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# On finite difference approximations in option pricing


István Gyöngy

School of Mathematics and Maxwell Institute
Edinburgh University


Based on joint works with N.V. Krylov, with D. Šiška and with M. Gerencsér

I. A multi-asset Black-Scholes market

Case 1. Smooth coefficients and smooth data

- Finite difference schemes. European options
- Rate of convergence,
- Richardson extrapolation

Case 2. Lipschitz continuous data

II. Finite diff. schemes for American options

## I. **A multi-asset Black-Scholes Market**

- Risky assets: $(S_t^1, ...., S_t^d)$, $t \in [0, T]$

$$dS_t^i = \lambda(t, S_t) S_t^i \, dt + \sum_{k=1}^{d_1} \gamma^{ik}(t, S_t) S_t^i \, dW_t^k, \qquad i = 1, ..., d$$

$$S_0^i > 0,$$

$W$ is a $d_1$-dimensional Wiener process, $\lambda = \lambda(t, x)$, $\gamma^{ij} = \gamma^{ij}(t, x)$ are bounded measurable functions on $[0, T] \times \mathbb{R}^d$.

- Bank account: $(B_t)$, $t \in [0, T]$

$$dB_t = \lambda(t, S_t) B_t \, dt, \qquad B_0 = 1,$$

$\lambda = \lambda(t, x)$ is non-negative measurable and bounded.

**Assumption 1.** For $f = (\lambda, \gamma)$
(i) there is a constant $C$ s.t.

$$|f(t,x)| \leq C \quad x \in \mathbb{R}^d, \quad t \in [0,T].$$

(ii) For each $R > 0$ there is a constant $C_R$ s.t.

$$|f(t,x) - f(t,y)| \leq C_R |x-y| \quad x,y \in \mathbb{R}^d, \quad t \in [0,T].$$

**Proposition 1.** Let Assumption 1 hold. Then there is a unique solution $(B_t, S_t)_{t\in[0,T]}$, and a.s. $B_t > 0$, $S_t^i > 0$ for all $t \in [0,T]$, $i = 1,...,d$.

*Proof:* By Itô's theorem there is a unique solution $(B,S)$, and by Itô's formula

$$B_t = \exp\left(\int_0^t \lambda_r \, dr\right), \quad S_t^i = S_0^i \exp\left(\int_0^t \rho_r^i \, dr + \int_0^t \gamma_r^{ik} \, dW_r^k\right),$$

where $\lambda_r = \lambda(r, S_r)$, $\gamma_r^{ik} = \gamma^{ik}(r, S_r)$, $\rho_r^i = \lambda_r - \frac{1}{2}\sum_k |\gamma_r^{ik}|^2$.
□

**European type option**:
A contract which can be exercised at $T$ by the holder and at time $T$ makes a loss of $h(S_T)$ to the seller, where $h = h(x)$, $x \in \mathbb{R}^d_+$ is a non-negative function, the *pay-off function*, given in the contract.
Examples: $d = 1$, $\lambda(t, x) = constant$, $K$ is a constant,

- European Call: $h(x) = (x - K)^+$,
- European Put: $h(x) = (K - x)^+$, $\quad x \in \mathbb{R}_+$.

**American type option**:
A contract exercised at any stopping time $\tau \leq T$ by the holder of the contract, and when it is exercised at $\tau$ it causes a loss of $h(S_\tau)$ to the seller of the contract.

**Aim:** Calculate numerically the 'fair price' of European and American type options.

## 2. Calculation of prices. Smooth data

Set $H_T = [0, T] \times \mathbb{R}^d$. Let $m \geq 0$ be and integer.
Assumption 2. $\lambda, \gamma \in C^{0,m}(H_T)$, $h \in C^m(\mathbb{R}^d)$, and there are constants $C$, $n \geq 0$ such that for $k = 0, 1, ..., m$

$$|D_x^k(\lambda, \gamma, h)| \leq C(1 + |x|^n) \quad t \in [0, T], x \in \mathbb{R}^d.$$

Theorem 2. Let Assumptions 1 and 2 hold with $m \geq 2$.
Then $C_t = v(t, S_t)$, where for $t \in [0, T]$

$$v(t, x) = E\{e^{-\int_t^T \lambda(r, S_r^{t,x})\, dr} h(S_T^{t,x})\}, \quad x \in \mathbb{R}_+^d.$$

$(S_r^{t,x})_{r \in [t,T]}$: solution with initial condition $S_t = x$.

