CREST Seminar: PART IV

Dirichlet Forms associated to Poisson Measures and Lévy Processes: The Lent Particle Method

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Based on joint works with N. Bouleau.

PART IV: Application of the lent particle to prove smoothness of density of Poisson functionals.

In this part, we shall:

- Define higher order Sobolev spaces
- give a criterion which ensures smoothness of density
- apply it to the case of sde's.

Definition and hypotheses on the bottom space

Let E be a Hilbert space. We denote by $\mathbf{d}(E)$ the completion of functions of the form

$$u = \sum_{i=1}^{k} \varphi_i e_i$$

with $e_1, \cdots, e_k$ in E and $\varphi_1, \cdots, \varphi_k$ in $\mathbf{d}$ w.r.t. the norm

$$\| u\|^2_{\mathbf{d}(E)} = \|u\|^2_{L^2(\nu)} + \|u^\flat\|^2_{L^2(\nu;L_0\otimes E)}. \qquad (1)$$

Here $u^\flat(r) = \sum_{i=1}^{k} \varphi_i^\flat e_i$.

Hypothesis (C): There exists a dense subvector space $\mathbf{d}_0 \subset \mathbf{d}$ such that each element u in $\mathbf{d}_0$ is such that:

1. $u \in \bigcap_{p \geqslant 2} L^p(\nu)$.
2. u is *infinitely differentiable* in the sense that $u^\flat \in \mathbf{d}(L_0)$, $u^{(2\flat)} = \left(u^\flat\right)^\flat \in \mathbf{d}(L_0^{\otimes 2}), \ldots, u^{((n+1)\flat)} = \left(u^{(n\flat)}\right)^\flat \in \mathbf{d}(L_0^{\otimes (n+1)})\ldots$
3. For all $n \in \mathbb{N}^*$, $u^{(n\flat)} \in \bigcap_{p \geqslant 2} L^p(\nu; L_0^{\otimes n})$.

$$\mathbf{d}_0(E) = \{u = \sum_{i=1}^{n} \varphi_i e_i | \, \varphi_i \in \mathbf{d}_0, \; i = 1, \cdots, n\}.$$

Definition

Let $n \in \mathbb{N}^*$, $p \geqslant 2$. We denote by $\mathbf{d}^{n,p}(E)$ the completion of $\mathbf{d}_0(E)$ w.r.t. the norm

$$\|u\|_{n,p} \;\; = \;\; \|u\|_{L^p(\nu;E)} + \|u^\flat\|_{L^p(\nu;L_0\otimes E)}\| + \cdots + \|u^{(n\flat)}\|_{L^p(\nu;L_0^{\otimes n}\otimes E)}.$$

And we set:

$$\mathbf{d}^\infty(E) = \bigcap_{n\in\mathbb{N}^*, p \geqslant 2} \mathbf{d}^{n,p}(E).$$

Definition

We denote by $\bar{\mathbf{d}}^{\infty}$ *the subvector space of elements* u *in* $\mathbf{d}^{\infty}$ *such that* u *belongs to* $\mathcal{D}(a)$ *and* $a(u) \in \mathbf{d}^{\infty}$ *and we consider*

$$\bar{\mathbf{d}}_0(E) = \{u = \sum_{i=1}^{n} \varphi_i e_i | \, \varphi_i \in \bar{\mathbf{d}}^{\infty}, \; i = 1, \cdots, n\}.$$

Let $n \in \mathbb{N}^*$, $p \geqslant 2$. *We denote by* $\bar{\mathbf{d}}^{n,p}(E)$ *the completion of* $\bar{\mathbf{d}}_0(E)$ *w.r.t. the norm*

$$\|u\|_{\bar{\mathbf{d}}^{n,p}} \;\; = \;\; \|u\|_{n,p} + \|a(u)\|_{n,p}$$

And we set:

$$\bar{\mathbf{d}}^{\infty}(E) = \bigcap_{n \in \mathbb{N}^*, p \geqslant 2} \bar{\mathbf{d}}^{n,p}(E).$$

