An Asymptotic Static Hedge of a Timing Risk and its Error

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Outline

By P. Carr and J. Picron , under a Black-Scholes environment, they tried to apply the semi-static hedging formula of barrier options to hedge a payment at a stopping time. Contrary to European claims where the payment occurs only at the prescribed time, there is an extra risk which they call **timing risk**. They found that an integral of the barrier option formula provides a static hedge of the timing risk. The integral (with respect to time) implies that the static hedge portfolio consists of (infinitesimal amount of) options with different (continuum of) maturities, which should be discretized in practice.

Outline

- **Timing risk** is a risk of uncertain dividend, especially of its payment time.
  Example: Defaultable bond, American option, Insurance, etc...
- In the paper by Carr and Picron (1999) they gave a static hedging of a timing risk under Black-Scholes economy.
- We introduce **Asymptotic Static Hedging** of a barrier option and its error.
- Asymptotic static hedging of barrier options can be applied to give that of timing risk.

Black-Scholes Model

Let X to be the solution to the stochastic differential equation

$$dX_t = rX_t \; dt + \sigma X_t \; dB_t.$$

$\boxed{\text{Carr and Picron's static hedging}}$

$$\mathbf{E}[e^{-r\tau}] = \text{the value of the simplest timing risk.}$$

where $\mathbf{E}$ is the expectation under the risk neutral measure $\mathbf{P}$ and

$$\tau = \inf\{t > 0 : X_t \leq K\}, \; (K > 0).$$

Carr and Picron's static hedging

$$
\begin{aligned}
\mathbf{E}[e^{-r\tau}] &= \int_0^\infty e^{-rt}\mathbf{P}(\tau \in dt) \\
&= [e^{-rt}\mathbf{P}(\tau < t)]_0^\infty + r\int_0^\infty e^{-rt}\mathbf{P}(\tau < t)dt \\
&= r\int_0^\infty e^{-rt}\mathbf{P}(\tau < t)dt \\
&= r\int_0^\infty e^{-rt}\mathbf{E}[(1+(\frac{X_t}{K})^{1-\frac{2r}{\sigma^2}})I_{\{X_t \le K\}}]dt.
\end{aligned}
$$

Carr and Picron's static hedging

$$\mathbf{P}(\tau < t|\mathcal{F}_\tau) = \mathbf{E}[(1 + (\frac{X_t}{K})^{1-\frac{2r}{\sigma^2}})I_{\{X_t \leq K\}}|\mathcal{F}_\tau]$$

$\boxed{\text{static hedging of a knock-in option}}$

$$\mathbf{P}(\tau < t) = \text{the value of a knock-in option},$$

Bowie and Carr (1994) showed that it is hedged by two European options with payoff

$$I_{\{X_t \leq K\}},\ (\frac{X_t}{K})^{1-\frac{2r}{\sigma^2}} I_{\{X_t \leq K\}}.$$

Two keys of their result are the reflection principle of Brownian motion and Cameron-Martin-Maruyama-Girsanov theorem.

Static hedging of a knock-in option

$\boxed{\text{static hedging formula}}$

$$\mathbf{P}(\tau < t|\mathcal{F}_\tau) = \mathbf{E}[(1 + (\frac{X_t}{K})^{1-\frac{2r}{\sigma^2}})I_{\{X_t \leq K\}}|\mathcal{F}_\tau]$$

$\boxed{(\frac{X_t}{K})^{1-\frac{2r}{\sigma^2}}}$

- If $\tau \leq t$, then at the hitting time τ the value of the derivative with payoff $I_{\{X_t > K\}}$ coinsids to the one of $(\frac{X_t}{K})^{1-\frac{2r}{\sigma^2}} I_{\{X_t \leq K\}}$. Hence we can exchange the portfolio each other. Then if $X_t > K$, payment becomes 1 and the other case, payment is zero.
- If $\tau \leq t$, then $X_t > K$. Therefore payment is zero.

$\boxed{I_{\{X_t \leq K\}}}$

- If $X_t \leq K$, then payment is 1.
- If $X_t \leq K$, payment is nothing.

