E_Q[\xi_{T^*}]}$ and define

$$\frac{dQ^*}{dQ} = \alpha \xi_{T^*}.$$

The density process Z_t^*, $0 \leq t \leq T^*$, satisfies

$$Z_t^* = \alpha E_Q[\xi_{T^*} | \mathcal{F}_t] = \alpha \xi_t P(t, T^*).$$

Therefore

$$Z_t^* \frac{P(t, T)}{P(t, T^*)} = \alpha E_Q[\xi_T | \mathcal{F}_t],$$

hence $\frac{P(t,T)}{P(t,T^*)}$, $0 \leq t \leq T$, is a martingale with respect to Q^*.

Using Markovianity of W and integrating over the transition density of squared Bessel processes one obtains the following explicit expression for $P(t, T)$.

$$P(t,T) = \frac{A(T)}{A(t)} E_Q \left[\frac{\| x + W_t \|^2}{\| x + W_T \|^2} \,\middle|\, \mathcal{F}_t \right] = \frac{A(T)}{A(t)} \left(1 - e^{-\frac{\|x+W_t\|^{-2}}{2(T-t)}} \right), \tag{16}$$

see work of Heath/Platen.

Assume that B_t^n is defined as in Definition 13, where we additionally assume that there exists a constant $K \geq 1$ such that, for all n, $\max_{1 \leq i \leq k_n} |t_i^n - t_{i-1}^n| \leq \frac{KT^*}{k_n}$. Then we get the following.

Lemma

$B_t^n \to B_t := \frac{A(0)}{A(t)}$ a.s., for $n \to \infty$.

By Theorem 15, $(B_t)_{0\leq t\leq T^*}$ discounted with respect to the numéraire $P(., T^*)$ is a Q^*–supermartingale. Lemma 21 and the definition of the measure Q^* moreover gives that $\left(\frac{B_t}{P(t,T^*)}\right)_{0\leq t\leq T^*}$ is a strict local martingale under Q^* as

$$Z_t^* \frac{B_t}{P(t, T^*)} = \alpha \xi_t B_t = \alpha \frac{A(0)}{\|x + W_t\|^2};$$

hence it does not qualify as strong nor as weak numéraire.

Besides the term structure given by $(P(t,T))_{0\leq t\leq T}$, we observe a second virtual term structure in correspondance with Definition: if there was enough liquidity such that the bank account qualifies as numéraire, one can price with the supermartingale deflator

$$\frac{1}{E_{Q^*}[\frac{B_T}{P(T,T^*)}]}\frac{B_T}{P(T,T^*)}\frac{1}{B_T},$$

which stems from changing measure by the supermartingale $\frac{B_T}{P(T,T^*)}$. Pricing 1 at time T with respect to this deflator, we obtain the virtual term structure

$$\begin{aligned}\tilde{P}(0,T) &= B_0 E_{Q^*}\Big[\frac{B_T}{P(T,T^*)}\frac{1}{E_{Q^*}[\frac{B_T}{P(T,T^*)}]}\frac{1}{B_T}\Big]\\ &= \frac{B_0}{B_T} = \frac{A(T)}{A(0)}.\end{aligned} \tag{17}$$

The illiquidity premium for the claim $Y\equiv 1$ at time 0 can be computed from (9) and we obtain

$$\begin{aligned}1-\Big(E_{Q^*}\big[\frac{B_T}{P(T,T^*)}\big]\Big)^{-1} &= 1-\frac{1}{A(0)\,E_{Q^*}\big[(A(T)P(T,T^*))^{-1}\big]}\\ &= 1-\frac{A(T^*)}{A(0)\,E_{Q^*}\Big[\Big(1-e^{-\frac{\|x+W_T\|^{-2}}{2(T^*-T)}}\Big)^{-1}\Big]},\end{aligned}$$

by (16). The expectation is given in terms of the transition density of Bessel processes.

Exotic bank account processes

In this section we elaborate on a question raised in Section 65, namely how much money can be lost by investing in the roll-over strategy. We construct an example by means of not uniformly integrable martingales in which the rollover reaches zero almost surely in finite time.