Sobolev spaces on the upper space

We follow the same construction as on the bottom space, starting from

$$\mathbb{D}_0 = \left\{\varphi(\tilde{N}(f_1), \cdots, \tilde{N}(f_k)) |\ k \in \mathbb{N}^*, \varphi \in C_c^\infty(\mathbb{R}^k), f_i \in \mathbf{d}_\infty\ i = 1, \cdots, k\right\}$$

$$\mathbb{D}_0(E) = \{\sum_{i=1}^k G_i e_i |\ k \in \mathbb{N}^*, G_i \in \mathbb{D}_0,\ e_i \in E\ \ i = 1, \cdots, k\}.$$

$X^{(n\sharp)}$: the derivate of $X^{((n-1)\sharp)} \in \mathbb{D}\left(L^2(\hat{\mathbb{P}}^{(n-1)}; E)\right)$ so it belongs to $\mathbb{D}\left(L^2(\hat{\mathbb{P}}^n; E)\right) \subset L^2(\mathbb{P} \times \hat{\mathbb{P}}^n; E)$.

Definition

Let $n \in \mathbb{N}^*, p \geqslant 2$ the Sobolev space $\mathbb{D}^{n,p}(E)$ is the closure of $\mathbb{D}_0(E)$ with respect to the norm

$$\|X\|_{n,p} = \|X\|_{L^p(\mathbb{P};E)} + \|X^\sharp\|_{L^p(\mathbb{P}\times\hat{\mathbb{P}};E)}\| + \cdots + \|X^{(n\sharp)}\|_{L^p(\mathbb{P}\times\hat{\mathbb{P}}^n;E)},$$

and $\mathbb{D}^\infty(E) = \bigcap_{n\in\mathbb{N}^*, p\geqslant 2} \mathbb{D}^{n,p}(E)$.

In the same way as in the previous subsection, for all $n \in \mathbb{N}^*, p \geqslant 2$ we consider first $\bar{\mathbb{D}}^\infty$, the subvector space of elements in $\mathbb{D}^\infty \bigcap \mathcal{D}(A)$ such that $A(X) \in \mathbb{D}^\infty$, then define in an obvious way $\bar{\mathbb{D}}_0(E)$ by

$$\bar{\mathbb{D}}_0(E) = \{\sum_{i=1}^k G_i e_i |\, k \in \mathbb{N}^*, G_i \in \bar{\mathbb{D}}^\infty,\ e_i \in E \ \ i = 1, \cdots, k\}.$$

and finally we construct space $\bar{\mathbb{D}}^{n,p}(E)$ which is the closure of $\bar{\mathbb{D}}_0(E)$ with repect to the norm

$$\|X\|_{\bar{\mathbb{D}}^{n,p}(E)} = \|X\|_{n,p} + \|A(X)\|_{n,p},$$

and put

$$\bar{\mathbb{D}}^\infty(E) = \bigcap_{n \in \mathbb{N}^*, p \geqslant 2} \bar{\mathbb{D}}^{n,p}(E).$$

Representation of the n-order derivative

For all $n \in \mathbb{N}^*$, we construct a random Poisson measure $N \odot \rho^n$ on $[0, +\infty[\times X \times R^n$ with compensator $dt \times \nu \times \underbrace{\rho \times \cdots \times \rho}_{n \ times}$ defined on the product probability space: $(\Omega, \mathcal{A}, \mathbb{P}) \times (\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})^{\mathbb{N}^*}$.

