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Shortfall risk based good-deal bounds for American derivatives

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1. Introduction

European derivative case

Consider an incomplete financial market.

Probability space: $(\Omega, \mathcal{F}, P; \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]})$
$T > 0$ is the maturity
Let the interest rate be given by 0.

H: a European derivative, which is an $\mathcal{F}_T$-measurable r.v.

$\mathcal{C}$: a convex set, including 0, of attainable claims with zero initial endowment.

Let $l : \mathrm{R} \to [0, \infty)$ be a continuous, non-decreasing, convex function with $l(x) = 0$ if $x \leq 0$, and $l(x) > 0$ if $x > 0$.

Recall that the shortfall risk with the loss function l for sellers is expressed by $E[l(-x - C + H)]$ when the price of a claim H and the hedging strategy are given by $x \in \mathrm{R}$ and $C \in \mathcal{C}$, respectively.

See Föllmer and Leukert (Fin. Stoch., 2000)

Fix $\delta > 0$. Define

$$\mathcal{A}_e^0 := \{Y \in \mathcal{L}^0 | E[l(-Y)] \leq \delta\}.$$

For any $X \in \mathcal{L}^0$, define

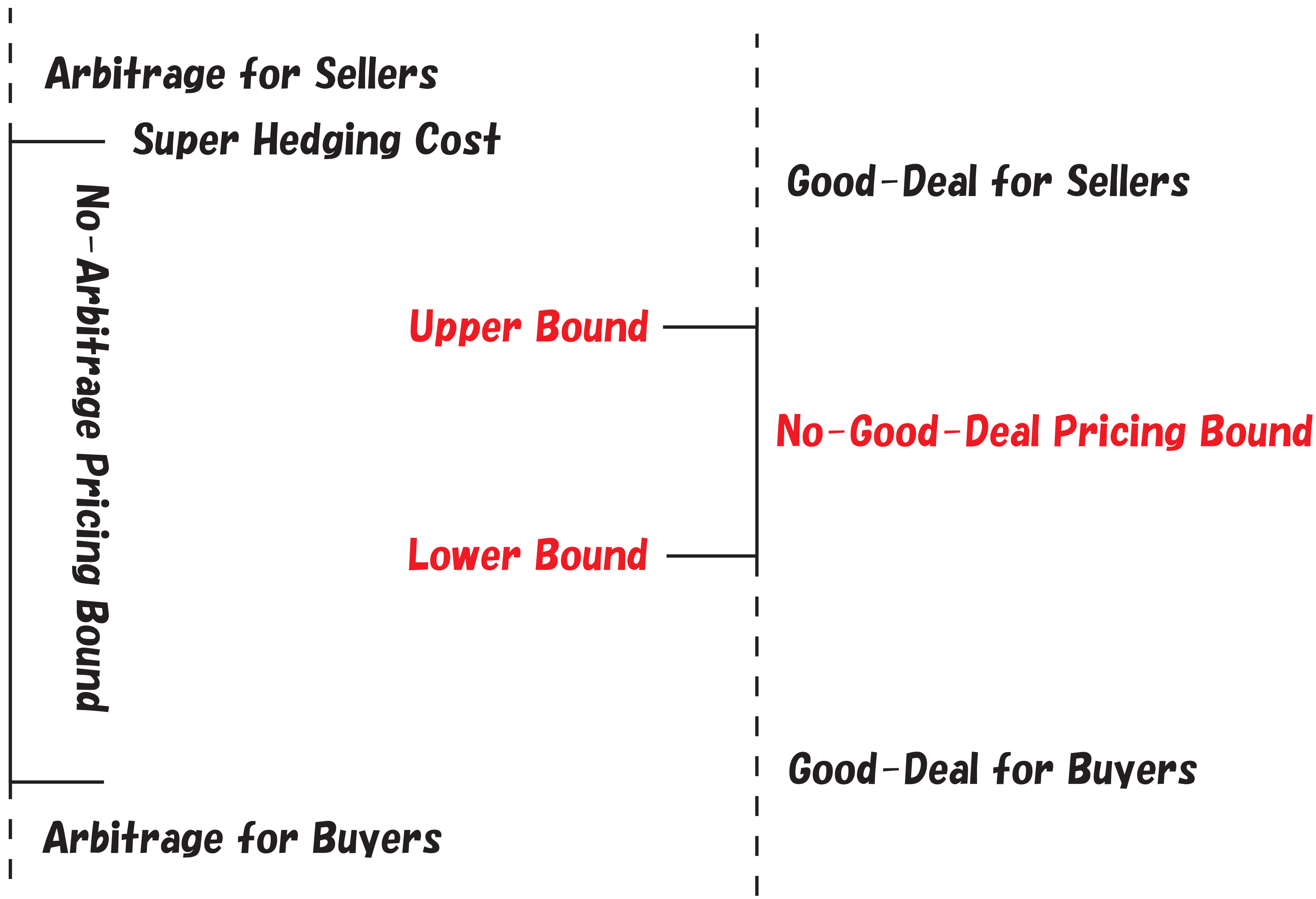
$$\rho_e(X) := \inf\{x \in \mathrm{R} | \exists C \in \mathcal{C} \text{ s.t. } x + C + X \in \mathcal{A}_e^0\}.$$

