# Dirichlet Forms associated to Poisson Measures and Lévy Processes: The Lent Particle Method

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Based on joint works with N. Bouleau.

# PART III: Application of the lent particle to prove existence of density of Poisson functionals.

In this part, we shall:

- Give some first examples;
- Apply the lent particle method to the case of sde's driven by Lévy measure.

# Adaptation of the lent particle method to the case of Lévy processes

If $(Y_t)$ is a $\mathbb{R}^d$-valued centered Lévy process without gaussian part, we have the representation

$$Y_t = \int_0^t \int_{\mathbb{R}^d} x \tilde{N}(ds, dx).$$

$\hookrightarrow$ we have to consider time-space Poisson measures.
So, we consider:

• $N$: a Poisson random measure on $[\mathbf{0}, +\infty[\times X$ with intensity $dt \times \nu(du)$ defined on the probability space $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ where $\Omega_1$ is the **configuration space**, $\mathcal{A}_1$ the $\sigma$-field generated by $N$ and $\mathbb{P}_1$ the law of $N$.

• $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$: a local symmetric Dirichlet structure which admits a carré du champ operator and satisfies property (EID).

# The lent particle method in this setting

The Dirichlet form we consider on the bottom space $[0,+\infty[\times X$ is the product form:

$$(L^2(dt),0)\otimes(\mathbf{d},e)$$

which satisfies $(EID)$.
Operators:

$$\varepsilon^+_{(t,u)}(w_1)=w_1\mathbf{1}_{\{(t,u)\in supp\, w_1\}}+(w_1+\varepsilon_{(t,u)}\})\mathbf{1}_{\{(t,u)\notin supp\, w_1\}}$$
$$\varepsilon^-_{(t,u)}(w_1)=w_1\mathbf{1}_{\{(t,u)\notin supp\, w_1\}}+(w_1-\varepsilon_{(t,u)}\})\mathbf{1}_{\{(t,u)\in supp\, w_1\}}.$$

$$\varepsilon^+H(w_1,t,u)=H(\varepsilon^+_{(t,u)}w_1,t,u)\quad \varepsilon^-H(w_1,t,u)=H(\varepsilon^-_{(t,u)}w_1,t,u).$$

# *Upper* Dirichlet structure

Following our construction we obtain a Dirichlet form $(\mathbb{D}, \mathcal{E})$ on $L^2(\Omega_1)$ which carré du champ $\Gamma$, gradient operator that we $\sharp$ and we know that for $F \in \mathbb{D}$:

- $F^\sharp = \int_0^{+\infty} \int_{X \times R} \varepsilon^-((\varepsilon^+ F(t, \cdot))^\flat(u, r))\, dN \odot \rho(dt, du, dr) \in L^2(\mathbb{P}_1 \times \hat{\mathbb{P}})$.
- $\forall F \in \mathbb{D}, \quad \Gamma[F] = \hat{\mathbb{E}}(F^\sharp)^2 = \int_0^{+\infty} \int_X \varepsilon^-(\gamma[\varepsilon^+ F])\, dN.$
- $(\mathbb{D}, \mathcal{E}, \Gamma)$ satisfies (EID).

# Let us recall the example

$(X, \mathcal{X}) = (\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$, $\nu = kdx$, $\xi = (\xi_{ij})_{1 \leq i,j \leq r}$. We assume that there exists an open set $O \subset \mathbb{R}^r$ and a function $\psi$ continuous on $O$ and null on $\mathbb{R}^r \setminus O$ such that

1. $k > 0$ on $O$ $\nu$-a.e. and is locally bounded on $O$
2. $\xi$ is locally bounded and locally elliptic on $O$.
3. $k \geq \psi > 0$ $\nu$-a.e. on $O$.
4. for all $i, j \in \{1, \cdots, r\}$, $\xi_{i,j}\psi$ belongs to $H^1_{loc}(O)$.