Lemma

Let $h \in L^2(\mathbb{R}^+, dt) \otimes \mathbf{d}^\infty$, then $\tilde{N}(h) = \int_0^{+\infty} \int_X h(t,u) \tilde{N}(ds, du)$ belongs to $\mathbb{D}^\infty$ and for all $n \in \mathbb{N}^*$:

$$\tilde{N}(h)^{(n\sharp)} = \int_0^{+\infty} \int_{X \times R^n} h^{(n\flat)}(t, u, r_1, \cdots, r_n) N \odot \rho^n (dt, du, dr_1, \cdots, r_n).$$

Properties of these Sobolev spaces

- $X \in \mathbb{D}^\infty$, $Y \in \mathbb{D}^\infty(E) \Rightarrow XY \in \mathbb{D}^\infty(E)$
- $X \in \mathbb{D}^\infty(E) \Rightarrow Y = \|X\|_E^2 \in \mathbb{D}^\infty$.
- $X \in \mathbb{D}^\infty \Rightarrow \Gamma[X] \in \mathbb{D}^\infty$.
- Let $X \in \mathbb{D}^\infty$ be positive and such that $\frac{1}{X} \in \bigcap_{p \geqslant 2} L^p(\mathbb{P})$, then

$$\frac{1}{X} \in \mathbb{D}^\infty.$$

Expression for the divergence

The operator $X \mapsto X^\sharp$, considered as an unbounded operator with domain $\mathbb{D} \subset L^2(\mathbb{P})$ and values in $L^2(\mathbb{P} \times \hat{\mathbb{P}})$, admits an adjoint operator that we denote by $\delta : \mathcal{D}(\delta) \subset L^2(\mathbb{P} \times \hat{\mathbb{P}}) \to L^2(\mathbb{P})$.

Lemma

Let $X \in \mathbb{D}^\infty$ and $Y \in \bar{\mathbb{D}}^\infty$ then $XY^\sharp$ belongs to $\mathcal{D}(\delta)$ and

$$\delta[XY^\sharp] = -2XAY - \Gamma[X,Y].$$

Proof.

Let $Z \in \mathbb{D}^\infty$ then $XZ \in \mathbb{D}^\infty$ and by definition of A, we have:

$$\mathbb{E}\hat{\mathbb{E}}[(ZX)^\sharp Y^\sharp] = \mathbb{E}[\Gamma[ZX,Y]] = E[ZX(-2AY)].$$

But $(ZX)^\sharp = Z^\sharp X + ZX^\sharp$ so that

$$\begin{aligned}\mathbb{E}\hat{\mathbb{E}}[Z^\sharp XY^\sharp] &= E[ZX(-2AY)] - E\hat{\mathbb{E}}[ZX^\sharp Y^\sharp] \\ &= E[ZX(-2AY)] - E[Z\Gamma[X,Y]].\end{aligned}$$

The main result

Proposition

Let $d \in \mathbb{N}^*$ and X be in $\left(\bar{\mathbb{D}}^\infty\right)^d$. If $(\Gamma[X])^{-1} \in \bigcap_{p \geqslant 2} L^p(\mathbb{P};\mathbb{R}^{d\times d})$, then X admits a density which belongs to $C_b^\infty(\mathbb{R}^d)$.

Idea of the proof: $d = 1$. Let $f \in C_c^\infty(\mathbb{R})$.

$$\Gamma[f(X), X] = f'(X)\Gamma[X, X] \Rightarrow f'(X) = (\Gamma[X])^{-1}\,\Gamma[f(X), X].$$

$$\begin{aligned}\mathbb{E}[f'(X)] &= \mathbb{E}[\Gamma[f(X), X]\,(\Gamma[X])^{-1}] = \mathbb{E}\hat{\mathbb{E}}[f(X)^\sharp X^\sharp\,(\Gamma[X])^{-1}] \\ &= \mathbb{E}\left[f(X)\delta[X^\sharp\,(\Gamma[X])^{-1}]\right]\end{aligned}$$

by iteration:

$$\begin{aligned}\mathbb{E}[f^{(n)}(X)] &= \\ &\mathbb{E}\left\{f(X)\delta\left[X^\sharp\,(\Gamma[X])^{-1}\,\delta\left[X^\sharp\,(\Gamma[X])^{-1}\,\delta\left[\cdots\delta\left[X^\sharp\,(\Gamma[X])^{-1}\right]\cdots\right]\right]\right]\right\}\end{aligned}$$