Monte Carlo, Multi-level Monte Carlo methods.
M. Giles, K. Ritter,..

Set $\mathcal{L} = x^i x^j \alpha^{ij}(t,x) D_{ij} + x^i \lambda(t,x) D_i - \lambda(t,x)$, $\alpha = \gamma\gamma^*/2$

**Theorem 3.** Let Assumptions 1-3 hold with $m \geq 2$. Then $v$ is the unique classical solution of

$$D_t v(t,x) + \mathcal{L}v(t,x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^d =: H_T \qquad (1)$$
$$v(T,x) = h(x), \qquad x \in \mathbb{R}^d. \qquad (2)$$

*Proof:* $v \in C^{1,2}(H_T)$, $|(v, D_x v, D_x^2 v)| \leq N(1 + |x|^p)$. By Itô's formula with $\tau = \tau_r = \inf\{r \geq t : |S_r^{t,x}| \geq r\} \wedge T$

$$E_{t,x} e^{-\int_t^\tau \lambda_s\,ds} v(\tau, S_\tau) = v(t,x) + E_{t,x} \int_t^\tau D_t v(s, S_s) + Lv(s, S_s)\,ds$$
$$= v(t,x)$$

Letting $r \to \infty$ gives $E_{t,x} e^{-\int_t^T \lambda_s\,ds} h(S_T) = u(t,x)$. $\square$

Use finite differences to solve (1)-(2).

Challenges:

- Growing coefficients: $\alpha^{ik}x^i x^j$, $\lambda x^i$
- Growing terminal data: $h$
- Infinite domain: $\mathbb{R}^d$
- Equation (1) may degenerate
- In important cases $h$ is only Lipschitz continuous

## • **Log transformation**

Consider the process $X_t = \log S_t := (\log S_t^1, ..., \log S_t^d)$. By Itô's formula

$$dX_t^i = (\lambda(t, S_t) - \alpha^{ii}(t, S_t))\,dt + \sum_k \gamma^{ik}(t, S_t)\,dW_t^k,$$

where $\alpha^{ii} = (\gamma\gamma^*)^{ii}/2$.
Hence $S_t = e^{X_t} := (e^{X_t^1}, e^{X_t^2}, ..., e^{X_t^d})$ and $u(t,x) = v(t, e^x)$, $e^x = (e^{x^1}, ..., e^{x^d})$, $x \in \mathbb{R}^d$, is the classical solution of

$$D_t u(t,x) + Lu(t,x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^d =: H_T \quad (3)$$

$$u(T,x) = g(x) := h(e^x), \qquad x \in \mathbb{R}^d, \quad (4)$$

where $L := a^{ij}D_{ij} + b^i D_i - c, \qquad \sigma^{ij}(t,x) := \gamma^{ij}(t, e^x),$

$a = \frac{1}{2}\sigma\sigma^*, \qquad b^i(t,x) := \lambda(t, e^x) - a^{ii}(t,x), \qquad c(t,x) := \lambda(t, e^x).$

Notice: $a^{ij}$, $b^i$, $c$ are bounded, $g$ is unbounded.

## • Truncation

Let $R > 0$, $\kappa = \kappa_R(x)$ 'smooth indicator' of $B_R$,
$\kappa(x) \in [0,1]$, $\kappa(x) = 1$ $x \in B_R = \{x \in \mathbb{R}^d : |x| \leq R\}$,
$\kappa(x) = 0$ for $|x| \geq R + 1$.