$(\mathbf{d}, e)$, the local Dirichlet form which satisfies (EID):

$$\forall f, g \in H,\ e(f,g) = \sum_{i,j=1}^{r} \int_O \xi_{i,j}(x)\partial_i f(x)\partial_j g(x)\psi(x)\, dx$$

$$\forall f \in \mathbf{d},\ \gamma(f)(x) = \sum_{i,j=1}^{r} \xi_{i,j}(x)\partial_i f(x)\partial_j f(x)\frac{\psi(x)}{k(x)}.$$

# Application to Lévy's stochastic area

We consider $X(t) = (X_1(t), X_2(t))$ a Lévy process with vakues in $\mathbb{R}^2$ and Lévy measure $\sigma$.
$\hookrightarrow$ The bottom structure is $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \nu, \mathbf{d}, e)$.
Assume that the gradient is given by the general formula:

$$\gamma[f](x) = \alpha_{11}(x) f_1'^2(x) + 2\alpha_{12}(x) f_1'(x) f_2'(x) + \alpha_{22}(x) f_2'^2(x),$$

where $f_i' = \dfrac{\partial f}{\partial x_i}$.
Let be

$$V(t) = \left( X_1(t), X_2(t), \int_0^t X_1(s_-) dX_2(s) - \int_0^t X_2(s_-) dX_1(s) \right).$$

# Let us apply the lent particle method...

We have for $0 < \alpha < t$ and $x = (x_1, x_2) \in \mathbb{R}^2$:
$\varepsilon^+_{(\alpha,x)} V = V +$

$$(x_1, x_2, X_1(\alpha_-)x_2 + x_1(X_2(t) - X_2(\alpha)) - X_2(\alpha_-)x_1 - x_2(X_1(t) - X_1(\alpha)))$$

$$= V + (x_1, x_2, x_1(X_2(t) - 2X_2(\alpha)) - x_2(X_1(t) - 2X_1(\alpha)))$$

because $\varepsilon^+ V$ is defined $\mathbb{P} \times \nu \times d\alpha$-a.e. and $\nu \times d\alpha$ is diffuse, so

$$\gamma[\varepsilon^+ V] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & A\alpha_{11} - B\alpha_{12} \\ \alpha_{12} & \alpha_{22} & A\alpha_{12} - B\alpha_{22} \\ A\alpha_{11} - B\alpha_{12} & A\alpha_{12} - B\alpha_{22} & A^2\alpha_{11} - 2AB\alpha_{12} + B^2\alpha_{22} \end{pmatrix}$$

denoting $A = (X_2(t) - 2X_2(\alpha))$ and $B = (X_1(t) - 2X_1(\alpha))$.

The operator $\varepsilon^-$ gives a functional defined $\mathbb{P}_N$-a.e. so that for example

$$\varepsilon^-_{(\alpha,x_1,x_2)}(X(t)) = X(t) - \Delta X_\alpha \qquad \mathbb{P}_N(d\alpha dx_1 dx_2)\text{-a.e.}$$

This yields

$$\varepsilon^- A = X_2(t) - \Delta X_2(\alpha) - 2X_2(\alpha_-) \qquad \text{let us denote it } \tilde{A}$$

$$\varepsilon^- B = X_1(t) - \Delta X_1(\alpha) - 2X_1(\alpha_-) \qquad \text{let us denote it } \tilde{B}$$

and eventually $\Gamma[V] = \sum_{\alpha \le t}$

$$\begin{pmatrix} \alpha_{11}(\Delta X_\alpha) & \alpha_{12}(\Delta X_\alpha) & \tilde{A}\alpha_{11}(\Delta X_\alpha) - \tilde{B}\alpha_{12}(\Delta X_\alpha) \\ \sim & \alpha_{22}(\Delta X_\alpha) & \tilde{A}\alpha_{12}(\Delta X_\alpha) - \tilde{B}\alpha_{22}(\Delta X_\alpha) \\ \sim & \sim & \tilde{A}^2\alpha_{11}(\Delta X_\alpha) - 2\tilde{A}\tilde{B}\alpha_{12}(\Delta X_\alpha) + \tilde{B}^2\alpha_{22}(\Delta X_\alpha \end{pmatrix}$$

the symbol $\sim$ denoting the symmetry of the matrix.