$$\rho_e(-X) := \inf\{x \in \mathrm{R} | \exists C \in \mathcal{C} \text{ s.t. } x + C - X \in \mathcal{A}_e^0\}.$$

We call ρ_e shortfall risk measure.

A price for a claim is called a good deal price, if investors are able to find a strategy to which the corresponding cashflow at the maturity suppresses its shortfall risk below a certain level.

What is "Good deal"?



The upper bound $\overline{B}$ of no good deal prices is given by

$$\overline{B} = \inf\{x \in \mathrm{R} | \exists C \in \mathcal{C} \text{ s.t. } x + C - H \in \mathcal{A}_e^0\}.$$

The lower bound $\underline{B}$ of no good deal prices is given by

$$\underline{B} = \sup\{x \in \mathrm{R} | \exists C \in \mathcal{C} \text{ s.t. } -x + C + H \in \mathcal{A}_e^0\}.$$

Note that $\overline{B} = \rho_e(-H)$ and $\underline{B} = -\rho_e(H)$.

Proposition 1.1

Assume $\rho_e(0) > -\infty$.
ρ_e is an R-valued convex risk measure on M^l satisfying:

$$\rho_e(X) = \max_{Q \in \mathcal{P}^{l^*}} \left\{ E_Q[-X] - \sup_{X^1 \in \mathcal{A}_e^1} E_Q[-X^1] - \inf_{\lambda > 0} \frac{1}{\lambda} \left\{ \delta + E \left[l^* \left(\lambda \frac{dQ}{dP} \right) \right] \right\} \right\},$$

where

$$\mathcal{P}^{l^*} := \{ Q \ll P | dQ/dP \in L^{l^*} \},$$
$$\mathcal{A}_e^1 := \{ X^1 \in M^l | \exists C \in \mathcal{C} \text{ s.t. } X^1 + C \geq 0 \}.$$

See Föllmer and Schied (Fin. Stoch., 2002), Arai (SIFIN (forthcoming), 2010).

Let l^* be the convex conjugate function of l.

Orlicz space $L^l := \{X \in \mathcal{L}^0 | E[l(c|X|)] < \infty \text{ for some } c > 0\}$,

Orlicz heart $M^l := \{X \in \mathcal{L}^0 | E[l(c|X|)] < \infty \text{ for any } c > 0\}$.

Luxemburg norm $\|X\|_l := \inf\left\{\lambda > 0 \middle| E\left[l\left(\left|\frac{X}{\lambda}\right|\right)\right] \leq 1\right\}$,

Orlicz norm $\|X\|_l^* := \sup\{E[XY] |\ Y \in L^{l^*}, \|Y\|_l \leq 1\}$.

Definition

A functional ρ defined on M^l is called a convex risk measure if it satisfies the following four conditions:

(a) Properness : $\rho(0) \in \mathrm{R}$ and ρ is $(-\infty, \infty]$-valued,

(b) Monotonicity : $\rho(X) \geq \rho(Y)$ for any $X, Y \in M^l$ such that $X \leq Y$,

(c) Translation invariance : $\rho(X + m) = \rho(X) - m$ for $X \in M^l$ and $m \in \mathrm{R}$,

(d) Convexity : $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for any $X, Y \in M^l$ and $\lambda \in [0, 1]$.

Moreover, if a convex risk measure ρ satisfies

(e) Positive homogeneity : $\rho(\lambda X) = \lambda\rho(X)$ for any $\lambda \geq 0$,

then ρ is called a coherent risk measure.

