Transition density of SDEs driven by α-stable Lévy process with Hölder continuous coefficients

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Introduction

We study the existence and regularities of the transition densities of the solution to

$$dX_s = \sigma(X_{s-})dZ_s$$

where σ are Hölder continuous functions, and Z is stable Lévy process with stable parameter in $(0, 1) \cup (1, 2)$.

Remark

We work in one dimension

- *Bass et al [BBC], no strong solution in general.*
- *Results on the existence of weak solutions is shown in Zanzotto [Z].*

Current literature

Debussche and Fournier [DF].

Assumption: The coefficient σ is bounded and Hölder continuous.

Result: The density to X exists and belongs to Besov space of certain order depending on the parameters.

Remark

The drawback of the their result is that there is only an existence result.

Current literature

In Konakov and Menozzi [KM].

Assumption:

- $\sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz with bounded derivatives.
- there exists constants $0 < \underline{\sigma} \leq \overline{\sigma} < \infty$ such that, for all $x \in \mathbb{R}$, $0 < \underline{\sigma} < \sigma(x) \leq \overline{\sigma}$.

Result: Parametrix method is used to derive an asymptotic expansion of the transition density.

Our Setup

We apply a variant of the parametrix method developed in a working paper of Bally and Kohatsu in [BK]

Assumption:

- $\sigma : \mathbb{R} \to \mathbb{R}$ is Hölder continuous.
- there exists constants $0 < \underline{\sigma} \leq \overline{\sigma} < \infty$ such that, for all $x \in \mathbb{R}$, $0 < \underline{\sigma} < \sigma(x) \leq \overline{\sigma}$

Result: Using the (backward method) parametrix method.

- The transition density exists and is jointly continuous.
- The transition density is once differentiable w.r.t to initial point.

1. Overview of Parametrix Method

Notations

For $X_t = X_0 + \int_{(0,t]} b(X_{s-})ds + \int_{(0,t]} \sigma(X_{s-})dZ_s$, we denote by

- $(P_t)_{t\geq 0}$, the semigroup and $\mathcal{L}$ the infinitesimal generator.
- $(P_t^*)_{t\geq 0}$ and $\mathcal{L}^*$ their dual operators.
- $p_t(x,y)$, the transition density of X (If exists)

For $X_t^z = x + b(z)t + \sigma(z)Z_t$ and denote by

- $(P_t^z)_{t\geq 0}$ the semigroup $\mathcal{L}^z$ the infinitesimal generator.
- $(P_t^{z,*})_{t\geq 0}$ and $\mathcal{L}^{z,*}$ their dual operators.
- $p_t^z(x,y)$, the transition density of X^z (If exists)

Remark

Note that in general, $\mathcal{L}^{z,*} \neq \mathcal{L}^{*,z}$.

Parametrix Methods - Forward Method

The *Forward Method*, we compute for $g \in C_c^\infty(\mathbb{R})$

$$P_T g(x) - P_T^z g(x) = \int_0^T \partial_t (P_{T-t}^z P_t) g(x) dt$$
$$= \int_0^T P_{T-t}^z (\mathcal{L} - \mathcal{L}^z) P_t g(x) dt$$

and by expanding again using $P_t g(x)$, we can derive a 'formal' expansion of the semigroup.

$$P_T g(x) = P_T^z g(x) + \sum_{n=1}^{N} J_T^n(g)(x) + R^N(x)$$

Parametrix Methods - Backward Method

For $g \in C_c^\infty(\mathbb{R})$. We present the 'Backward Method'.

Step 1. The *Backward Method* is the expansion of the dual operator

$$
\begin{aligned}
P_T^* g(y) - P_T^{y,*} g(y) &= \int_0^T \partial_t (P_{T-t}^{y,*} P_t^*) g(y) dt \\
&= \int_0^T P_{T-t}^{y,*} (\mathcal{L}^* - \mathcal{L}^{y,*}) P_t^* g(y) dt
\end{aligned}
$$

and by expanding again using $P_t^* g(x)$, we can derive a 'formal' expansion of the dual semigroup.