# First case: $\sigma = \nu = k(x)dx$

Let us consider the case $\alpha_{12} = 0$, that $\nu$ possesses a density $k$ and that we assume hypotheses of Lemma ▸, and under these assumptions, with same notation, we may choose $\alpha_{11}(x) = \alpha_{22}(x) = (x_1^2 + x_2^2)\frac{\psi(x)}{k(x)}$. We have

$$\Gamma[V] = \sum_{\alpha \le t} |\Delta X_\alpha|^2 \frac{\psi(\Delta X_\alpha)}{k(\Delta X_\alpha)} \begin{pmatrix} 1 & 0 & \tilde{A} \\ 0 & 0 & 0 \\ \tilde{A} & 0 & \tilde{A}^2 \end{pmatrix} + |\Delta X_\alpha|^2 \frac{\psi(\Delta X_\alpha)}{k(\Delta X_\alpha)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \tilde{B} \\ 0 & \tilde{B} & \tilde{B}^2 \end{pmatrix}$$

Hence $X$ has a density if the dimension of the vector space spanned by

$$\left( \begin{pmatrix} 1 \\ 0 \\ X_2(t) - \Delta X_2(\alpha) - 2X_2(\alpha-) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ X_1(t) - \Delta X_1(\alpha) - 2X_1(\alpha-) \end{pmatrix} \right.$$

is equal to 3, where $JT = \{\alpha \in [0, t],\ \Delta X_\alpha \in O\}$.

## Proposition

*Let us suppose that moreover* $\nu(O) = +\infty$, *then* $V$ *has a density.*

# A case where the Lévy measure is carried by a graph

Assume that the Lévy measure of $X_1$ is $k(x_1)dx_1$ and that it satisfies hypotheses of Lemma .
Take $X_2 = [X_1]$.
$\hookrightarrow$ The Lévy measure of $(X_1, [X_1])$ is carried by the curve $x_2 = x_1^2$.
We put $\lambda(x_1, x_2) = 2x_1$, $\alpha_{11} = X_1^2 \frac{\psi(x)}{k(x)}$ , $\alpha_{12} = \lambda\alpha_{11}$ and $\alpha_{22} = \lambda^2\alpha_{11}$. We have

$$\Gamma[V] = \sum_{\alpha \leq t} \alpha_{11}(\Delta X_\alpha) \begin{pmatrix} 1 & \lambda & \tilde{A} - \lambda\tilde{B} \\ \lambda & \lambda^2 & \lambda\tilde{A} - \lambda^2\tilde{B} \\ \tilde{A} - \lambda\tilde{B} & \lambda\tilde{A} - \lambda^2\tilde{B} & (\tilde{A} - \lambda\tilde{B})^2 \end{pmatrix}.$$

$V$ has a density as soon as

$$\dim \mathcal{L}\left( \begin{pmatrix} 1 \\ \lambda \\ \tilde{A} - \lambda \tilde{B} \end{pmatrix}, \quad \alpha \in JT \right) = 3 \tag{1}$$

with $\tilde{A} - \lambda\tilde{B} = -X_2(\alpha_-) + \lambda(\Delta X(\alpha))X_1(\alpha_-) + X_2(t) - X_2(\alpha) - \lambda(\Delta X(\alpha))(X_1(t) - X_1(\alpha))$.

## Proposition

*$V = \left(X_1(t), [X_1]_t, \int_0^t X_1(s_-)d[X_1](s) - \int_0^t [X_1](s_-)dX_1(s)\right)$ has a density as soon as the Lévy measure of $X_1$ satisfies hypotheses of Lemma given at the beginning with $\nu(O) = +\infty$.*

# The SDE we consider

We consider another probability space $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ on which an $\mathbb{R}^n$-valued semimartingale $Z = (Z^1, \cdots, Z^n)$ is defined, $n \in \mathbb{N}^*$.
Assumption on $Z$: There exists a positive constant $C$ such that for any square integrable $\mathbb{R}^n$-valued predictable process $h$:

$$\forall t \geq 0, \ \mathbb{E}[(\int_0^t h_s dZ_s)^2] \leq C^2 \mathbb{E}[\int_0^t |h_s|^2 ds]. \qquad (2)$$