Remark ($\mathcal{C}$ is linear)

Define

$$\mathcal{M}^{l^*} := \{Q \in \mathcal{P}^{l^*} | E_Q[C] = 0 \text{ for any } C \in \mathcal{C}\}.$$

Then, we have

$$\rho_e(X) = \max_{Q \in \mathcal{M}^{l^*}} \left\{ E_Q[-X] - \inf_{\lambda > 0} \frac{1}{\lambda} \left\{ \delta + E\left[l^* \left(\lambda \frac{dQ}{dP} \right) \right] \right\} \right\},$$

that is,

$$\sup_{X^1 \in \mathcal{A}_e^1} E_Q[-X^1] = \begin{cases} 0 & \text{if } Q \in \mathcal{M}^{l^*} \\ +\infty & \text{if } Q \notin \mathcal{M}^{l^*} \end{cases}$$

Corollary 1

In the above Remark, when $\mathcal{C}$ is cone, we have

$$\sup_{X^1 \in \mathcal{A}_e^1} E_Q[-X^1] = \begin{cases} 0, & \text{if } Q \in \mathcal{M}^{l^*}, \\ \infty, & \text{if } Q \notin \mathcal{M}^{l^*}, \end{cases}$$

where $\mathcal{M}^{l^*} := \{Q \in \mathcal{P}^{l^*} | E_Q[C] \leq 0 \text{ for any } C \in \mathcal{C}\}$.

Reminder:

$$\mathcal{P}^{l^*} := \{Q \ll P | dQ/dP \in L^{l^*}\},$$
$$\mathcal{A}_e^1 := \{X^1 \in M^l | \exists C \in \mathcal{C} \text{ s.t. } X^1 + C \geq 0\}.$$

Corollary 2

Under the same assumptions as the above, for any $Q \in \mathcal{M}^{l^*}$, if we find a $\widehat{\lambda}_Q > 0$ satisfying $\delta = E\left[\Phi\left(I\left(\widehat{\lambda}_Q \frac{dQ}{dP}\right)\right)\right]$, then we have

$$\rho_e(X) = \max_{Q \in \mathcal{M}^{l^*}} \left\{ E_Q[-X] - E_Q\left[I\left(\widehat{\lambda}_Q \frac{dQ}{dP}\right)\right]\right\},$$

Recall that I is the right-continuous inverse of the right-derivative l'. Note that we can find such a $\widehat{\lambda}_Q$ at least when I is continuous.

American derivative case

Two big differences:

1. Shortfall risk measures are defined on a space of stochastic processes

2. Upper and lower bounds are discussed separately

$\boldsymbol{H = \{H_t\}_{t\in[0,T]}}$: **a payoff process of an American derivative, which is an adapted càdlàg process**

$\boldsymbol{\mathcal{C}}$: **a convex set, including 0, of attainable claims with zero initial endowment, which is a set of adapted càdlàg processes**

$\boldsymbol{\mathcal{T}}$: **the set of all stopping times on $[0, T]$**

$\boldsymbol{\mathcal{X}}$: **the set of all adapted càdlàg processes**

The least price which the seller can accept is

$$\inf\{x \in \mathrm{R} | \exists C \in \mathcal{C} \text{ s.t. } E[l(-x - C_\tau + H_\tau)] \leq \delta \text{ for } \forall \tau \in \mathcal{T}\}$$

The greatest price which the buyer can accept is

$$\sup\{x \in \mathrm{R} | \exists C \in \mathcal{C}, \exists \tau \in \mathcal{T} \text{ s.t. } E[l(x - C_\tau - H_\tau)] \leq \delta\}$$

We prepare two classes of adapted càdlàg processes:

$$\mathcal{S}^l := \{X \in \mathcal{X} | \{l(|X_\tau|)\}_{\tau \in \mathcal{T}} \text{ is uniformly integrable}\},$$

and

$$\mathcal{R}^l := \{X \in \mathcal{X} | X^* \in M^l\},$$

where $X^* := \sup_{t \in [0,T]} |X_t|$.