Static hedging of a knock-in/knock-out option

Under a general diffusion model, can we get a static hedging formula?

$$\mathbf{P}(\tau < t|\mathcal{F}_\tau) = \mathbf{E}[F_{IN}(X_t)|\mathcal{F}_\tau]$$
$$\mathbf{P}(\tau > t|\mathcal{F}_\tau) = \mathbf{E}[F_{OUT}(X_t)|\mathcal{F}_\tau]$$

$\boxed{\text{Answer}}$:

Static hedging of a knock-in/knock-out option

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$\boxed{\text{Answer}}$: no hope!!

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$\boxed{\text{Answer}}$: no hope!!
But, we have a hope to get an (asymptotic) approximation:

$$\mathbf{P}(\tau < t) = \sum_n \int_0^t \mathbf{E}[F_{IN}^n(X_t)]dt$$
$$\mathbf{P}(\tau > t) = \sum_n \int_0^t \mathbf{E}[F_{OUT}^n(X_t)]dt$$

What is the error of asymptotic static hedging?

The error of asymptotic static hedging

On $\{\tau \le t\}$,

$\boxed{\text{Black-Scholes model}}$

$$\mathbf{E}[(1+(\frac{X_t}{K})^{1-\frac{2r}{\sigma^2}})I_{\{X_t \le K\}}\Big|\mathcal{F}_\tau] - \mathbf{P}(\tau < t|\mathcal{F}_\tau)$$
$$= -\mathbf{E}[(\frac{X_t}{K})^{1-\frac{2r}{\sigma^2}} I_{\{X_t \le K\}}\Big|\mathcal{F}_\tau] + \mathbf{P}(\tau < t,\ X_t > K|\mathcal{F}_\tau)$$
$$= 0,$$

$$\mathbf{E}[I_{\{X_t > K\}} - (\frac{X_t}{K})^{1-\frac{2r}{\sigma^2}} I_{\{X_t \le K\}}\Big|\mathcal{F}_\tau] - \mathbf{P}(\tau > t|\mathcal{F}_\tau) = 0.$$

$\boxed{\text{the error of asymptotic static hedging}}$

$$\sum_{n \le N} \int_0^t \mathbf{E}[F_{IN}^n(X_t)|\mathcal{F}_\tau]dt - \mathbf{P}(\tau < t|\mathcal{F}_\tau) = Err_{IN}^N,$$

$$\sum_{n \le N} \int_0^t \mathbf{E}[F_{OUT}^n(X_t)|\mathcal{F}_\tau]dt - \mathbf{P}(\tau > t|\mathcal{F}_\tau) = Err_{OUT}^N.$$

A Generalized Timing Risk

- Carr and Picron's result gives a static hedge of the most simplest timing risk under Black-Scholes model.

A Generalized Timing Risk

- Carr and Picron's result gives a static hedge of the most simplest timing risk under Black-Scholes model.
- In the following, we give an asymptotic static hedging of knock-out/knock-in option under a general diffusion model, which enables the (asymptotic) static hedge of generalized timing risk by Carr and Picron's technique.
- Our asymptotic static hedging is based on the **"parametrix"**.

Parametrix

D: a domain in $\mathbf{R}^d$,
T_t, S_t : semigroups acting on $C_0(D)$ with the generators$L$, M, respectively.
Assume that $S_t : C_0(D) \to \mathcal{D}(M) \cap \mathcal{D}(L)$.

$\boxed{\text{Parametrix}}$
If

$$\sum_{n=0}^{\infty} S * ((L-M)S)_t^{*n}$$

is absolutely convergent, then it is the solution of (integral) equation

$$T_t = S_t + T_t * (L-M)S_t.$$

Here "$*$" is the convolution of semigroups defined by
$\phi(t) * \psi(t) f(x) = \int_0^t \phi(s)\psi(t-s) f(x) ds$.

Parametrix

Parametrix formula

Under suitable conditions, for any $N \in \mathbf{N}$ and $f \in C_0(D)$, it holds that

$$T_t f = \sum_{n=0}^{N-1} S_t * ((L-M)S_t)^{*n} f + T_t * ((L-M)S_t)^{*N} f.$$

Let us discuss the above parametrix formula later.

Asymptotic Static Hedging

$\mathcal{L}$: the infinitesimal generator of d-dim diffusion process X,
$\mathcal{L}^{OUT}$: the one with absorbed at the halfspace,
$H_K := \{(x_1, \cdots, x_d) \in \mathbf{R}^d : x_1 \leq K\}$.