We shall work on the product probability space:
$(\Omega, \mathcal{A}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \times \mathbb{P}_2)$.
Let $d \in \mathbb{N}^*$, we consider the following SDE :

$$X_t = x + \int_0^t \int_X c(s, X_{s^-}, u) \tilde{N}(ds, du) + \int_0^t \sigma(s, X_{s^-}) dZ_s \qquad (3)$$

where $x \in \mathbb{R}^d$, $c : \mathbb{R}^+ \times \mathbb{R}^d \times X \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times n}$ satisfy the set of hypotheses below denoted (R).

# Hypotheses (R)

For simplicity, we fix a finite terminal time $T > 0$.

1. There exists $\eta \in L^2(X, \nu)$ such that:

a) for all $t \in [0, T]$ and $u \in X$, $c(t, \cdot, u)$ is differentiable with continuous derivative and

$$\forall u \in X, \quad \sup_{t \in [0,T], x \in \mathbb{R}^d} |D_x c(t, x, u)| \leq \eta(u),$$

b) $\forall (t, u) \in [0, T] \times U, \; |c(t, 0, u)| \leq \eta(u)$,

c) for all $t \in [0, T]$ and $x \in \mathbb{R}^d$, $c(t, x, \cdot) \in \mathbf{d}$ and

$$\sup_{t \in [0,T], x \in \mathbb{R}^d} \gamma[c(t, x, \cdot)](u) \leq \eta(u),$$

d) for all $t \in [0, T]$, all $x \in \mathbb{R}^d$ and $u \in X$, the matrix $I + D_x c(t, x, u)$ is invertible and

$$\sup_{t \in [0,T], x \in \mathbb{R}^d} \left| (I + D_x c(t, x, u))^{-1} \right| \leq \eta(u).$$

2. For all $t \in [0, T]$ , $\sigma(t, \cdot)$ is differentiable with continuous derivative and

$$\sup_{t \in [0,T], x \in \mathbb{R}^d} |D_x \sigma(t, x)| < +\infty.$$

3. As a consequence of hypotheses 1. and 2. above, it is well known that equation (3) admits a unique solution $X$ such that $\mathbb{E}[\sup_{t \in [0,T]} |X_t|^2] < +\infty$. We suppose that for all $t \in [0, T]$, the matrix $(I + \sum_{j=1}^{n} D_x \sigma_{\cdot,j}(t, X_{t^-}) \Delta Z_t^j)$ is invertible and its inverse is bounded by a deterministic constant uniformly with respect to $t \in [0, T]$.

# Derivation of the equation

$\mathcal{H}_{\mathbb{D}}$ : the set of real valued processes $(H_t)_{t\in[0,T]}$, which belong to $L^2([0,T];\mathbb{D})$ i.e. such that

$$\|H\|^2_{\mathcal{H}_{\mathbb{D}}} = \mathbb{E}[\int_0^T |H_t|^2 dt] + \int_0^T \mathcal{E}(H_t)dt < +\infty.$$

## Proposition

*The equation* $(3)$ *admits a unique solution* $X$ *in* $\mathcal{H}^d_{\mathbb{D}}$*. Moreover, the gradient of* $X$ *satisfies:*

$$\begin{aligned} X^\sharp_t &= \int_0^t \int_U D_x c(s, X_{s-}, u) \cdot X^\sharp_{s-} \tilde{N}(ds, du) \\ &+ \int_0^t \int_{X\times R} c^\flat(s, X_{s-}, u, r) N \odot \rho(ds, du, dr) \\ &+ \int_0^t D_x\sigma(s, X_{s-}) \cdot X^\sharp_{s-} dZ_s \end{aligned}$$

# Derivative of the flow generated by $X$

Let us define the $\mathbb{R}^{d\times d}$-valued processes $U$ and $K$ by

$$dU_s = \sum_{j=1}^{n} D_x\sigma_{.,j}(s, X_{s-})dZ_s^j.$$

$$K_t = I + \int_0^t \int_X D_x c(s, X_{s-}, u)K_{s-}\tilde{N}(ds, du) + \int_0^t dU_s K_{s-}$$