We consider a market with NAFL and denote by Q^* the measure in Theorem 12. Our starting point are bond prices of the form

$$P(t, T) = E_{Q^*}\left[\frac{N_t}{N_T}|\mathcal{F}_t\right] \tag{18}$$

with a finite time horizon $T^* = 2$ and the numéraire N, chosen as follows:

Let $\tau : [0,1) \to \mathbb{R}_{\geq 0}$ be an increasing, differentiable time transformation with $\tau(0) = 0$ and $\tau(t) \to \infty$ as $t \to 1$. The numéraire N is given by

$$N(t) := P(t, T^*) = \begin{cases} \exp(W^2_{\tau(t)} - \tau(t)^2) & 0 \leq t < 1 \\ 1 & t \in [1,2]. \end{cases}$$

with Q^*-Brownian motion W.

Note that N is càdlàg: for any $\epsilon > 0$

$$\begin{aligned} Q^*(N(t) \leq \epsilon) &= Q^*(\tau(t)\xi^2 \leq \log \epsilon + \tau(t)^2) \\ &= 2\Phi(\sqrt{\tau(t) + \epsilon\tau(t)^{-1}}) - 1 \end{aligned}$$

for a standard normal random variable ξ. The last expression converges to 1 as $\tau(t) \to \infty$ and existence of left limits of N follows. However, N is not uniformly integrable. The filtration is given by $\mathcal{F}_t := \sigma(W_{\tau(s)} : 0 \leq s \leq t),\ t \in [0, 2]$, with the usual augmentation by null sets.

Bond prices now can be computed from (18): Note that

$$\begin{aligned}
E_{Q^*}[e^{-W_T^2+W_t^2}|\mathcal{F}_t] &= E_{Q^*}[e^{-(W_T-W_t)^2-2W_t(W_T-W_t)}|W_t] \\
&= E_{Q^*}[e^{-(T-t)\xi^2-2W_t\sqrt{T-t}\,\xi}|W_t] \\
&=: \exp(W_t^2 f(t,T) - g(t,T))
\end{aligned}$$

where ξ is standard normal, independent of W_t; we obtain $f(t,T) = 2(T-t)\,(1+2(T-t))^{-1}$ and $g(t,T) = \frac{1}{2}\log(1+2(T-t))$. Hence,

$$P(t,T) = \exp(W_{\tau(t)}^2 f_\tau(t,T) - g_\tau(t,T) + \tau^2(T) - \tau^2(t)),$$

$0 \le t \le T < 1$, where we set $f_\tau(t,T) := f(\tau(t),\tau(T)))$ and similarly for g_τ. For $t \ge 1$ the term structure is flat, i.e. $P(t,T) = 1$.

Now we turn to the limit of the roll-over account. Fix $T < 1$ and consider $t_i^n := t_i = \tau^{-1}(iT/n)$. Then

$$B_{t_n}^n = \exp(-\sum_{i=1}^{n} f_\tau(t_{i-1}, t_i) W_{\tau(t_{i-1})}^2 + \sum_{i=1}^{n} g_\tau(t_{i-1}, t_i) - \tau^2(t_n)).$$

The discounted limit of the roll-over account turns out to be

$$V(T) = P(T, T^*)^{-1}B(T) = \exp(-W^2_{\tau(T)} - 2\int_0^{\tau(T)} W_s^2 ds + \tau(T))$$
$$= Z(\tau(T)),$$

letting

$$Z(T) := \exp(T - 2\int_0^T W_s^2 ds - W_T^2). \tag{19}$$

We are interested in

$$V(1) = \lim_{T\to 1} Z(\tau(T)) = \lim_{T\to\infty} Z(T).$$

The following lemma shows that $\lim_{T\to\infty} Z(T) = 0$, hence $B(1) = 0$. It turns out that investing in the roll-over strategy leads to the total loss of invested money such that the classical risk-free investment strategy becomes highly risky in this example.

Lemma

Consider Z as in (19). Then Z converges to 0 Q^*-almost surely.