Set $(\sigma_R, b_R, c_R, g_R) = \kappa(\sigma, b, c, g)$ and consider

$$D_t u(t,x) + L_R u(t,x) = 0, \quad (t,x) \in H_T \tag{5}$$

$$u(T,x) = g_R(x), \qquad x \in \mathbb{R}^d, \tag{6}$$

where

$$L_R = a_R^{ij} D_{ij} + b_R^i D_i - c_R, \qquad a_R = \frac{1}{2}\sigma_R \sigma_R^*.$$

Then (5)-(6) has a unique classical solution $u_R$.

- **Localisation error**

Theorem 4. There is $\nu = \nu(K,T)$ such that for $r = \nu R$

$$\sup_{t \in [0,T]} \sup_{|x| \le r} |u(t,x) - u_R(t,x)| \le N e^{-\frac{1}{3}R^2 + p\nu R}$$

with $N = N(K,p,T)$.

*Proof:*

$$u(t,x) = E\{e^{-\int_t^T c(s,X_s^{t,x})\,dr} g(X_T^{t,x})\} =: EU,$$

$$u_R(t,x) = E\{e^{-\int_t^T c_R(s,Y_s^{t,x})\,dr} g_R(Y_T^{t,x})\} =: EU_R,$$

where $(X^{t,x}, Y_s^{t,x})_{s\in[t,T]}$ solve

$$dX_s = b(s,X_s)\,ds + \sigma(s,X_s)\,dW_s,$$

$$dY_s = b_R(s,Y_s)\,ds + \sigma_R(s,Y_s)\,dW_s.$$

with $X^{t,x} = Y^{t,x} = x$. Notice that $X_t^{t,x} = Y_t^{t,x}$ for $t \le \tau_R$, where

$$\tau_R = \inf\{s \in [t,T] : |X_s^{t,x}| \ge R\}.$$

Hence

$$|u(t,x) - u_R(t,x)| \le 2E(U\mathbf{1}_{\tau_R \le T}) = 2E(U\mathbf{1}_{\sup_{s\in[t,T]}|X_s^{t,x}|\ge R}).$$

$E(U\mathbf{1}_{\sup_{s\in[t,T]}|X_s^{t,x}|\geq R})$ can be estimated by the help of the following lemma.

Lemma. Consider $dZ_t = \beta_t\,dt + \sigma_t\,dW_t$, where $|\beta| \leq K$, $|\sigma| \leq K$. Then there is $\nu = \nu(K,T)$ such that

$$E \sup_{t\in[0,T]} e^{\nu Z_t^2} \leq N E e^{Z_0^2}$$

with $N = N(K)$. If $X_t$ is one-dimensional, then

$$E \sup_{t\leq T} e^{Z_t} \leq N E e^{Z_0}$$

with $N = N(K,T)$.

- **Finite difference approximations**

We consider finite difference schemes for $u := u_R$,

$$D_t u + Lu = 0 \tag{7}$$
$$u(T,x) = g(x), \tag{8}$$

where $g := g_R$, $L := L_R = a^{ij}D_{ij} + b^i D_i - c$, $(a,b,c) := (a_R, b_R, c_R)$.

For simplicity of presentation we consider finite difference schemes in the spatial variable $x \in \mathbb{R}^d$.

Let $\Lambda_1 \subset \mathbb{R}^d \setminus 0$, $\Lambda := \Lambda_1 \cup -\Lambda_1$. For $h \neq 0$ define

$$G_h = \{h(\lambda_1 + \lambda_2 + ... + \lambda_n) : \lambda_i \in \Lambda, n = 1, 2, ...\}.$$

and the difference operators

$$\delta_{h,\lambda}\varphi(x) = (\varphi(x + h\lambda) - \varphi(x))/h$$

for $\lambda \in \Lambda$. Consider

$$D_t u_h(t,x) + L_h u_h = 0, \quad t \in [0,T], x \in G_h \tag{9}$$

$$u_h(T,x) = g(x) \quad x \in G_h, \tag{10}$$

where $L_h$ is a differential operator with coefficients vanishing for $|x| \geq R$.

Notice that (9) is a finite system of ODEs for $(u_h(\cdot,x))_{x\in G_h}$.