More precisely, we have for all $n \in \mathbb{N}^*$:

$$\mathbb{E}\left[f^{(n)}(X)\right] \qquad = \mathbb{E}\left[f(X) Z_n\right], \qquad (2)$$

where $(Z_n)_n$ is defined inductively by :

$$\begin{cases} Z_1 &= \delta[X^\sharp (\Gamma[X])^{-1}] \\ Z_n &= \delta[X^\sharp (\Gamma[X])^{-1} Z_{n-1}], \quad n \in \mathbb{N}^*. \end{cases}$$

So that

$$Z_n = -2A[X] (\Gamma[X])^{-1} Z_{n-1} - \Gamma[X, \Gamma[X] Z_{n-1}].$$

Z_n belongs to $\mathbb{D}^\infty$ hence in $L^1(\mathbb{P})$.
So, equality (2) implies that for all $n \in \mathbb{N}^*$ and all $f \in C_c^\infty(\mathbb{R}^d)$:

$$\mathbb{E}[|f^{(n)}(X)|] \leqslant \|f\|_\infty \mathbb{E}[|Z_n|].$$

Application to SDE's

$$X_t = x + \int_0^t \int_X c(s, X_{s^-}, u)\tilde{N}(ds, du) + \int_0^t \sigma(s, X_{s^-})dZ_s \qquad (3)$$

where $x \in \mathbb{R}^d$, $c : \mathbb{R}^+ \times \mathbb{R}^d \times X \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times n}$. We assume that there exists a positive constant C such that for any square integrable $\mathbb{R}^n$-valued predictable process h:

$$\forall t \geqslant 0,\ \mathbb{E}[(\int_0^t h_s dZ_s)^2] \leqslant C^2\mathbb{E}[\int_0^t |h_s|^2 ds]. \qquad (4)$$

For simplicity, we fix all along this article a finite terminal time $T > 0$.

Hypotheses

Assumption $(\bar{\mathrm{R}})$: 1. There exists $\eta \in \bigcap_{p \geqslant 2} L^p(X, \nu)$ such that:
a) for all $t \in [0, T]$ and $u \in X$, $c(t, \cdot, u)$ is infinitely differentiable and

$$\forall \alpha \in \mathbb{N}^*, \quad \sup_{t \in [0,T], x \in \mathbb{R}^d} |D_x^\alpha c(t, x, \cdot)| \in \bigcap_{p \geqslant 2} L^p(X, \nu),$$

b) $\forall (t, u) \in [0, T] \times X, \ |c(t, 0, u)| \leqslant \eta(u)$,
c) for all $t \in [0, T]$, $\alpha \in \mathbb{N}$ and $x \in \mathbb{R}^d$, $D_x^\alpha c(t, x, \cdot) \in \bar{\mathbf{d}}^\infty$ and

$$\forall n \in \mathbb{N}^*, \ \forall p \geqslant 2, \quad \sup_{t \in [0,T], x \in \mathbb{R}^d} \|D_x^\alpha c(t, x, \cdot)\|_{\bar{\mathbf{d}}^{n,p}} < +\infty.$$

2. For all $t \in [0, T]$, $\sigma(t, \cdot)$ is infinitely differentiable and

$$\forall \alpha \in \mathbb{N}^* \quad \sup_{t \in [0,T], x \in \mathbb{R}^d} |D_x^\alpha \sigma(t, x)| < +\infty.$$

3. As a consequence of hypotheses 1. and 2. above, it is well known that equation (3) admits a unique solution X such that $\mathbb{E}[\sup_{t\in[0,T]}|X_t|^2] < +\infty$. We suppose that for all $t \in [0, T]$, the matrix $(I + \sum_{j=1}^n D_x\sigma_{\cdot,j}(t, X_{t^-})\Delta Z_t^j)$ is invertible and its inverse is bounded by a deterministic constant uniformly with respect to $t \in [0, T]$.