We define the following order: for any $X^1, X^2 \in \mathcal{X}$,

$$\begin{aligned} X^1 \preceq X^2 &\iff X^1_t \leq X^2_t \text{ a.s. for } \forall t \in [0,T] \\ &\iff P(X^1_t \leq X^2_t \text{ for } \forall t \in [0,T]) = 1 \\ (&\iff X^1_\tau \leq X^2_\tau \text{ a.s. for } \forall \tau \in \mathcal{T}) \\ &\Longrightarrow (X^1)^* \leq (X^2)^* \text{ a.s.} \end{aligned}$$

Define

$$\mathcal{A}^0_{rv} := \{Y \in \mathcal{L}^0 | E[l(-Y)] \leq \delta\}.$$

and, for any $X \in \mathcal{X}$,

$$\rho^s(X) := \inf\{x \in \mathrm{R} | \exists C \in \mathcal{C} \text{ s.t. } x + C_\tau + X_\tau \in \mathcal{A}^0_{rv} \text{ for } \forall \tau \in \mathcal{T}\}$$

Then, the upper bound of the good deal bound for H is

$$\text{Upper bound } = \rho^s(-H)$$

Define, for any $X \in \mathcal{X}$,

$$\rho^b(X) := \inf\{x \in \mathrm{R} \mid \exists C \in \mathcal{C}, \exists \tau \in \mathcal{T} \text{ s.t. } x + C_\tau + X_\tau \in \mathcal{A}^0_{rv}\}$$

Then, the lower bound of the good deal bound for H is

$$\text{Lower bound } = -\rho^b(H)$$

2. Representations as functionals on $\mathcal{S}^l$

Assumption 2.1 $\mathcal{C}$ is a convex set on $\mathcal{S}^l$

2.1 On ρ^s

We regard ρ^s as a functional defined on $\mathcal{S}^l$.

$$\rho^s(X) := \inf\{x \in \mathrm{R} | \exists C \in \mathcal{C} \text{ s.t. } x + C_\tau + X_\tau \in \mathcal{A}^0_{rv} \text{ for } \forall \tau \in \mathcal{T}\}$$

Define

$$\mathcal{A}^0_{pr} := \{X \in \mathcal{X} | X_\tau \in \mathcal{A}^0_{rv} \text{ for } \forall \tau \in \mathcal{T}\}$$

Reminder:

$$\mathcal{S}^l := \{X \in \mathcal{X} | \{l(|X_\tau|)\}_{\tau \in \mathcal{T}} \text{ is uniformly integrable}\},$$

$$\mathcal{A}^0_{rv} := \{Y \in \mathcal{L}^0 | E[l(-Y)] \leq \delta\}.$$

Proposition 2.1

Assume $\rho^s(0) > -\infty$.
Then, ρ^s is a convex risk measure on $\mathcal{S}^l$.
That is, ρ^s satisfies the following four conditions:

(a) ρ^s is R-valued,

(b) $\rho^s(X) \geq \rho^s(Y)$ for any $X, Y \in \mathcal{S}^l$ whenever $X \preceq Y$,

(c) $\rho^s(X + m) = \rho^s(X) - m$ for $X \in \mathcal{S}^l$ and $m \in \mathrm{R}$,

(d) $\rho^s(\lambda X + (1-\lambda)Y) \leq \lambda\rho^s(X) + (1-\lambda)\rho^s(Y)$
for any $X, Y \in \mathcal{S}^l$ and $\lambda \in [0, 1]$.

ρ^s is said **shortfall risk measure for the seller**.

The dual space of $\mathcal{S}^l$ is not described concretely.

We prepare a functional on $\mathcal{L}^0$ as follows:

$$\rho^0(Y) := \{y \in \mathrm{R} | y + Y \in \mathcal{A}^0_{rv}\}$$

Proposition 2.2

For any $X \in \mathcal{S}^l$, we have

$$\rho^s(X) = \inf_{C \in \mathcal{C}} \sup_{\tau \in \mathcal{T}} \rho^0(C_\tau + X_\tau)$$

2.2 On ρ^b

ρ^b does not have the convexity!

$$\rho^b(X) := \inf\{x \in \mathrm{R} | \exists C \in \mathcal{C}, \exists \tau \in \mathcal{T} \text{ s.t. } x + C_\tau + X_\tau \in \mathcal{A}^0_{rv}\}$$

Example 2.1
For simplicity, set $\mathcal{C} = \{0\}$, $\delta = 1$, $l(x) = (x \vee 0)^2$.
Then,

$$\rho^b(X) = \inf\{x \in \mathrm{R} | \exists \tau \in \mathcal{T} \text{ s.t. } E[l(-x - X_\tau)] \leq 1\}$$

Consider two processes as follows:

$$X_t^1 := \begin{cases} 0, & t < T/2, \\ 1, & t \geq T/2, \end{cases}$$

and

$$X_t^2 := \begin{cases} 1, & t < T/2, \\ 0, & t \geq T/2. \end{cases}$$

Now, $\rho^b(X^1) = \rho^b(X^2) = -2$.
On the other hand, we have

$$\rho^b\left(\frac{X^1 + X^2}{2}\right) = -\frac{3}{2}.$$

Hence, ρ^b **is not convex.** □

Proposition 2.3
For any $X \in \mathcal{S}^l$, we have $\rho^b(X) = \inf_{\tau \in \mathcal{T}} \inf_{C \in \mathcal{C}} \rho^0(C_\tau + X_\tau)$.