Under our hypotheses, for all $t \geq 0$, the matrix $K_t$ is invertible and it inverse $\bar{K}_t = (K_t)^{-1}$ satisfies:

$$\begin{aligned}\bar{K}_t &= I - \int_0^t \int_X \bar{K}_{s-}(I + D_x c(s, X_{s-}, u))^{-1} D_x c(s, X_{s-}, u)\tilde{N}(ds, du)\\ &\quad - \int_0^t \bar{K}_{s-}dU_s + \sum_{s\leq t} \bar{K}_{s-}(\Delta U_s)^2(I + \Delta U_s)^{-1}\\ &\quad + \int_0^t \bar{K}_s d < U^c, U^c >_s .\end{aligned}$$

# Obtaining the carré du champ matrix

## Theorem

*For all* $t \in [0, T]$,

$$\Gamma[X_t] = K_t \int_0^t \int_X \bar{K}_s \gamma[c(s, X_{s-}, \cdot)] \bar{K}_s^* \, N(ds, du) K_t^* .$$

*Proof*: Let $(\alpha, u) \in [0, T] \times X$. We put $X_t^{(\alpha,u)} = \varepsilon^+_{(\alpha,u)} X_t$.

$$\begin{aligned} X_t^{(\alpha,u)} &= x + \int_0^\alpha \int_X c(s, X_{s^-}^{(\alpha,u)}, u') \tilde{N}(ds, du') \\ &+ \int_0^\alpha \sigma(s, X_{s^-}^{(\alpha,u)}) dZ_s + c(\alpha, X_{\alpha^-}^{(\alpha,u)}, u) \\ &+ \int_{]\alpha,t]} \int_X c(s, X_{s^-}^{(\alpha,u)}, u') \tilde{N}(ds, du') + \int_{]\alpha,t]} \sigma(s, X_{s^-}^{(\alpha,u)}) dZ_s . \end{aligned}$$

Let us remark that $X_t^{(\alpha,u)} = X_t$ if $t < \alpha$ so that, taking the gradient with respect to the variable $u$, we obtain:

$$\begin{aligned} (X_t^{(\alpha,u)})^\flat &= (c(\alpha, X_{\alpha^-}^{(\alpha,u)}, u))^\flat \\ &+ \int_{]\alpha,t]} \int_X D_x c(s, X_{s^-}^{(\alpha,u)}, u') \cdot (X_{s^-}^{(\alpha,u)})^\flat \tilde{N}(ds, du') \\ &+ \int_{]\alpha,t]} D_x \sigma(s, X_{s^-}^{(\alpha,u)}) \cdot (X_{s^-}^{(\alpha,u)})^\flat dZ_s. \end{aligned}$$

Let us now introduce the process $K_t^{(\alpha,u)} = \varepsilon^+_{(\alpha,u)}(K_t)$ which satisfies the following SDE:

$$K_t^{(\alpha,u)} = I + \int_0^t \int_X D_x c(s, X_{s^-}^{(\alpha,u)}, u') K_{s-}^{(\alpha,u)} \tilde{N}(ds, du') + \int_0^t dU_s^{(\alpha,u)} K_{s-}^{(\alpha,u)}$$

and its inverse $\bar{K}_t^{(\alpha,u)} = (K_t^{(\alpha,u)})^{-1}$. Then, using the flow property, we have:

$$\forall t \geq 0, \; (X_t^{(\alpha,u)})^\flat = K_t^{(\alpha,u)} \bar{K}_\alpha^{(\alpha,u)} (c(\alpha, X_{\alpha^-}, u))^\flat.$$

Now, we calculate the carré du champ and then we take back the particle:

$$\forall t \geq 0,\ \varepsilon^{-}_{(\alpha,u)}\gamma[(X_t^{(\alpha,u)})] = K_t\bar{K}_\alpha\gamma[c(\alpha, X_{\alpha^-}, \cdot)]\bar{K}_\alpha^* K_t^*$$