**(a) Monotone schemes**

Assume $\Lambda_1 = -\Lambda_1$. Consider

$$L_h = \sum_{\lambda\in\Lambda_1} q_\lambda \delta_{-h,\lambda}\delta_{h,\lambda} + \sum_{\lambda\in\Lambda_1} p_\lambda \delta_{h,\lambda} - c$$

with some functions $q_\lambda$, $p_\lambda$ on $H_T = [0,T]\times\mathbb{R}^d$.

Assumption 1. (consistency) $q_\lambda = p_\lambda = 0$ for $|x| \geq R$,

$$a^{ij} = \sum_{\lambda\in\Lambda_1} q_\lambda \lambda^i\lambda^j, \quad b^i = \sum_{\lambda\in\Lambda_1} p_\lambda\lambda^i, \quad i,j = 1,..,d,$$

Assumption 2. (regularity) $q_\lambda, p_\lambda, c, g \in C^{0,m}([0,T]\times\mathbb{R}^d)$,

$$\sum_{\lambda\in\Lambda_1}(|D^j q_\lambda|^2 + |D^j p_\lambda|^2) + |D^j c|^2 + |D^j g|^2 \leq K, \quad j \leq m$$

Assumption 3. $q_\lambda + hp_\lambda \geq 0$ for all $h \in (0,h_0]$, $\lambda \in \Lambda_1$.

## • Rate of convergence, Richardson extrapolation

Aim: For $k \geq 0$, $h \in (0, h_0]$

$$u_h = u^{(0)} + \sum_{j=1}^{k} \frac{h^j}{j!} u^{(j)} + h^{k+1} r_h, \tag{11}$$

for $x \in \mathbb{G}_h$, $t \in [0, T]$, where $u^{(0)}$ is the solution of (7)-(8), $u^{(1)}, \ldots, u^{(k)}$ and $r^h$ are some functions on $[0, T] \times \mathbb{R}^d$ $u^{(1)}, \ldots, u^{(k)}$, are independent of $h$ and

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{G}_h} |r_h| \leq N |g|_m \tag{12}$$

with a constant $N$ independent of $h$.
Here

$$|g|_m^2 = \sup_{x \in \mathbb{R}^d} \sum_{j=0}^{m} |D^j g|^2.$$

This implies

(i) $k = 0$ gives $\sup_{t\in[0,T]} \sup_{x\in G_h} |u_h - u| \le Nh$

(ii) $k \ge 1$ gives Richardson extrapolation: take mesh-sizes $h, h/2, ..., h/2^k$, calculate $u_h$, $u_{h/2}$,...,$u_{h/2^k}$ and set

$$\bar{u}_h = \sum_{j=0}^{k} \lambda_j u_{h/2^j},$$

where $(\lambda_0, ..., \lambda_k) = (1, 0, ...0)V^{-1}$, $V^{ij} = 2^{-(i-1)(j-1)}$. Then $\sup_{t\in[0,T]} \sup_{x\in G_h} |\bar{u}_h - u| \le Nh^{k+1}$

(iii) If $u^{(j)} = 0$ for odd $j \le k$, we set $\tilde{u}_h = \sum_{j=0}^{\tilde{k}} \tilde{\lambda}_j u_{h/2^j}$, where $\tilde{k} = [(k-1)/2]$, $(\tilde{\lambda}_0, ..., \tilde{\lambda}_k) = (1, ..., 0)\tilde{V}^{-1}$, $\tilde{V}^{ij} = 4^{-(i-1)(j-1)}$. Then

$$\sup_{t\in[0,T]} \sup_{x\in G_h} |\tilde{u}_h - u| \le Nh^{k+1}.$$

**Theorem 5.** (I.G & N.V. Krylov 2011) Let Assumptions (1)-(3) hold. Let $k \geq 0$. Then

(a) expansion (11) holds provided $m \geq 2k+3$,

(b) if $k$ is odd, $m \geq 2k+2$ and $p_{-\lambda} = -p_\lambda$ for $\lambda \in \Lambda_1$, then expansion (11) holds and $u^{(j)} = 0$ for odd $j \leq k$.