$\boxed{\text{the price of barrier options}}$

$$\begin{aligned} e^{t\mathcal{L}^{OUT}} f(x) &= \mathbf{E}[f(X_t) I_{\{\tau_B > t\}} | X_0 = x], \\ \text{``} e^{t\mathcal{L}^{IN}} \text{''} f(x) &= \mathbf{E}[f(X_t) I_{\{\tau_B \leq t\}} | X_0 = x] \\ &= e^{t\mathcal{L}} f(x) - e^{t\mathcal{L}^{OUT}} f(x), \end{aligned}$$

for $x \in H_K^c$, where $\tau_B = \inf\{t \geq 0 : X_t \in H_K\}$.

Asymptotic Static Hedging

Under Black-Scholes model with generator $\mathcal{M}$, for any $x \in H_B^c$,

$$e^{t\mathcal{M}^{OUT}} f(x) = e^{t\mathcal{M}} \pi f(x),$$
$$e^{t\mathcal{M}^{IN}} f(x) = e^{t\mathcal{M}} \eta f(x),$$

where

$$\pi f(x) = f(x) - \left(\frac{x_1}{K}\right)^{1-\frac{2r}{\sigma^2}} f\left(\left(\frac{K^2}{x_1}, x_2, \cdots, x_d\right)\right)$$

and

$$\eta f(x) = f(x) - \pi f(x).$$

Asymptotic Static Hedging of Knock-Out Option

$\mathcal{L}$: the infinitesimal generator of X,
$\mathcal{M}$: the infinitesimal generator of Geometric Brownian motion (Black-Scholes model),
π, η are already defined in the above slide for Black-Scholes model.

Theorem

For any N, the error of an asymptotic static hedging of knock-out option with maturity t and function $f \in C_0 := \{f \in C_0(\mathbf{R}^d) : \partial_i f = 0,\ i = 2, 3 \cdots, d\}$ is

$$\left\{ e^{(t-\tau)\mathcal{L}} \pi f - e^{(t-\tau)\mathcal{L}} * \eta \sum_{n=1}^{N-1} ((\mathcal{L} - \mathcal{M}) e^{(t-\tau)\mathcal{M}^{OUT}})^{*n} f \right\} - e^{(t-\tau)\mathcal{L}^{OUT}} f$$

$$= e^{(t-\tau)\mathcal{L}^{IN}} * ((\mathcal{L} - \mathcal{M}) e^{(t-\tau)\mathcal{M}^{OUT}})^{*N} f(K).$$

Sketch of the Asymptotic Static Hedging

- $e^{t\mathcal{M}^{OUT}} f(x) = e^{t\mathcal{M}} \pi f(x)$,
- $e^{t\mathcal{M}^{IN}} f(x) = e^{t\mathcal{M}} \eta f(x)$,

Sketch of the Asymptotic Static Hedging

- $e^{t\mathcal{M}^{OUT}} f(x) = e^{t\mathcal{M}} \pi f(x)$,
- $e^{t\mathcal{M}^{IN}} f(x) = e^{t\mathcal{M}} \eta f(x)$,
- $$\begin{aligned} e^{t\mathcal{L}^{OUT}} f &= e^{t\mathcal{M}^{OUT}} f + e^{t\mathcal{L}^{OUT}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}^{OUT}} f \\ &= e^{t\mathcal{M}} \pi f + e^{t\mathcal{L}^{OUT}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}} \pi f. \end{aligned}$$

Sketch of the Asymptotic Static Hedging

- $e^{t\mathcal{M}^{OUT}} f(x) = e^{t\mathcal{M}} \pi f(x)$,
- $e^{t\mathcal{M}^{IN}} f(x) = e^{t\mathcal{M}} \eta f(x)$,
- $$\begin{aligned} e^{t\mathcal{L}^{OUT}} f &= e^{t\mathcal{M}^{OUT}} f + e^{t\mathcal{L}^{OUT}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}^{OUT}} f \\ &= e^{t\mathcal{M}} \pi f + e^{t\mathcal{L}^{OUT}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}} \pi f. \end{aligned}$$
- $e^{t\mathcal{L}} \pi f = e^{t\mathcal{M}} \pi f + e^{t\mathcal{L}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}} \pi f$.

