Precautionary Measures for Credit Risk Management in Jump Models

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Motivation

Reflecting the financial crisis of 2008, banks may want to detect an appropriate time to undertake actions of raising capital:

- Especially useful when we observe declining financial markets.
- Not too early– False alarm is costly.
- Not too late – Do not want to violate the capital requirements.
- Use jump models to incorporate unexpected and abrupt decline in the asset value.

$\Rightarrow$ Optimal stopping problem in a Lévy type model.

Model

Let $(\Omega, \mathcal{F}, \mathbb{P}^x)$ be a probability space hosting a Lévy process $X = \{X_t : 0 \leq t < \infty\}$ of the form

$$X_t = x + ct + \sigma B_t + J_t, \quad X_0 = x$$

where

- X represents bank's net worth allocated to its credit business with the initial net worth x,
- B being non-default fluctuations (due to interest rate changes) and
- J being default events.

We want to choose alarm time, τ optimally in the following sense:

Violation Risk and Regret

Let the violation time θ be the first time X goes below zero:

$$\theta := \inf\{t \geq 0 : X_t < 0\}.$$

Associated with each alarm time τ, the risk of violation is quantified by

(1) $$R_x^{(q)}(\tau) := \mathbb{E}^x\left[e^{-q\theta}1_{\{\tau \geq \theta\}}\right],$$

which measures the cost of "too late" (violation risk).

On the other hand, the opportunity cost associated with each alarm time τ is

(2) $$H_x^{(q,h)}(\tau) := \mathbb{E}^x\left[1_{\{\tau < \infty\}}\int_\tau^\theta e^{-qt}h(X_t)\mathrm{d}t\right]$$

where $h : (0, \infty) \mapsto \mathbb{R}$ is a monotone increasing and continuous function. This measures the cost of "too early" (regret).

Problem

Consider the linear combination of (1) and (2) with a given weight $\gamma > 0$:

$$U_x^{(q,h)}(\tau, \gamma) := R_x^{(q)}(\tau) + \gamma H_x^{(q,h)}(\tau), \quad x \geq 0.$$

Since the violation time is observable and the game is over once it happens, we just consider a set of stopping times

$$\mathcal{S} := \{\tau \ \text{ stopping time} : \tau \leq \theta \ \text{ a.s.}\}.$$

Then the objective is too choose the alarm time τ^* such that

$$\tau^* \in \arg\min_{\tau \in \mathcal{S}} U_x^{(q,h)}(\tau, \gamma).$$

Reduction to Optimal Stopping

Taking advantage of the set $\mathcal{S}$, the problem reduces to

$$(3) \qquad U_x^{(q,h)}(\tau,\gamma) = \mathbb{E}^x[e^{-q\tau}G(X_\tau)1_{\{\tau<\infty\}}], \quad x \geq 0$$

where

$$(4) \qquad G(x) := 1_{\{x<0\}} + \gamma\mathbb{E}^x\left[\int_0^\theta e^{-qt}h(X_t)\mathrm{d}t\right]1_{\{x\geq 0\}}, \quad x \in \mathbb{R}.$$

Our objective function is motivated by the Bayes risk in sequential analysis; it is modeled in terms of linear combination of

- the false alarm probability and expected detection delay in change-point detection (e.g. Shiryaev(1976)),

- the misdiagnosis probability and expected sample observation size in sequential hypothesis testing (e.g., Wald and Wolfowitz (1950)).

Spectrally Negative Lévy Case

The Laplace exponent of X

$$\psi(\beta) := \log \mathbb{E}^0[e^{\beta X_1}], \quad \beta \in \mathbb{R}.$$

becomes

$$\text{(5)} \quad \psi(\beta) = c\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(-\infty,0)} (e^{\beta x} - 1 - \beta x 1_{\{x>-1\}})\Pi(\mathrm{d}x).$$

It is zero at the origin, convex on $\mathbb{R}_+$ and has the right-continuous inverse:

$$\text{(6)} \qquad \zeta_q := \sup\{\lambda > 0 : \psi(\lambda) = q\} \quad \text{for} \quad q \geq 0.$$

There exists a monotone increasing differentiable scale function:
$W^{(q)} : [0, \infty) \mapsto \mathbb{R}$ for $q \geq 0$,

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\beta) - q}, \quad \beta > \zeta_q,$$

which gives

$$\mathbb{E}^x\left[e^{-q\tau_a^+} 1_{\left\{\tau_a^+ < \tau_0\right\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}.$$

where τ_a^+ is the first time that X goes above $a > x$. Furthermore, if X has a Gaussian component, $W^{(q)} \in C^2(0, \infty)$ (Chan et al. (2010)).