*Proof:* We consider finite difference schemes

$$D_t u_h(t,x) + L_h u_h + f = 0, \quad t \in [0,T], x \in \mathbb{R}^d$$
$$u_h(T,x) = g(x) \quad x \in \mathbb{R}^d,$$

The key step in the proof is to prove the following estimate: there is $N = N(K, d, T, m, \Lambda_1)$ such that

$$|u_h|_m \leq N(|f|_m + |g|_m),$$

where $|u_h|_m := \sum_{k=0}^{m} \sup_{t\in[0,T],x\in\mathbb{R}^d} |D^k u_h|$.

- **Other schemes**

$\lambda \in \Lambda_1 \subset \mathbb{R}^d \setminus \{0\}$, $\delta^h_\lambda = (\delta_{h,\lambda} + \delta_{-h,\lambda})/2$,

$$L_h = \sum_{\lambda,\mu\in\Lambda_1} \mathfrak{a}^{\lambda\mu}\delta^h_\lambda + \sum_{\lambda\in\Lambda_1} \mathfrak{b}^\lambda \delta^h_\lambda - c$$

$$D_t u_h(t,x) + L_h u_h = 0, \quad t \in [0,T], x \in G_h \tag{13}$$

$$u_h(T,x) = g(x) \quad x \in G_h, \tag{14}$$

Assumptions:

(i) $\mathfrak{a}^{\lambda\mu} = \mathfrak{b}^\lambda = 0$ for $|x| \geq R$, $\quad \lambda, \mu \in \Lambda_1$

$$a^{ij} = \textstyle\sum_{\lambda\in\Lambda_1} \mathfrak{a}^{\lambda\mu}\lambda^i\lambda^j, \qquad b^i = \sum_{\lambda\in\Lambda_1} \mathfrak{b}^\lambda \lambda^i$$

(ii) $|D^i \mathfrak{a}^{\lambda\mu}| \leq K$, $|D^j \mathfrak{b}^\lambda| \leq K$, $|D^l c| \leq K$, $|D^l g| \leq K$
for $i \leq \max(m,2)$, $\max(m,1)$, $l \leq m$.

(iii) $|g|_m := |g|_{H^m} < \infty$

Theorem 6. (I.G. 2013) Let $k \geq 0$. If Assumptions (i)-(iii) hold with with $m > 2k + 3 + d/2$ then expansion (11) holds. If $k$ is odd and Assumptions (i)-(iii) hold with $m > 2k + 2 + d/2$, then expansion (11) holds and $u^{(j)} = 0$ for odd $j \leq m$.

## • Rate of convergence. Lipschitz continuous data

Monotone schemes, $\Lambda_1 = -\Lambda_1$

$$L_h = \sum_{\lambda \in \Lambda_1} q_\lambda \delta_{-h,\lambda}\delta_{h,\lambda} + \sum_{\lambda \in \Lambda} p_\lambda \delta_{h,\lambda} - c$$

Assumptions

(1) Consistency

(2) for $f = (\sqrt{q_\lambda}, p_\lambda, g, c)$ we have $|f| \le K$,

$$|f(x) - f(y)| \le K|x-y| \quad \text{for } t \in [0,T] \text{ and } x \in \mathbb{R}^d$$

(3) $q_\lambda \ge 0$, $p_\lambda \ge 0$ for all $\lambda \in \Lambda_1$

Theorem 7. Assumptions (1), (2') and (3) hold. then

$$\sup_{t\in[0,T]} \sup_{x\in G_h} |u_h - u| \le N h^{1/2} \quad \text{for all } h > 0,$$

where $N = N(K, d, T, \Lambda)$ and

$$u(t,x) := E\{e^{-\int_t^T c(s, Y_s^{t,x})\,dr} g(Y_T^{t,x})\}$$

where $(Y_s^{t,x})_{s\in[t,T]}$ solves

$$dY_s = b(s, Y_s)\,ds + \sigma(s, Y_s)\,dW_s.$$

with $Y_t^{t,x} = x$.