Reminder: $\rho^0(Y) := \{y \in \mathrm{R} | y + Y \in \mathcal{A}^0_{rv}\}$.

Fixing $\tau \in \mathcal{T}$, we consider a representation of $\inf_{C \in \mathcal{C}} \rho^0(C_\tau + X_\tau)$.

$$\widehat{\mathcal{S}}^l := \{X \in \mathcal{S}^l | X_\tau \in M^l \text{ for } \forall \tau \in \mathcal{T}\}.$$

Remind $\mathcal{S}^l := \{X \in \mathcal{X} | \{l(|X_\tau|)\}_{\tau \in \mathcal{T}} \text{ is uniformly integrable}\}$.

Assumption 2.2
(1) $\inf_{C\in\mathcal{C}} \rho^0(C_\tau) > -\infty$,
(2) $\mathcal{C}$ is cone.

Denoting

$$\mathcal{C}_\tau := \{C_\tau | C \in \mathcal{C}\},$$

we have

$$\inf_{C\in\mathcal{C}} \rho^0(C_\tau + X_\tau) = \inf\{x \in \mathrm{R} | \exists C_\tau \in \mathcal{C}_\tau \text{ s.t. } x + C_\tau + X_\tau \in \mathcal{A}^0_{rv}\}$$

The RHS is regarded as the shortfall risk measure of X_τ in the European derivative case.

Since $\mathcal{C}_\tau$ is cone, we have

$$\inf_{C\in\mathcal{C}} \rho^0(C_\tau + X_\tau) = \max_{Q\in\mathcal{M}^*_\tau} \left\{ E_Q[-X_\tau] - \inf_{\lambda>0} \frac{1}{\lambda} \left\{ \delta + E\left[l^*\left(\lambda \frac{dQ}{dP} \right) \right] \right\} \right\},$$

where $\mathcal{M}^*_\tau := \left\{ Q \ll P \middle| \frac{dQ}{dP} \in L^{l^*}, E_Q[C_\tau] \leq 0 \text{ for } \forall C_\tau \in \mathcal{C}_\tau \right\}$.

Hence, for any $X \in \widehat{\mathcal{S}}^l$, we have

$$\rho^b(X) = \inf_{\tau\in\mathcal{T}} \max_{Q\in\mathcal{M}^*_\tau} \left\{ E_Q[-X_\tau] - \inf_{\lambda>0} \frac{1}{\lambda} \left\{ \delta + E\left[l^*\left(\lambda \frac{dQ}{dP} \right) \right] \right\} \right\}.$$

3. Representations as functionals on $\mathcal{R}^l$

Reminder:

$$\mathcal{R}^l := \{X \in \mathcal{X} | X^* \in M^l\},$$
$$M^l := \{X \in \mathcal{L}^0 | E[l(c|X|)] < \infty \text{ for any } c > 0\}.$$
$$\rho^s(X) := \inf\{x \in \mathrm{R} | \exists C \in \mathcal{C} \text{ s.t. } x + C_\tau + X_\tau \in \mathcal{A}^0_{rv} \text{ for } \forall \tau \in \mathcal{T}\}.$$

Corollary 3.1
Defining ρ^s on $\mathcal{R}^l$ and assuming $\rho^s(0) > -\infty$, ρ^s is a convex risk measure on $\mathcal{R}^l$.

Assumption 3.1 (1) $\rho^s(0) > -\infty$
(2) $\mathcal{C} \subset \mathcal{R}^l$
(3) $\lim_{x\to\infty} \frac{l(x)}{x} = +\infty$

Define

$$q_l := \inf_{x>0} \frac{xl'(x)}{l(x)},$$

and

$$p'_{l^*} := \frac{q_l}{q_l - 1}, \quad p_{l^*} := \sup_{y>0} \frac{yI(y)}{l^*(y)},$$

where I is the right-continuous inverse of l'.