Rewriting in terms of the Scale function:

For given $q > 0$ and $0 \le A \le x$, try to represent the stopping value $G(\cdot)$ in (4) in terms of the scale function. For this purpose, use Bertoin (1997) to obtain

(7) $$\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt}h(X_t)\mathrm{d}t\right] = W^{(q)}(x-A)\int_A^\infty e^{-\zeta_q(y-A)}h(y)\mathrm{d}y - \int_A^x W^{(q)}(x-y)h(y)\mathrm{d}y,$$

and compute the derivatives (when X has a Gaussian component)

(8)
$$\frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt}h(X_t)\mathrm{d}t\right]\Big|_{x=A+} = \frac{2}{\sigma^2}\int_0^\infty e^{-\zeta_q y}h(y+A)\mathrm{d}y, \quad A \ge 0,$$

(9)
$$\frac{\partial}{\partial A}\mathbb{E}^x\left[\int_0^{\tau_A} e^{-qt}h(X_t)\mathrm{d}t\right] = -W'_{\zeta_q}(x-A)e^{\zeta_q x}\int_A^\infty e^{-\zeta_q y}h(y)\mathrm{d}y > 0, \quad 0 \le A \le x.$$

Note that the second term of the stopping value $G(\cdot)$ in (4) is obtained by setting $A = 0$ in (7).

Exponential Jump Diffusion Case

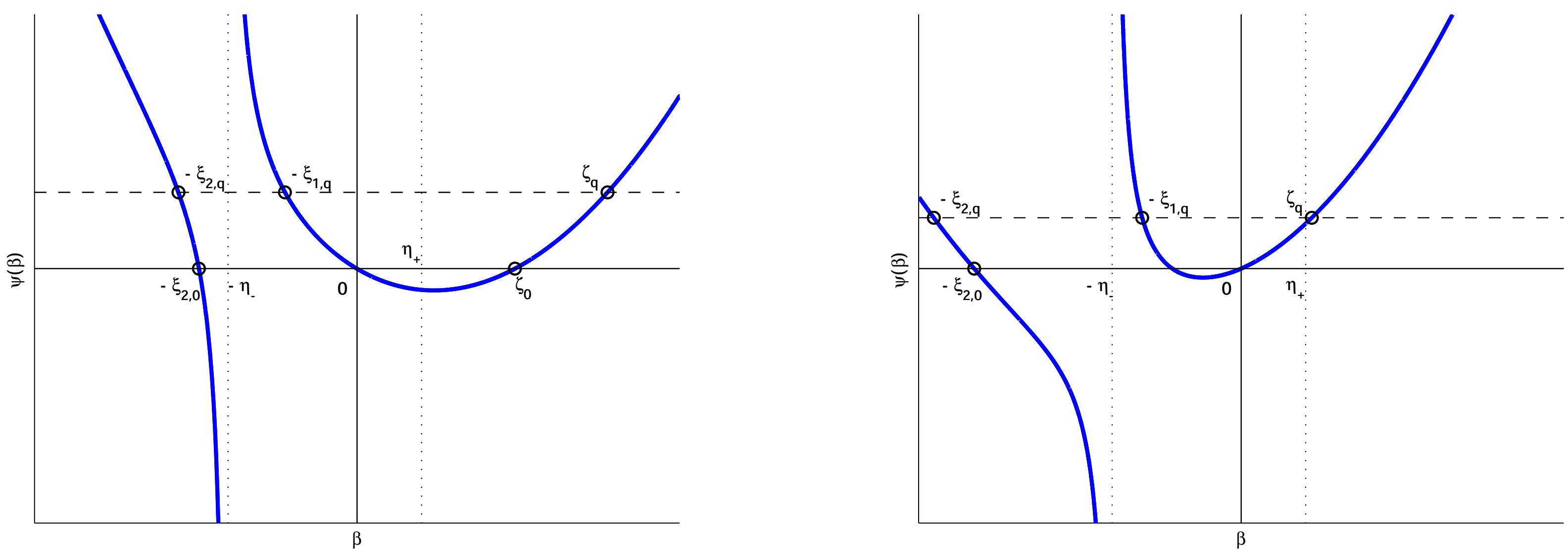
In this case, the Laplace exponent is, by using the jump size parameter $\eta > 0$,

$$\psi(\beta) = c\beta + \frac{1}{2}\sigma^2\beta^2 + \lambda\left(\frac{\eta}{\eta+\beta} - 1\right).$$

The overall drift of X is $\bar{u} := c - \frac{\lambda}{\eta}$ and there are three roots to

$$\psi(\beta) = q$$

(the Cramér-Lundberg equation).