Spaces of processes

- $\mathcal{H}_{\bar{\mathbb{D}}^{n,p},\mathcal{P}}$: the set of predictable real valued processes which belong to $L^2([0,T];\bar{\mathbb{D}}^{n,p})$.
- $\mathcal{H}_{\bar{\mathbb{D}}^{n,p}\otimes\bar{\mathbf{d}}^{n,p},\mathcal{P}}$: the set of real valued processes H defined on $[0,T]\times\Omega\times X$ which are predictable and belong to $L^2([0,T];\bar{\mathbb{D}}^{n,p}\otimes\bar{\mathbf{d}}^{n,p})$.

In a natural way, we set

$$\mathcal{H}_{\bar{\mathbb{D}}^{\infty},\mathcal{P}} = \bigcap_{n\in\mathbb{N}^*,p\geqslant 2} \mathcal{H}_{\bar{\mathbb{D}}^{n,p},\mathcal{P}},\ \mathcal{H}_{\bar{\mathbb{D}}^{\infty}\otimes\bar{\mathbf{d}}^{\infty},\mathcal{P}} = \bigcap_{n\in\mathbb{N}^*,p\geqslant 2} \mathcal{H}_{\bar{\mathbb{D}}^{n,p},\mathcal{P}}.$$

Functional calculus related to stochastic integrals

Proposition

Let $H \in \mathcal{H}_{\bar{\mathbb{D}}^\infty \otimes \bar{\mathbf{d}}^\infty, \mathcal{P}}$ *then for all* $t \in [0, T]$

$$X_t = \int_0^t \int_X H(s,u)\tilde{N}(ds, du)$$

belongs to $\mathbb{D}^\infty$ *and we have:*

$$\begin{aligned} X_t^\sharp(w, w_1) &= \int_0^t \int_X H^\sharp(s,u)(w, w_1)\tilde{N}(ds, du)(w) \\ &+ \int_0^t \int_{X \times R} H^\flat(s, u, r_1)(w) N \odot \rho(ds, du, dr_1)(w, w_1), \end{aligned}$$

$$
\begin{aligned}
X_t^{(2\sharp)}(w,w_1,w_2) &= \int_0^t \int_X H^{2\sharp}(s,u)(w,w_1,w_2)\tilde{N}(ds,du)(w) \\
&+ \int_0^t \int_{X\times R} H^{\sharp,\flat}(s,u,r_1)(w,w_1) N\odot\rho(ds,du,dr_1)(w,w_2) \\
&+ \int_0^t \int_{X\times R} H^{\sharp,\flat}(s,u,r_1)(w,w_2) N\odot\rho(ds,du,dr_1)(w,w_1) \\
&+ \int_0^t \int_{X\times R^2} H^{(2\flat)}(s,u,r_1,r_2)(w) N\odot\rho^2(ds,du,dr_1,dr_2)(w,w_1,w_2).
\end{aligned}
$$

etc......

How does the generator operate on stochastic integrals?

Proposition

Let $H \in \mathcal{H}_{\bar{\mathbb{D}}^\infty \otimes \bar{\mathbf{d}}^\infty, \mathcal{P}}$ *then for all* $t \in [0, T]$

$$X_t = \int_0^t \int_X H(s,u) \tilde{N}(ds, du)$$

belongs to $\bar{\mathbb{D}}_\infty$ *and*

$$A[X_t] = \int_0^t \int_X \left(A[H(s,u)] + a[H(s,\cdot)](u) \right) \tilde{N}(ds, du).$$