Assumption 3.2 $p_{l^*} < \infty$ (l^* is moderate)

Remark $p'_{l^*} \leq p_{l^*}$

Example 3.1 Examples of l satisfying all conditions:

(a) $l(x) = x^p/p$, $p > 1$

(b) $l(x) = e^x - x - 1$ We can say easily $p'_{l^*} \leq p_{l^*} \leq 2$.

Example 3.2 Examples of l not satisfying all conditions:

(a) $l(x) = x$

(b) $l(x) = e^x - 1$ We can see that $p'_{l^*} = +\infty$.

(c) $l(x) = (x+1)\log(x+1) - x$, (d) $l(x) = x - \log(x+1)$

Define, for any $X \in \mathcal{R}^l$,

$$\|X\|_{\mathcal{R}^l} := \|X^*\|_l.$$

Reminder:
$\|X\|_l := \inf\left\{\lambda > 0 \middle| E\left[l\left(\left|\frac{X}{\lambda}\right|\right)\right] \leq 1\right\}$.

Proposition 3.1
Banach space $(\mathcal{R}^l, \|\cdot\|_{\mathcal{R}^l})$ is a locally convex Fréchet lattice under the order $\preceq$.

Let $(\mathcal{R}^l)'$ be the dual space of $\mathcal{R}^l$, that is, the space of all continuous linear functionals on $\mathcal{R}^l$.

$$\begin{aligned}\mathcal{D}^* := \{(D^-, D^+) | & D^\pm \text{ are right-continuous,} \\ & \text{integrable variation processes,} \\ & V := \int_{(0,T]} |dD_t^-| + \int_0^T |dD_t^+| \in L^{l^*}, \\ & D^- \text{ is a predictable process with } D_0^- = 0, \\ & D^+ \text{ is a purely discontinuous optional process}\}\end{aligned}$$

Theorem 3.1

(a) For any $(D^-, D^+) \in \mathcal{D}^*$, defining a functional J as

$$J(X) := E\left[\int_{(0,T]} X_{t-} dD_t^- + \int_{[0,T)} X_t dD_t^+\right] \text{ for } \forall X \in \mathcal{R}^l, \qquad (1)$$

we have $J \in (\mathcal{R}^l)'$.

Moreover, $\|J\| \leq 2\|V\|_{l^*}$, where

$$\|J\| := \sup_{X \in \mathcal{R}^l, \|X\|_{\mathcal{R}^l} \leq 1} |J(X)|,$$

$$V := \int_{(0,T]} |dD_t^-| + \int_0^T |dD_t^+|$$

(b) For any $J \in (\mathcal{R}^l)'$, there exists $(D^-, D^+) \in \mathcal{D}^*$ uniquely, which satisfies (1).
Furthermore, we have $\|V\|_{l^*} \leq p'_{l^*}\|J\|$.

Sketch of proof of (b):
Fix $J \in (\mathcal{R}^l)'_+$. There exist a predictable process D^- and an optional process D^+ such that

$$J(X) := E\left[\int_{(0,T]} X_{t-} dD^-_t + \int_{[0,T)} X_t dD^+_t\right]$$

for any bounded càdlàg process X.

We try to see that $(D^-, D^+) \in \mathcal{D}^*$.

It suffices that, denoting $V := D^- + D^+$, we prove

$$\|V\|_{l^*} \leq p'_{l^*}\|J\|.$$

Note that we have

$$\|V\|_{l^*} \leq \|V\|_l^* \leq \sup_{\xi \in L_1^l} E[\xi V],$$

where $L_1^l := \{\xi \in L_+^l | \|\xi\|_l \leq 1\}$.

Reminder:
$\|X\|_l := \inf\left\{\lambda > 0 | E\left[l\left(\left|\frac{X}{\lambda}\right|\right)\right] \leq 1\right\}$,
$\|X\|_l^* := \sup\{E[XY] |\ Y \in L^{l^*}, \|Y\|_l \leq 1\}$.

For $\xi \in L_1^l$, we have $E[\xi V] = \lim_{n\to\infty} E[\xi_n V] \leq \sup_{\xi \in L_1^\infty} E[\xi V]$, where $L_1^\infty := L_1^l \cap L^\infty$, and $\xi_n := \xi \wedge n$.

For $\xi \in L_1^\infty$, we define $X_t := E[\xi|\mathcal{F}_t]$.