(a) $\bar{u} < 0$ (b) $\bar{u} > 0$

Obtaining the Scale Function

For exponential jump diffusion, we can use some results from Kou and Wang (2003). Indeed, we have

$$\zeta_q = \frac{2q\eta}{\sigma^2 \xi_{1,q}\xi_{2,q}}, \quad q > 0 \quad \text{and} \quad \zeta_0 = \frac{2|\bar{u}|\eta}{\sigma^2 \xi_{2,0}}, \quad q = 0.$$

The scale function, for every $q \geq 0$, is

$$W^{(q)}(x) = (c_1 + c_2)e^{\zeta_q x} - c_1 e^{-\xi_{1,q}x} - c_2 e^{-\xi_{2,q}x}, \quad x \geq 0 \tag{10}$$

where

$$c_1 := \frac{2}{\sigma^2(\xi_{2,q} - \xi_{1,q})}\frac{\eta - \xi_{1,q}}{\xi_{1,q} + \zeta_q} \quad \text{and} \quad c_2 := \frac{2}{\sigma^2(\xi_{2,q} - \xi_{1,q})}\frac{\xi_{2,q} - \eta}{\xi_{2,q} + \zeta_q}$$

Solution procedure

For a specific *threshold* level $0 \leq A$, we associate the corresponding value in the continuation region $U_x^{(q,h)}(\tau_A, \gamma)$ and the differential over the stopping value $G(x)$,

$$\delta_A(x) := U_x^{(q,h)}(\tau_A, \gamma) - G(x) = R_x^{(q)}(\tau_A) - \gamma \mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} h(X_t) \mathrm{d}t \right] \tag{11}$$

1. Choose A^* such that $\delta'_A(A^*+) = 0$ (the smooth-fit) to find the optimal threshold level, and

2. Prove that this strategy is indeed optimal.

This can be done since we know the representation of (11) by (7), (10) and the violation risk (1) at level A:

$$R_x^{(q)}(\tau_A) = \frac{e^{-\eta A}}{\eta} \frac{(\xi_{2,q} - \eta)(\eta - \xi_{1,q})}{\xi_{2,q} - \xi_{1,q}} \left[e^{-\xi_{1,q}(x-A)} - e^{-\xi_{2,q}(x-A)} \right],$$

Recall:

$$R_x^{(q)}(\tau_A) = \mathbb{E}^x \left[e^{-q\theta} 1_{\{\tau_A \geq \theta\}} \right] = \mathbb{E}^x \left[e^{-q\tau_A} 1_{\{X_{\tau_A} < 0, \tau_A < \infty\}} \right].$$

Threshold Level for Arbitrary h Function

There exists a unique threshold level A^* that attains the smooth-fit as a solution to this equation: for $q > 0$,

$$e^{-\eta A}\frac{(\eta - \xi_{1,q})(\xi_{2,q} - \eta)}{\eta} = \frac{2\gamma}{\sigma^2}\int_0^\infty e^{-\eta_q y} h(y + A)\mathrm{d}y, \tag{12}$$

and for $q = 0$ and $\bar{u} < 0$,

$$e^{-\eta A}(\xi_{2,0} - \eta) = \frac{2\gamma}{\sigma^2}\int_0^\infty e^{-\eta_q y} h(y + A)\mathrm{d}y. \tag{13}$$

We shall show that

$$\tau^* = \inf\{t \geq 0 : X_t \leq A^* \vee 0\}$$

is indeed optimal. Define the corresponding value for strategy τ^* by

$$\phi(x) := \mathbb{E}^x\left[e^{-q\tau^*}G(X_{\tau^*})1_{\{\tau^* < \infty\}}\right], \quad x \geq 0. \tag{14}$$

Verification of Optimality

$$\xrightarrow{\hspace{20em}} x$$

0 Stop($\phi = G$) A^* Continue($\phi > G$)

Our candidate value function $\phi(x)$ satisfies the following:

Lemma 1 *For every $x \geq 0$, we have $\phi(x) \leq G(x)$.*

(Proof) By differentiating (11) with respect to A to find the derivative is non-negative if and only if $A \geq A^*$.

Lemma 2 *The process $M = \{M_t := e^{-q(t \wedge \tau_0)}\phi(X_{t\wedge\tau_0}); t \geq 0\}$ is a submartingale.*

(Proof) We can prove (directly) that $(\mathcal{L} - q)\phi(x) = 0$ on $x \in (A^*, \infty)$ and $(\mathcal{L} - q)\phi(x) = -\gamma h(x) + \lambda e^{-\eta x}$ on $x \in (0, A^*)$, where

$$\mathcal{L}\phi(x) = c\phi'(x) + \frac{1}{2}\sigma^2\phi''(x) + \lambda \int_{\mathbb{R}} [\phi(x+z) - \phi(x)] f(z) \mathrm{d}z.$$

Moreover, we have $\delta''_{A^*}(A^*+) < 0$, which leads to

$$(\mathcal{L} - q)\phi(x) > \lim_{x \uparrow A^*}(\mathcal{L} - q)\phi(x) = \lim_{x \downarrow A^*}(\mathcal{L} - q)\phi(x) - \frac{1}{2}\sigma^2\phi''(A^*+) > \lim_{x \downarrow A^*}(\mathcal{L} - q)\phi(x) = 0$$

on $x \in [0, A^*)$.