$$
P_t^* g = \sum_{n=0}^{N} I_t^n(g) + R^N(g)
$$

Backward Method - Continued

Step 2. To obtain an expansion of the semigroup itself, one use *duality* and write

$$
\begin{aligned}
\langle P_T h, g\rangle &= \langle h, P_T^* g\rangle \\
&= \sum_n^N \langle h, I_T^n(g)\rangle + \langle h, R_T^N(g)\rangle \\
(Fubini + Duality) &= \sum_n^N \langle I_T^{n,*}(h), g\rangle + \langle R_T^{N,*}(h), g\rangle
\end{aligned}
$$

Remark

In the series converges, then $\langle P_T h, g\rangle = \sum_n^\infty \langle I_T^{n,*}(h), g\rangle$*, gives only the form of* P_T *in the 'weak sense'.*

Backward Method - Continued

Step 4. The form of the density function is retrieved by identification.

$$p_t(x,y) = \sum_n^{\infty} I_t^{n,*}(x,y)$$

Step 5. The continuity and differentiability of the density follow from showing the uniform convergence of the following sums.

$$\sum_n^{\infty} I_t^{n,*}(x,y) \quad \text{and} \quad \sum_n^{\infty} \partial_x I_t^{n,*}(x,y)$$

1. Representation of The Semigroup

Setup and Notation

Notation:

- $\mu(dc) = \frac{1}{c^{1+\alpha}} dc$, $\alpha \in (0,1)$.
- $q(c,x)$ is the gaussian density with mean 0 and variance c
- $\mathcal{N}(dc, dx, ds)$ is an Poisson random measure such that

$$\mathcal{N}(dc, dx, ds) - q(c,x) dx \mu(dc) ds$$

  is a martingale.
- The auxiliary process V,

$$V_t := \int_{\mathbb{R}_+ \times \mathbb{R} \times (0,t]} c\mathcal{N}(dc, dx, ds),$$

  V is a stable subordinator with parameter between $(0,1)$.

Process Z

Our methodology:

- If $\alpha \in (0, \frac{1}{2})$, we set the driving process Z to be

$$Z_t = \int_{(0,t]\times\mathbb{R}_+\times\mathbb{R}} x\mathcal{N}(dc, dx, ds)$$

- If $\alpha \in (\frac{1}{2}, 1)$, we set the driving process Z to be

$$Z_t = \int_{(0,t]\times\mathbb{R}_+\times\mathbb{R}} x\left[\mathcal{N}(dc, dx, ds) - \mathbb{1}_{[-1,1]}(x)q(c, x)dx\mu(dc)ds\right]$$

Process Z

To see the connection between Z and the stable processes. Apply Fubini's theorem and power mix equality. E.g. in the case, $\alpha \in (0, \frac{1}{2})$,

$$\mathbb{E}_{\mathbb{P}}\left(e^{i\theta Z_t}\right) = \exp\left(t\int_{\mathbb{R}\times\mathbb{R}_+}(e^{i\langle\theta,x\rangle}-1)q(c,x)dx\mu(dc)\right)$$
$$= \exp\left(Ct\int_{\mathbb{R}_+}(e^{i\langle\theta,x\rangle}-1)\frac{1}{x^{1+2\alpha}}dx\right).$$

The process Z corresponds to a true 2α-stable Lévy process where $2\alpha \in (0,2)\setminus\{1\}$.

Lemma (Schröenbergs theorem)

Power mixing equality, $\frac{K}{x^{1+2\alpha}} = \int_{\mathbb{R}} q(c,x)\frac{1}{c^{1+\alpha}}$.

Notation

For $g \in C_c^\infty(\mathbb{R})$, we set

$$\widehat{S}_t^* g(x) := \int_{\mathbb{R}} g(y) p_t^y(x,y) \widehat{\theta}_t(x,y) dy$$
$$Q_t^* g(x) := \int g(y) p_t^y(x,y) dy.$$

and $\widehat{\theta}_t(x,y)$ satisfies

$$(\mathcal{L}^{y_1} - \mathcal{L}^y)(p_t^y(\cdot,y))(x)\Big|_{y_1=x} = p_t^y(x,y)\widehat{\theta}_t(x,y)$$