Since D^- is a natural process, $E[X_T D_T^-] = E\left[\int_{(0,T]} X_{t-} D_t^-\right]$.

Moreover, we have $X_T D_T^+ = \int_0^T D_{t-}^+ dX_t + \int_0^T X_t dD_t^+$,

and $\int_0^t D_{s-}^+ dX_s$ is a martingale.

Thus, we have $E[X_T D_T^+] = E\left[\int_0^T X_t dD_t^+\right]$.

$$\sup_{\xi\in L_1^\infty} E[\xi V] = \sup_{X\in\mathcal{M}_1^l} E\left[\int_{(0,T]} X_{t-}D_t^- + \int_0^T X_t dD_t^+\right]$$
$$= \sup_{X\in\mathcal{M}_1^l} J(X) \leq \sup_{X\in\mathcal{M}_1^l} \|J\|\|X^*\|_l,$$

where

$$\mathcal{M}_1^l := \{X : X_t := E[\xi|\mathcal{F}_t] \text{ for some } \xi\in L_1^\infty\}$$

By the Doob type inequality for Orlicz spaces,

$$\|X^*\|_l \leq p'_{l^*}\|X_T\|_l \leq p'_{l^*}\|\xi\|_l.$$

Thus,

$$\sup_{X\in\mathcal{M}_1^l} \|J\|\|X^*\|_l \leq p'_{l^*}\|J\|.$$

□

Notation:

$$(\mathcal{R}^l)'_1 := \{J \in (\mathcal{R}^l)'_+ | J(1) = 1\}$$

$$a(J) := \sup_{X \in \mathcal{R}^l} \{J(-X) - \rho^s(X)\} \text{ for } \forall J \in (\mathcal{R}^l)'_1$$

$$\mathcal{A}_{\rho^s} := \{X \in \mathcal{R}^l | \rho^s(X) \leq 0\}$$

$$\widetilde{\mathcal{A}} := \{X \in \mathcal{R}^l | \exists C \in \mathcal{C} \text{ s.t. } X + C \succeq Y \text{ for some } Y \in \mathcal{A}^0_{pr}\}$$

Extended Namioka-Klee Theorem (Biagini-Frittelli(2009)) implies

$$\rho^s(X) = \max_{J \in (\mathcal{R}^l)'_1} \{J(-X) - a(J)\} \text{ for } \forall X \in \mathcal{R}^l.$$

Lemma 3.1 $a(J) = \sup_{X \in \widetilde{\mathcal{A}}} J(-X)$ for $\forall J \in (\mathcal{R}^l)'_1$.

Denoting

$$\mathcal{A}^1_{pr} := \{X \in \mathcal{X} | \exists C \in \mathcal{C} \text{ s.t. } C + X \succeq 0\},$$

we have

$$\widetilde{\mathcal{A}} := \{X^0 + X^1 | X^0 \in \mathcal{A}^0_{pr}, X^1 \in \mathcal{A}^1_{pr}\}.$$

Thus,

$$a(J) = \sup_{X^0 \in \mathcal{A}^0_{pr}} J(-X^0) + \sup_{X^1 \in \mathcal{A}^1_{pr}} J(-X^1) \tag{2}$$

Reminder: $\mathcal{A}^0_{pr} := \{X \in \mathcal{X} | X_\tau \in \mathcal{A}^0_{rv} \text{ for } \forall \tau \in \mathcal{T}\}$.

Remark

If $\mathcal{C}$ is cone, we have, for any $J \in (\mathcal{R}^l)'_1$,

$$\sup_{X \in \mathcal{A}^1_{pr}} J(-X) = \begin{cases} 0, & \text{if } J(C) \leq 0 \text{ for } \forall C \in \mathcal{C} \\ +\infty, & \text{otherwise.} \end{cases}$$

We focus on the first term of the RHS of (2).

Any $(D^-, D^+) \in \mathcal{D}^*$ corresponding to $J \in (\mathcal{R}^l)'_1$ is increasing, and satisfies $E[D^-_T + D^+_T] = 1$.

Denote $V := D^-_T + D^+_T$ and $K^* := \inf_{\lambda > 0} \frac{1}{\lambda}\{\delta + E[l^*(\lambda V)]\}$.

Theorem 3.2 $K^* \leq \sup_{X \in \mathcal{A}^0_{pr}} J(-X)$.

Proof:
Suppose that $\sup_{X \in \mathcal{A}^0_{pr}} J(-X) < \infty$ and I is continuous.