Solution

Proposition 1 *These lemmas give us the quasi-variational inequalities required and hence*

$$\phi(x) = \inf_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-q\tau} G(X_\tau) 1_{\{\tau < \infty\}} \right]$$

and τ^* is optimal:

$$\tau^* = \inf\{t \geq 0 : X_t \leq A^* \vee 0\}$$

with A^* solves (12) for $q > 0$, or (13) for $q = 0$ and $\bar{u} < 0$.

Note:

$$\phi(x) = \begin{cases} U_x^{q,h}(\tau_{A^*}, \gamma), & A^* \leq x, \\ G(x), & 0 \leq x < A^*, \\ 1, & x < 0. \end{cases}$$

Numerical example

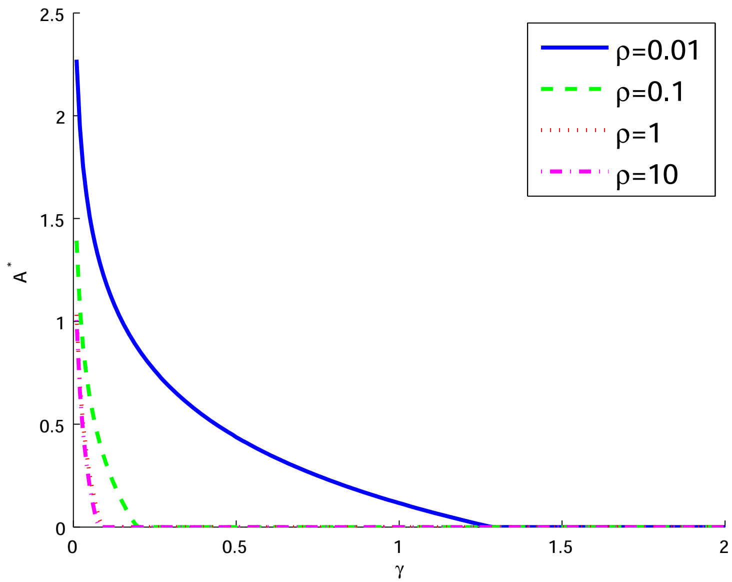
We use the exponential utility with $\rho > 0$,

$$h(x) = 1 - e^{-\rho x}, \quad x \geq 0.$$

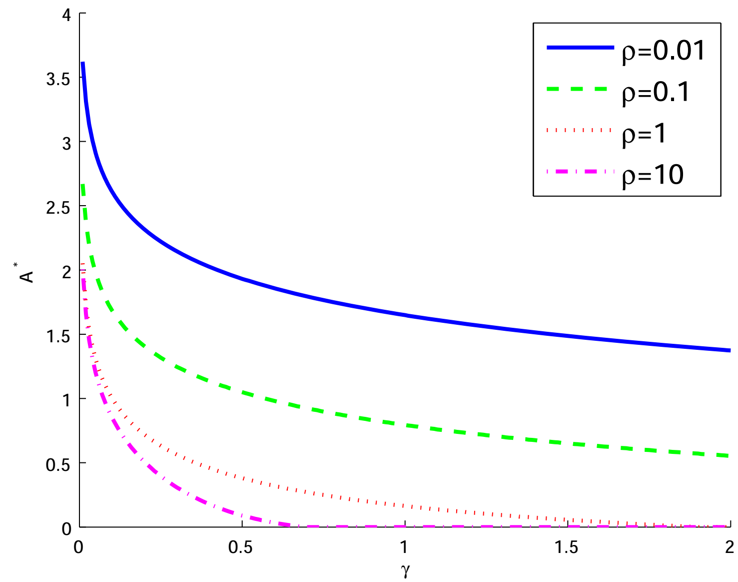
The first order condition reduces to

$$e^{-\eta_- A} \frac{(\eta_- - \xi_{1,q})(\xi_{2,q} - \eta_-)}{\eta_-} + e^{-\rho A} \frac{2\gamma}{\sigma^2(\zeta_q + \rho)} = \frac{2\gamma}{\sigma^2 \zeta_q}.$$

The optimal threshold level A^* with various values of coefficients of absolute risk aversion ρ: (a) $q = 0.1$ and $\overline{u} = 0.5$ ($c = 1$, $\sigma = 1$, $\eta_- = 2$, $\lambda = 1$) and (b) $q = 0$ and $\overline{u} = -1.5$ ($c = -1$, $\sigma = 1$, $\eta_- = 2$, $\lambda = 1$).