Notation

We can derive the formal expansion

$$P_t(g) = \sum_n I_t^{n,*}(g) + R_t^{n,*}(g)$$

where the terms of the expansion can written into

$$I_t^{n,*}(g) = \int_0^t dt_1 \cdots \int_0^{t_{N-1}} dt_N \, Q_{t_n} \widehat{S}^*_{t_{N-1}-t_N} \ldots \widehat{S}^*_{t_0-t_1} g(x)$$

$$R_t^{n,*}(g) = \int_0^t dt_1 \cdots \int_0^{t_{N-1}} dt_N \, P_{t_n} \widehat{S}^*_{t_{N-1}-t_N} \ldots \widehat{S}^*_{t_0-t_1} g(x)$$

Lemma

The transition probability of the process X^z is given by

$$p_t^z(x,y) = \mathbb{E}\left(q(V_t\sigma^2(z), x-y)\right)$$

for $t > 0$.

Lemma

In our case

$$p_t^y(x,y)\widehat{\theta}_t(x,y) = \int_{\mathbb{R}_+} \mathbb{E}\left(q(\Sigma(x), x-y) - q(\Sigma(y), x-y)\right)\mu(dc)$$

where $\Sigma(x) := \sigma^2(x)c + \sigma^2(y)V_t$.

Lemma

For $g \in C_c^\infty(\mathbb{R})$, there exists $\widehat{\gamma}$. such that $0 \le \widehat{\gamma} < \alpha$

$$|R_{t_0}^{N,*}(g)(x)| \le C \int_0^{t_0} dt_1 \cdots \int_0^{t_{N-1}} dt_N \prod_{i=0}^{N-1} \left[(t_i - t_{i+1})^{-\frac{\widehat{\gamma}}{\alpha}} + 1 \right]$$

Lemma

If for all $j = 1, \ldots, m$, $0 \le b < 1$, $0 \le \gamma_j < 1$, then

$$\int_0^{t_1} dt_0 \ldots \int_0^{t_{N-1}} dt_N \; t_N^b \prod_{i=0}^{N-1} \left(\sum_{j=1}^{m} (t_i - t_{i+1})^{-\gamma_j} \right)$$

$$= \sum_{|k|=N} \binom{N}{k_1 \ldots k_l} \frac{\left[\Gamma(1-\gamma_j) t_0^{-\gamma_j} \right]^{k_j}}{\Gamma(1 + b + \sum_{j=1}^{l} k_j (1-\gamma_j))}$$

Convergence

To see the series converges, we note that

$$R^N(g) \leq \frac{\left[\sum_{j=1}^{m} \Gamma(1-\gamma_j) t_0^{-\gamma_j}\right]^N}{\lfloor (1-\gamma_{j^*})N \rfloor !} \approx \frac{c^N}{N!}$$

where $\gamma_{j^*} = \max(\gamma_j)$.

Stochastic Representation

Lemma

Let N be a standard Poisson process with jumps $(\tau_i)_{i\in\mathbb{N}}$ and τ_T is the last jump before T. where $X^{\pi,y}$ is the Euler scheme with grids $(\tau_i)_{i\in\mathbb{N}}$ and initial point y.

$$P_t^* g(y) = e^T \mathbb{E}_{\mathbb{P}} \left(g(X_T^{\pi,y}) \prod_{j=0}^{N_T-1} \widehat{\theta}_{\tau_{j+1}-\tau_j}(X_{\tau_{j+1}}^{\pi,y}, X_{\tau_j}^{\pi,y}) \right)$$

By duality, for $h \in L^1(\mathbb{R})$ and Z an independent random variable with density h, then

$$P_t h(x) = \mathbb{E} \left(p_{T-\tau_T}^{X_{\tau_T}^{\pi,Z}}(X^{\pi,Z}, x) \prod_{i=0}^{N_T-1} \widehat{\theta}_{\tau_{j+1}-\tau_j}(X_{\tau_{j+1}}^{\pi,Z}, X_{\tau_j}^{\pi,Z}) \right)$$

Conclusion

Comments:

- Similar representation of the density can be obtained
- The proof of convergence uses similar idea, accept one step iteration does not work.

Future work:

- Z is multidimensional.
- Z is multidimensional symmetric stable Lévy.
- Simulation.

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Thank you all for listening!