For any sufficient large $c > 0$, we can find a constant $\lambda_c > 0$ satisfying

$$E[l(I(\lambda_c V))1_{\{V \leq c\}}] = \delta.$$

For any sufficient large $c > 0$, define a martingale X^c as

$$X^c_t := E[-I(\lambda_c V)1_{\{V \leq c\}} | \mathcal{F}_t],$$

that is, X^c is a bounded càdlàg martingale.

For any $\tau \in \mathcal{T}$, we have

$$E[l(-X^c_\tau)] \leq E[l(I(\lambda_c V))1_{\{V \leq c\}}] = \delta,$$

which means $X^c \in \mathcal{A}^0_{pr}$.
Moreover,

$$\begin{aligned}
\sup_{X \in \mathcal{A}^0_{pr}} J(-X) &\geq -J(X^c) = -E\left[\int_{(0,T]} X^c_{t-} dD^-_t + \int_0^T X^c_t dD^+_t\right] \\
&= -E[X^c_T V] = \frac{1}{\lambda_c} E[l(\lambda_c V)\lambda_c V 1_{\{V \leq c\}}] \\
&= \frac{1}{\lambda_c}\left\{E[l(I(\lambda_c V))1_{\{V \leq c\}}] + E[l^*(\lambda_c V)1_{\{V \leq c\}}]\right\} \\
&= \frac{1}{\lambda_c}\left\{\delta + E[l^*(\lambda_c V)1_{\{V \leq c\}}]\right\} \geq \frac{\delta}{\lambda_c}
\end{aligned}$$

Thus, λ_c tends to a positive number, denoted by λ_∞, as $c \to \infty$.

Consequently, we obtain

$$\begin{aligned}\sup_{X \in \mathcal{A}^0_{pr}} J(-X) &\geq \liminf_{c \to \infty} \frac{1}{\lambda_c} \left\{\delta + E[l^*(\lambda_c V) 1_{\{V \leq c\}}]\right\} \\ &\geq \frac{1}{\lambda_\infty}\{\delta + E[l^*(\lambda_\infty V)]\} \\ &\geq K^*\end{aligned}$$

Recall $K^* := \inf_{\lambda > 0} \frac{1}{\lambda}\{\delta + E[l^*(\lambda V)]\}$. □

Conclusion

1. ρ^s is a convex risk measure on $\mathcal{S}^l$ and $\mathcal{R}^l$, where
   $\mathcal{S}^l := \{X \in \mathcal{X} | \{l(|X_\tau|)\}_{\tau\in\mathcal{T}}$ is uniformly integrable$\}$,
   $\mathcal{R}^l := \{X \in \mathcal{X} | X^* \in M^l\}$.
2. ρ^b is not convex.
3. $\rho^s(X) = \inf_{C\in\mathcal{C}} \sup_{\tau\in\mathcal{T}} \rho^0(C_\tau + X_\tau)$,
   where $\rho^0(Y) := \{y \in \mathrm{R} | y + Y \in \mathcal{A}^0_{rv}\}$.
4.
$$\begin{aligned}\rho^b(X) &= \inf_{\tau\in\mathcal{T}} \inf_{C\in\mathcal{C}} \rho^0(C_\tau + X_\tau) \\ &= \inf_{\tau\in\mathcal{T}} \max_{Q\in\mathcal{M}^*_\tau} \left\{ E_Q[-X_\tau] - \inf_{\lambda>0} \frac{1}{\lambda} \left\{ \delta + E\left[l^*\left(\lambda \frac{dQ}{dP} \right) \right] \right\} \right\}.\end{aligned}$$

5. If ρ^s is defined on $\mathcal{R}^l$ and $\mathcal{C}$ is cone, then

$$\rho^s(X) = \max_{J \in (\mathcal{R}^l)'_2} \{J(-X) - \sup_{X \in \mathcal{A}^0_{pr}} J(-X)\},$$

where $(\mathcal{R}^l)'_2 = \{J \in (\mathcal{R}^l)'_+ | J(1) = 1, J(C) \leq 0$ for any $C \in \mathcal{C}\}$.

6.

$$\inf_{\lambda > 0} \frac{1}{\lambda} \left\{\delta + E[l^*(\lambda(D^-_T + D^+_T))]\right\} \leq \sup_{X \in \mathcal{A}^0_{pr}} J(-X).$$

Thank you for your attention