(a) $q > 0$



(b) $q = 0$ and $\overline{u} < 0$

Beyond the Exponential Jump Diffusion

In our paper, we discuss a double exponential diffusion (Kou and Wang (2003)) with $h = 1$, but here let us consider spectrally negative further.

Consider a continuous-time Markov chain $Y = \{Y_t; t \geq 0\}$ with finite state space $\{1, \ldots, m\} \cup \{\Delta\}$ where $1, \ldots, m$ are transient and Δ is absorbing. Its initial distribution is given by $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_m]$ such that $\alpha_i = \mathbb{P}\{Y_0 = i\}$ for every $i = 1, \ldots, m$. The intensity matrix $\boldsymbol{Q}$ is partitioned into the m transient states and the absorbing state Δ, and is given by

$$\boldsymbol{Q} := \begin{bmatrix} \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{0} & 0 \end{bmatrix}.$$

Here $\boldsymbol{T}$ is an $m \times m$-matrix called the phase-type generator, and $\boldsymbol{t} = -\boldsymbol{T}\mathbf{1}$ where $\mathbf{1} = (1, \ldots, 1)'$ (because each row sums up to zero).

A distribution is called phase-type with $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ if it is the distribution of the absorption time to Δ.

$$\mathbb{P}(Y \leq z) = F(z) = 1 - \boldsymbol{\alpha} e^{\boldsymbol{T}z}\mathbf{1} \quad \text{and} \quad \mathbb{P}(Y \in \mathrm{d}z) = \boldsymbol{\alpha} e^{\boldsymbol{T}z}\boldsymbol{t}\,\mathrm{d}z, \quad z \geq 0.$$

For any spectrally negative Lévy process X, there exists a sequence of spectrally negative Lévy processes with phase-type jumps $X^{(n)}$ conversing to X; i.e., $X_1^{(n)} \to X_1$ in distribution (Asmussen et al. (2004)).

Convergence of Scale Functions

The scale function of any spectrally negative Lévy process X can be approximated arbitrarily close by those of phase-type Lévy:

1. $\int_I W_n^{(q)}(y)\mathrm{d}y \to \int_I W^{(q)}(y)\mathrm{d}y$ for any interval I (by the continuity theorem).

2. If $W^{(q)} \in C^1(0,\infty)$ (which is true whenever the jump distribution has no atoms), then
$$W_n^{(q)}(x) \to W^{(q)}(x), \quad x \geq 0, \quad n \to \infty.$$

3. Suppose that, by denoting $W_{\zeta_q}(x) := e^{-\zeta_q x} W^{(q)}(x)$,

   (a) $W^{(q)} \in C^2(0,\infty)$ (which holds, for example, $\sigma > 0$),

   (b) $W'_{\zeta_q}(0+) < \infty$ (i.e., $\sigma > 0$ or $\Pi(-\infty, 0) < \infty$), and

   (c) $W''_{\zeta_q}(x) \leq 0$ for every $x \geq 0$ (which holds, for example, when the jump distribution is completely monotone),

   then we have
$$W_n^{'(q)}(x) \to W^{'(q)}(x), \quad x \geq 0.$$

Phase-type Model

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process of the form

$$X_t - X_0 = \mu t + \sigma B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \leq t < \infty$$

where $Z = \{Z_n : n = 1, 2, \ldots\}$ is an i.i.d. sequence of phase-type distributed random variables with $(m, \boldsymbol{\alpha}, \boldsymbol{T})$. Its Laplace exponent is

$$\psi(s) := \log \mathbb{E}\left[e^{sX_1}\right] = \mu s + \frac{1}{2}\sigma^2 s^2 + \lambda\left(\boldsymbol{\alpha}(s\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t} - 1\right), \tag{15}$$

To find the scale function of phase-type (spectrally negative) Lévy, we go back to the roots of

$$\psi(x) = q$$

for a fixed $q > 0$.