Finally integrating with respect to $N$ we get

$$\forall t \geq 0,\ \Gamma[X_t] \ = \ K_t \int_0^t \int_X \bar{K}_s\gamma[c(s, X_{s^-}, \cdot)](u)\bar{K}_s^* N(ds, du)K_t^*.$$

# First application

## Proposition

*Assume that $X$ is a topological space, that the intensity measure $ds \times \nu$ of $N$ is such that $\nu$ has an infinite mass near some point $u_0$ in $X$. If the matrix $(s, y, u) \to \gamma[c(s, y, \cdot)](u)$ is continuous on a neighborhood of $(0, x, u_0)$ and invertible at $(0, x, u_0)$, then the solution $X_t$ of $(3)$ has a density for all $t \in ]0, T]$.*

## Proof.

Let us fix $t \in ]0, T]$. As $\nu$ has infinite mass near $u_0$, as $X$ is right continuous and $\gamma[c]$ continuous, $N$ admits almost surely a jump at time $s \in ]0, t]$ with size $u \in X$ such that $\gamma[c(s, X_{s-}, \cdot)](u)$ is invertible. As a consequence,

$$\Gamma[X_t] \geq K_t \bar{K}_s \gamma[c(s, X_{s-}, \cdot)](u) \bar{K}_s^* K_t^*.$$

As $\Gamma[X_t]$ dominates an invertible matrix, it is also invertible and this permits to conclude. □

# Equation driven by a Lévy process

- $Y$: Lévy process with values in $\mathbb{R}^d$ , independent of $X_0$.
- $a$: $\mathbb{R}^+ \times \mathbb{R}^k \mapsto \mathbb{R}^{k\times d}$ .

We consider the equation:

$$X_t = X_0 + \int_0^t a(s, X_{s-})\, dY_s.$$

# A condition which ensures that $X_t * \mathbb{P} \ll \lambda^k$

## Proposition

*We assume that:*

1. *The Lévy measure, $\nu$, of $Y$ satisfies hypotheses of the example given at the beginning with $\nu(O) = +\infty$ and $\xi_{i,j}(x) = x_i^2 \delta_{i,j}$. Then we may choose the operator $\gamma$ to be*

$$\gamma[f] = \frac{\psi(x)}{k(x)} \sum_{i=1}^{d} x_i^2 \sum_{i=1}^{d} (\partial_i f)^2 \quad \text{for } f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$$

2. *$a$ is $\mathcal{C}^1 \cap Lip$ with respect to the first variable uniformly in $s$ and $\sup_{t,x} |(I + D_x a)^{-1}(x,t).u| \leq \eta(u)$*
3. *$a$ is continuous with respect to the second variable at 0, and such that the matrix $aa^*(X_0, 0)$ is invertible;*

*then for all $t > 0$ the law of $X_t$ is absolutely continuous w.r.t. the Lebesgue measure.*

# Idea of the proof ($d = 1$)

$$\gamma[f] = \frac{\psi(x)}{k(x)} x^2 f'^2(x).$$

We have the representation: $Y_t = \int_0^t \int_{\mathbb{R}} u\tilde{N}(ds, du)$, so that

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} a(s, X_{s-})u \, \tilde{N}(ds, du).$$

The lent particle method yields:

$$\Gamma[X_t] = K_t^2 \int_0^t \int_X \bar{K}_s^2 a^2(s, X_{s-})\gamma[j](u)N(ds, du)$$

where $j$ is the identity application: $\gamma[j](u) = \frac{\psi(u)}{k(u)} u^2$.

$$\begin{aligned}\Gamma[X_t] &= K_t^2 \int_0^t \int_X \bar{K}_s^2 a^2(s, X_{s-})\frac{\psi(u)}{k(u)} u^2 N(ds, du) \\ &= K_t^2 \sum_{\alpha < t} \bar{K}_s^2 a^2(s, X_{s-})\frac{\psi(\Delta Y_s)}{k(\Delta Y_s)} \Delta Y_s^2.\end{aligned}$$