The case of integrals w.r.t. Z

Proposition

Let $G \in \mathcal{H}_{\bar{\mathbb{D}}^\infty, \mathcal{P}}$ then for all $t \in [0, T]$

$$X_t = \int_0^t \int_X G_s \, dZ_s$$

belongs to $\bar{\mathbb{D}}^\infty$, and for all $n \in \mathbb{N}^*$:

$$X_t^{(n\sharp)} \;=\; \int_0^t G_s^{(n\sharp)} \, dZ_s.$$

Moreover,

$$A[X_t] = \int_0^t \int_X A[G_s] \, dZ_s.$$

The criterion

Applying the Picard iteration, we get

Proposition

Under hypotheses $(\bar{R})$, the equation (3) *admits a unique solution, X, in $(\mathcal{H}_{\mathbb{D}^\infty,\mathcal{P}})^d$ and we have for all $t \in [0,T]$ and all $i \in \{1,\cdots,d\}$:*

$$A[X_{i,t}] = \int_0^t \int_X a[c_i(s,X_{s^-},\cdot)](u)\,\tilde{N}(ds,du) +$$

$$\int_0^t \int_X \left(\frac{\partial c_i}{\partial x_j}(s,X_{s^-},u) A[X_{j,s^-}] + \frac{1}{2}\frac{\partial^2 c_i}{\partial x_j \partial x_k}(s,X_{s^-},u)\Gamma[X_{j,s^-},X_{k,s^-}] \right) \tilde{N}(ds,du$$

$$+ \int_0^t \left(\frac{\partial \sigma_i}{\partial x_j}(s,X_{s^-}) A[X_{j,s^-}] + \frac{1}{2}\frac{\partial^2 \sigma_i}{\partial x_j \partial x_k}(s,X_{s^-})\Gamma[X_{j,s^-},X_{k,s^-}] \right) dZ_s$$

Remember that:

$$\Gamma[X_t] \;=\; K_t \int_0^t \int_X \bar{K}_s \gamma[c(s, X_{s-}, \cdot)] \bar{K}_s^* \, N(ds, du) K_t^*,$$

Theorem

Assume hypotheses $(\bar{R})$. Let $t > 0$, if

$$\left(\int_0^t \int_X \bar{K}_s \gamma[c(s, X_{s-}, \cdot)] \bar{K}_s^* \, N(ds, du) \right)^{-1} \in \bigcap_{p \geqslant 2} L^p(\mathbb{P}, \mathbb{R}^{d \times d})$$

then X_t admits a density which belongs to $C_b^\infty(\mathbb{R}^d)$.

Application: the regular case

Proposition

Assume hypotheses ($\bar{R}$), that ν has an infinite mass near some point u_0 in X. Assume that the matrix $(s, y, u) \to \gamma[c(s, y, \cdot)](u)$ is continuous on a neighborhood of $(0, x, u_0)$ and invertible at $(0, x, u_0)$. Assume moreover that it satisfies the following (local) ellipticity assumption:

$$\forall (s', x, u) \in]0, s] \times \mathbb{R}^d \times \mathcal{O},\ \gamma[c(s', x, u] \geqslant \frac{1}{1 + |x|^\delta} \Theta(u) I_d,$$

Where $\delta, s > 0$ are constant, $\mathcal{O}$ is a neighborhood of u_0 and $\Theta > 0$ s.t.

$$\left(\int_0^t \int_{\mathcal{O}} \Theta(u)\, N(ds, du) \right)^{-1} \in \bigcap_{p \geqslant 2} L^p(\mathbb{P}). \quad (*)$$

Then, for all $t \geqslant s$ the solution X_t of (3) admits a density in $C_b^\infty(\mathbb{R}^d)$.

Lemma

Consider $\mathcal{O}$ *and* Θ *as above and assume that there exists* $\alpha \in (0,1)$ *such that the limit*

$$r_1 = \lim_{\lambda \to +\infty} \frac{1}{\lambda^\alpha} \int_{\mathcal{O}} (e^{-\lambda \Theta(u)} - 1)\, \nu(du)$$

exits and belongs to $(-\infty, 0)$ *then hypothesis* $(*)$ *of the previous proposition is fulfilled.*