Define the set of (the absolute values of) negative zeros:

$$\mathcal{I}_q := \{i : \psi(-\xi_{i,q}) = q \text{ and } \mathcal{R}(\xi_{i,q}) > 0\}.$$

and the set of poles:

$$\mathcal{J}_q := \left\{j : \frac{q}{q - \psi(-\eta_j)} = 0 \text{ and } \mathcal{R}(\eta_j) > 0\right\}.$$

The elements in $\mathcal{I}_q$ and $\mathcal{J}_q$ may not be distinct, and, in this case, we take each as many times as its multiplicity. By Asmussen et al.(2004), we have

$$|\mathcal{I}_q| = \begin{cases} |\mathcal{J}_q| + 1, & \text{for Case 1: } \sigma > 0, \\ |\mathcal{J}_q|, & \text{for Case 2: } \sigma = 0 \text{ and } \mu > 0. \end{cases}$$

First Passage Time via the Wiener-Hopf Factorization

In particular, if the representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ is minimal for the distribution function F (i.e., there exists no number $k < m$, k-vector $\boldsymbol{b}$ and $k\times k$-matrix $\boldsymbol{G}$ such that $F(x) = 1-\boldsymbol{b}e^{\boldsymbol{G}x}\mathbf{1}$)

$$\left\{ \begin{array}{llll} |\mathcal{I}_q| = m+1 & \text{and} & |\mathcal{J}_q| = m, & \text{for Case 1} \\ |\mathcal{I}_q| = m & \text{and} & |\mathcal{J}_q| = m, & \text{for Case 2} \end{array} \right\},$$

Let κ_q be an independent exponential r.v. with parameter q and express the first passage time to level $a \in [0, x)$

$$\mathbb{E}^x[e^{-q\tau_a}1_{\{\tau_a<\infty\}}] = \mathbb{E}[e^{-q\tau_{a-x}}1_{\{\tau_{a-x}<\infty\}}] = \mathbb{P}\{\tau_{a-x} < \kappa_q\} = \mathbb{P}\{\underline{X}_{\kappa_q} < a-x\} = \mathbb{P}\{-\underline{X}_{\kappa_q} > x-a\}$$

where $\underline{X}_t = \inf_{0\le s<t} X_s, \quad t \geq 0$ has the Wiener-Hoph factor, for every s with $\mathcal{R}(s) > 0$,

$$\varphi_q^-(s) := \mathbb{E}[\exp(s\underline{X}_{\kappa_q})] = \frac{\prod_{j\in\mathcal{J}_q}(s+\eta_j)}{\prod_{j\in\mathcal{J}_q}\eta_j}\frac{\prod_{i\in\mathcal{I}_q}\xi_{i,q}}{\prod_{i\in\mathcal{I}_q}(s+\xi_{i,q})}.$$

By inverting via partial fraction expansion,

$$\mathbb{P}\{-\underline{X}_{\kappa_q} \in \mathrm{d}x\} = \sum_{i\in\mathcal{I}_q} A_{i,q}\xi_{i,q}e^{-\xi_{i,q}x}\mathrm{d}x, \quad x > 0$$

where $\{A_{i,q}; i \in \mathcal{I}_q\}$ are the partial fraction coefficients of the expansion,

$$\varphi_q^-(s) - \varphi_q^-(\infty) = \sum_{i\in\mathcal{I}_q} A_{i,q}\frac{\xi_{i,q}}{\xi_{i,q}+s};$$

Combining, we have one representation:

(16) $$\mathbb{E}^x\left[e^{-q\tau_a}1_{\{\tau_a<\infty\}}\right] = \sum_{i\in\mathcal{I}_q} A_{i,q}e^{-\xi_{i,q}(x-a)}.$$

First Passage Time via the Scale Function

From Kyprianou (2006), for every $q > 0$ and $0 \leq a < x$,

$$(17)\quad \mathbb{E}^x[e^{-q\tau_a} 1_{\{\tau_a < \infty\}}] = \left(1 + \int_0^{x-a} W^{(q)}(y)\mathrm{d}y\right) - \frac{q}{\zeta_q} W^{(q)}(x-a),$$

from which we obtain for every $q > 0$,

$$\frac{\partial}{\partial a}\mathbb{E}^x\left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}}\right] = \frac{q}{\zeta_q} e^{\zeta_q (x-a)} W'_{\zeta_q}(x-a), \quad 0 \leq a < x,$$

$$\frac{\partial}{\partial x}\mathbb{E}^x\left[e^{-q\tau_0} 1_{\{\tau_0 < \infty\}}\right]\Big|_{x=0+} = -\frac{q}{\zeta_q}\nu,$$

with

$$\nu := -\zeta_q W^{(q)}(0) + W^{(q)\prime}(0+) = \left\{ \begin{array}{ll} \frac{2}{\sigma^2}, & \text{for Case 1} \\ -\frac{\zeta_q}{\mu} + \frac{q+\lambda}{\mu^2}, & \text{for Case 2} \end{array} \right\}.$$

Comparing the two representations (16) and (17), we obtain

Proposition 2 *For every $q > 0$,*

1. *For Case 1, we have*

$$W^{(q)}(x) = \frac{2}{\sigma^2 \varrho_q} \sum_{i \in \mathcal{I}_q} \frac{\xi_{i,q} A_{i,q}}{\zeta_q + \xi_{i,q}} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \right], \quad x \geq 0.$$