Sketch of the Asymptotic Static Hedging

- $e^{t\mathcal{M}^{OUT}} f(x) = e^{t\mathcal{M}} \pi f(x)$,
- $e^{t\mathcal{M}^{IN}} f(x) = e^{t\mathcal{M}} \eta f(x)$,
- $$\begin{aligned} e^{t\mathcal{L}^{OUT}} f &= e^{t\mathcal{M}^{OUT}} f + e^{t\mathcal{L}^{OUT}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}^{OUT}} f \\ &= e^{t\mathcal{M}} \pi f + e^{t\mathcal{L}^{OUT}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}} \pi f. \end{aligned}$$
- $e^{t\mathcal{L}} \pi f = e^{t\mathcal{M}} \pi f + e^{t\mathcal{L}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}} \pi f$.
- $$\begin{aligned} & e^{t\mathcal{L}} \pi f - e^{t\mathcal{L}^{OUT}} f \\ & \quad = e^{t\mathcal{L}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}} \pi f - e^{t\mathcal{L}^{OUT}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}} \pi f \\ & \quad = e^{t\mathcal{L}^{IN}} * (\mathcal{L} - \mathcal{M}) e^{t\mathcal{M}^{OUT}} f \end{aligned}$$

The Error Estimate

The Error Estimate via Parametrix

$$T_t f = \sum_{n=0}^{N-1} S_t * ((L-M)S_t)^{*n} f + T_t * ((L-M)S_t)^{*N} f.$$

$$\begin{aligned}
& S_t * ((L-M)S_t)^{*n} f(x) \\
& \equiv \int \cdots \int f(y_n) dy_1 \cdots dy_n \\
& \qquad \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} q_{s_1}(x, y_1)(L_{y_1} - M_{y_1}) q_{s_2 - s_1}(y_1, y_2) \cdots \\
& \qquad\qquad \cdots (L_{y_{n-1}} - M_{y_{n-1}}) q_{t-s_n}(y_n, y)\, ds_1 ds_2 \cdots ds_n,
\end{aligned}$$

The Error Estimate

The solution is given by

$$T_t f = \sum_{n=0}^{\infty} S * ((L - M)S)_t^{*n} f,$$

if the right-hand-side converges uniformly. A partial sum still gives the error estimate;

$$T_t f = \sum_{n=0}^{N} S * ((L - M)S)_t^{*n} f + T_t * ((L - M)S_t)^{*N} f$$

The Error Estimate

What we need is the integrability in $(s_1, \cdots, s_n, y)$ of the integral

$$I_n(s_1, \cdots, s_n, x, y)$$
$$= \int \cdots \int dy_1 \cdots dy_n$$
$$q_{s_1}(x, y_1)(L_{y_1} - M_{y_1}) q_{s_2 - s_1}(y_1, y_2) \cdots$$
$$\cdots (L_{y_n} - M_{y_n}) q_{t - s_n}(y_n, y)\, ds_1 ds_2 \cdots ds_n,$$

which gives the validity of the expression

$$S * ((L - M)S)_t^{*n} f(x) = \int \cdots \int dy_1 \cdots dy_{n-1}$$
$$\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} q_{s_1}(x, y_1)(L_{y_1} - M_{y_1}) q_{s_2 - s_1}(y_1, y_2) \cdots$$
$$\cdots (L_{y_n} - M_{y_n}) q_{t - s_n}(y_n, y)\, ds_1 ds_2 \cdots ds_n f(y) dy.$$

The Error Estimate

If the following is satisfied:

$$\sup_n \sup_{x\in\mathbf{R}^d} |(\frac{\sigma^L(x)-\sigma^M(x)}{x^2})^{(n)}| < \infty, \ \sup_{x\in\mathbf{R}^d} |\frac{\mu^L(x)-\mu^M(x)}{x}| < \infty,$$

Then we have

$$|I_n| \leq \text{Const.}\frac{1}{t^n},$$

and then

$$|\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} I_n(s_0,\cdots,s_n)ds_n \cdots ds_1| \leq \text{Const.}\frac{1}{n!}.$$

Therefore we have the validity of the parametrix:

$$T_t = \sum_{n=0}^{\infty} S * ((L-M)S)_t^{*n}.$$

Concluding Remark

- The hedging error caused by the hedge of a timing risk by a static portfolio is estimated by a “distance” between the model and the Black-Scholes one.
- We can also get a practical static hedging strategy by the discretization of the integral.

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Thank you!!