(H. Dong and N.V. Krylov (2005))

## II. **Finite difference schemes for American options**

For the price $A_t$ of an American type option with pay-off function we have $A_t = w(t, S_t)$, where

$$w(t,x) = \sup_{\tau \in \mathcal{T}_t^T} E\{e^{-\int_t^\tau \lambda(r, S_r^{t,x})\, dr} h(S_\tau^{t,x})\}, \qquad x \in \mathbb{R}_+^d.$$

where $\mathcal{T}_t^T$ is the set of stopping time $\tau \in [t, T]$. After the log transformation we get

$$u(t,x) := w(t, e^x) = E \sup_{\tau \in \mathcal{T}_t^T} E\{e^{-\int_t^\tau c(r, X_r^{t,x})\, dr} g(X_\tau^{t,x})\}$$

where, as before, $X_s^{t,x} = \log S_s^{t,e^x}$, satisfies for $s \in [t, T]$

$$dX_s = b(s, X)\, dt + \sigma(s, X_s)\, dW_s, \qquad X_t^{t,x} = x.$$

**Truncation:** $(\sigma_R, b_R, c_R, g_R) = \kappa_R(\sigma, b, c, g)$.

$$u_R(t,x) := E \sup_{\tau \in \mathcal{T}_t^T} E\{e^{-\int_t^\tau c_R(r, Y_r^{t,x})\,dr} g_R(Y_\tau^{t,x})\},$$

where $Y^{t,x}$ satisfies

$$dY_s = \sigma_R(s, Y_s)\,ds + b_R(s, X_s)\,dW_s, \qquad Y_t^{t,x} = x.$$

**Localisation error:**

There is $\nu = \nu(K,T)$ such that for $r = \nu R$

$$\sup_{t \in [0,T]} \sup_{|x| \le r} |u(t,x) - u_R(t,x)| \le N e^{-\frac{1}{3}R^2 + p\nu R}$$

with $N = N(K,p,T)$.

## Finite difference approximations for $u_R$

Notation: $f := (\sigma, b, c, g) := (\sigma_R, b_R, c_R, g_R)$.

Let $\tau > 0$. Set $T_\tau := \{i\tau \wedge T : i = 0, 1, 2...\}$.
Define $\delta_\tau$ by

$$\delta_\tau \varphi(t,x) = (\varphi(t+\tau, x) - \varphi(t,x))/\tau \quad \text{if } t + \tau < T$$

$$\delta_\tau \varphi(t,x) = (\varphi(T, x) - \varphi(t,x))/\tau \quad \text{if } t + \tau \geq T$$

Monotone fully discretised scheme:

$$\max[\delta_\tau u_{\tau,h} + L_h u_{\tau,h} g - u_{\tau,h}] = 0, \quad t \in T_\tau,\ x \in G_h$$
$$u_{\tau,h}(T,x) = g(x) \quad x \in G_h,$$

where

$$L_h = \sum_{\lambda \in \Lambda_1} q_\lambda \delta_{-h,\lambda} \delta_{h,\lambda} + \sum_{\lambda \in \Lambda_1} p_\lambda \delta_{h,\lambda} - c$$

Assumptions:

(1) Consistency: $a^{ij} = \sum_\lambda q_\lambda \lambda^i \lambda^j$, $b^i = \sum_\lambda p_\lambda \lambda^i$

(2) Regularity: for $f = (\sqrt{q}_\lambda, p_\lambda, c, g)$ we have $|f| \leq K$,

$$|f(t,x) - f(s,x)| \leq |t-s|^{1/2}, \quad |f(t,x) - f(t,y)| \leq K|x-y|$$

(2) $q_\lambda \geq 0$, $\quad p_\lambda \geq 0$.

Theorem 8. Let Assumptions (1)-(3) hold. Then

$$\max_{t \in T_\tau} \max_{x \in G_h} |u_{\tau,h}(t,x) - u(t,x)| \leq N(\tau^{1/4} + h^{1/2}).$$

The proof is an adaptation of N.V. Krylov method from Krylov (2005). See I.G. & D. Šiška (2009), D. Šiška (2012).

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