2. *For Case 2, we have*

$$W^{(q)}(x) = \frac{1}{\varrho_q} \left(-\frac{\zeta_q}{\mu} + \frac{q+\lambda}{\mu^2} \right) \sum_{i \in \mathcal{I}_q} \frac{\xi_{i,q} A_{i,q}}{\zeta_q + \xi_{i,q}} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \right] + \frac{1}{\mu} e^{\zeta_q x}, \quad x \geq 0.$$

In the above,

$$\varrho_q := \sum_{i \in \mathcal{I}_q} A_{i,q} \xi_{i,q}, \quad q > 0.$$

In this case, the function $W_{\zeta_q}(x) = e^{-\zeta_q x} W^{(q)}(x)$ is completely monotone;

$$(-1)^{n-1} W_{\zeta_q}^{(n)}(x) \geq 0, \quad x \geq 0.$$

Addendum: Obtaining the Coefficients

For the case of distinct roots and minimal representation, we obtain $A_{i,q}$ by solving

(18)
$$\boldsymbol{H}^T \boldsymbol{w} = \mathbf{1}.$$

for $\boldsymbol{w}$ where

$$\boldsymbol{H} = \begin{bmatrix} \widetilde{\boldsymbol{f}}(-\xi_{1,q}) \\ \vdots \\ \widetilde{\boldsymbol{f}}(-\xi_{|\mathcal{I}_q|,q}) \end{bmatrix}$$

and whose entries are

$$\widetilde{\boldsymbol{f}}(-\xi_{i,q}) = \left\{ \begin{array}{ll} \left[1 \quad \left((-\xi_{i,q}\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t}\right)'\right], & \text{for Case 1} \\ \left((-\xi_{i,q}\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t}\right)', & \text{for Case 2} \end{array} \right\}, \quad i \in \mathcal{I}_q.$$

For the hyper-exponential, with some $0 < \eta_1 < \cdots < \eta_m < \infty$ such that $\sum_{i=1}^m \alpha_i = 1$, the density is $f(z) = \sum_{i=1}^m \alpha_i \eta_i e^{-\eta_i z}, \quad z \in \mathbb{R}$. In Case 1,

$$\widetilde{\boldsymbol{f}}(s) = \left[1, \frac{\eta_1}{\eta_1 + s}, \ldots \frac{\eta_m}{\eta_m + s}\right]$$

and hence

$$\boldsymbol{H}^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{\eta_1}{\eta_1 - \xi_{1,q}} & \frac{\eta_1}{\eta_1 - \xi_{2,q}} & \cdots & \frac{\eta_1}{\eta_1 - \xi_{m+1,q}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\eta_m}{\eta_m - \xi_{1,q}} & \frac{\eta_m}{\eta_m - \xi_{2,q}} & \cdots & \frac{\eta_m}{\eta_m - \xi_{m+1,q}} \end{bmatrix}.$$

Numerical Examples:

- If the jump distribution is completely monotone, the converging sequence can be chosen to be the ones with hyper-exponential distributions:

$$f(z) = \sum_{i=1}^{m} \alpha_i \eta_i e^{-\eta_i z}, \quad z \in \mathbb{R},$$

  for some $0 < \eta_1 < \cdots < \eta_m < \infty$ such that $\sum_{i=1}^{m} \alpha_i = 1$.

- Feldmann and Whitt (1998) showed an algorithm for fitting hyper-exponential distributions to a general completely monotone distribution.

- We use their results to obtain the scale function when jumps are Weibull (c, a)

$$F(t) = 1 - e^{-(t/a)^c}, \quad t \geq 0,$$

  and Pareto (a, b)

$$F(t) = 1 - (1 + bt)^{-a}, \quad t \geq 0$$

  distributed.

Fitted Parameters for Weibull/Pareto

Taken from Feldmann and Whitt (1998),

i	α_i	η_i
1	0.029931	676.178
2	0.093283	38.7090
3	0.332195	4.27400
4	0.476233	0.76100
5	0.068340	0.24800
6	0.000018	0.09700

(i) Weibull(0.6,0.665)

i	α_i	η_i	i	α_i	η_i
1	8.37E-11	8.3E-09	8	0.000147	0.0020
2	7.18E-10	6.8E-08	9	0.001122	0.0100
3	5.56E-09	3.9E-07	10	0.008462	0.0570
4	4.27E-08	2.2E-06	11	0.059768	0.3060
5	3.27E-07	1.2E-05	12	0.307218	1.5460
6	2.50E-06	6.5E-05	13	0.533823	6.5160
7	1.92E-05	3.5E-04	14	0.089437	23.304

(ii) Pareto(1.2,5)

Weibull Distribution

The scale function of X with $\mu = 0$, $\sigma = 0.01$, $\lambda = 0.1$ and Weibull $(0.6, 0.665)$ can be approximated by

$$W^{(q)}(x) = \sum_{i \in \mathcal{I}_q} C_{i,q} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \right], \quad x \geq 0.$$

with

$$C_{i,q} := \frac{\nu}{\varrho_q} \frac{\xi_{i,q} A_{i,q}}{\zeta_q + \xi_{i,q}}$$

i	$\xi_{i,q}$	$A_{i,q}$	$C_{i,q}$
1	0.0969990705796	0.000010213932094	0.0000049474290
2	0.2387406362121	0.042207380215956	0.0502261535296
3	0.6178972697386	0.181977148543284	0.5577059566440
4	3.7980930145449	0.087513431726977	1.5832236290145
5	37.160241923152	0.051804177184444	6.4758763022306
6	78.497115144071	0.636476073284678	123.22078286003
7	676.26768636481	0.000011575112568	0.0039733229452

Weibull Distribution Error Analysis

We compare the LHS and RHS of the relationship

$$\sum_{i\in\mathcal{I}_q} C_{i,q}\left(\frac{1}{\beta}-\frac{1}{\zeta_q+\xi_{i,q}+\beta}\right)+\frac{W_{\zeta_q}(0)}{\beta}=\frac{1}{\psi(\beta+\zeta_q)-q}, \qquad \beta>0.$$

β	LHS	RHS	absolute error
0.05	2636.956051024141	2636.956050828426	1.9571e-007
0.10	1318.038418276205	1318.038418225931	5.0274e-008
0.50	262.9064571661544	262.9064571636729	2.4815e-009
1.00	131.0176256611043	131.0176256603400	7.6429e-010
10.0	12.36501436360020	12.36501436357116	2.9040e-011
100	0.792975691767121	0.792975691766349	7.7205e-013

Pareto Distribution

The scale function of X with $\mu = 0$, $\sigma = 0.01$, $\lambda = 0.1$ and Pareto $(1.2, 5)$ can be approximated by

$$W^{(q)}(x) = \sum_{i \in \mathcal{I}_q} C_{i,q} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \right], \quad x \geq 0.$$

i	$\xi_{i,q}$	$A_{i,q}$	$C_{i,q}$
1	0.000000008235156	0.007813458670425	0.000000000003246
2	0.000000067941699	0.000849779495869	0.000000000002913
3	0.000000389921386	0.000199758818815	0.000000000003929
4	0.000002199944736	0.000024894827628	0.000000000002763
5	0.000011999924064	0.000006271675736	0.000000000003797
6	0.000064999898911	0.000001541426092	0.000000000005055
7	0.000349996670502	0.000009429635617	0.000000000166497
8	0.001999852975897	0.000072915691854	0.000000007356267
9	0.009994374942093	0.000559383483826	0.000000282006005
10	0.056756368788265	0.004300742193124	0.000012305122823
11	0.296725362741112	0.031481003607834	0.000469432459232
12	1.335002927170950	0.139603262110976	0.009241025235379
13	5.355731274613405	0.150622194007917	0.038035855476823
14	22.605546762702495	0.026354651553805	0.023204214950600
15	78.642108436899349	0.638100712800481	1.248839677422900

Pareto Distribution Error Analysis

We compare the LHS and RHS of the relationship

$$\sum_{i\in\mathcal{I}_q} C_{i,q}\left(\frac{1}{\beta}-\frac{1}{\zeta_q+\xi_{i,q}+\beta}\right)+\frac{W_{\zeta_q}(0)}{\beta}=\frac{1}{\psi(\beta+\zeta_q)-q}, \qquad \beta > 0.$$

β	LHS	RHS	absolute error
0.05	2638.718948463406	2638.718947228851	1.2346e-006
0.10	1318.916455287831	1318.916454977145	3.1069e-007
0.50	263.0766634379456	263.0766634248073	1.3138e-008
1.00	131.0994239282779	131.0994239247748	3.5031e-009
10.0	12.36839081902370	12.36839081895586	6.7841e-011
100	0.792429248707535	0.792429248706064	1.4709e-012

Summary

- We constructed an "alarm" by modeling the tradeoff b/w cost of delay and premature capital raising.

- We solved the induced optimal stopping problem explicitly in a jump diffusion model by using the scale function in its explicit form.

- We obtained the scale function of phase-type distribution that can be utilized to approximate that of other jump distributions.

- Various extensions may be possible in terms of the regret function h, jump types, and so on.

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