Density Approach in Credit Risk Modelling

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Introduction

- Classically, the pricing of credit derivatives is considered before the default, that is, on the set $\{\tau > t\}$ since the product no longer exists after the default.
- There exist two main approaches in the credit risk modelling — the structural approach and the intensity approach.
- To analyze the credit risks in a much larger context such as the counterparty risks and the contagious defaults phenomenon, etc., we need to understand what goes on after the default, i.e. on the set $\{\tau \leq t\}$
- Motivated by the "after-default" studies, we propose a new approach — the density approach which is based on the density of the conditional distribution of τ.

Information and filtration

- The progressive enlargement of filtration plays an essential role in the credit risk modelling
- On the market $(\Omega, \mathcal{A}, \mathbb{P})$, the default information $\sigma(\tau \wedge t)$ is described as an exogenous source of information compared to the default-free information $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.
- The global information $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$.
- Remark: any $\mathcal{G}_t$-measurable random variable $Y_t^{\mathbb{G}}$ is written in the form

$$Y_t^{\mathbb{G}} = Y_t 1_{\{t<\tau\}} + Y_t(\tau) 1_{\{t \geq \tau\}},$$

where Y_t is $\mathcal{F}_t$-measurable and $Y_t(\theta)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

Before-default pricing

- For single credit name, the pricing of a default sensitive claim consists of computing the conditional expectation w.r.t. $\mathbb{G}$ under some risk-neutral probability.
- Most credit derivative ceases to exist once the default occurs. So classically, the pricing is on the set $\{\tau > t\}$.
- The idea (e.g. Bielecki-Jeanblanc-Rutkowski) is to establish an explicit relationship between the $\mathbb{G}$ and the $\mathbb{F}$ conditional expectations
- Key Lemma (Jeulin-Yor): for any $\mathcal{A}$-measurable r.v. Y,

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}[Y|\mathcal{G}_t] = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}[Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]}{\mathbb{P}(\tau > t|\mathcal{F}_t)} \quad a.s. \qquad (1)$$

on the set $A = \{S_t := \mathbb{P}(\tau > t|\mathcal{F}_t) > 0\}$.

Default density approach

- From now on, we are interested in what happens after a default, i.e. on $\{\tau \leq t\}$.
- Similar to Jacod's hypothesis in the initial enlargement of filtration, we introduce:

Density Hypothesis

For any $t \geq 0$, there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable r.v. $\alpha_t(\theta)$ s.t. for any bounded Borel function f on $\mathbb{R}_+$,

$$\mathbb{E}[f(\tau)|\mathcal{F}_t] = \int_0^\infty f(\theta)\alpha_t(\theta)d\theta \quad a.s.$$

- In finance, such type of hypotheses appeared in the insider information problems and in the interest rate modellings (Brody-Hughston).

Several simple properties

- The notion of density can be generalized to any non-negative non-atomic measure instead of the Lebesgue measure.
- The conditional distribution of τ is characterized by the survival probability defined by

$$S_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty \alpha_t(u)du$$

- In particular, the survival process $S_t = S_t(t) = \int_t^\infty \alpha_t(u)du$.
- For any $\theta \geq 0$, $(S_t(\theta), t \geq 0)$ and $(\alpha_t(\theta), t \geq 0)$ are $\mathbb{F}$-martingales: for any $\theta \geq t$, $S_t(\theta) = \mathbb{E}[S_\theta | \mathcal{F}_t]$ and $\alpha_t(\theta) = \mathbb{E}[\alpha_\theta(\theta) | \mathcal{F}_t]$.
- For any $t \geq 0$, $\int_0^\infty \alpha_t(u)du = 1$.

The "after-default" pricing

Theorem

Let $Y_T(\theta)$ be an integrable $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$-measurable random variable, then for any $t < T$,

$$E[Y_T(\tau)|\mathcal{G}_t] = Y_t^{\mathrm{bd}} 1\!\!1_{\{t<\tau\}} + Y_t^{\mathrm{ad}}(T,\tau) 1\!\!1_{\{\tau \le t\}} \quad a.s.$$

where

$$Y_t^{\mathrm{bd}} = \frac{\mathbb{E}\left[\int_t^\infty Y_T(u)\alpha_T(u)du|\mathcal{F}_t\right]}{S_t}$$

$$Y_t^{\mathrm{ad}}(T,\theta) = \frac{\mathbb{E}\left[Y_T(\theta)\alpha_T(\theta)\middle|\mathcal{F}_t\right]}{\alpha_t(\theta)}$$

We observe the after-default density $\alpha_t(\theta)$ where $t \ge \theta$.

- The proof is obtained by a simple verification.
  Any $\mathcal{G}_t$-measurable random variable can be written on the set $\{\tau \leq t\}$ as $H_t(\tau)1\!\!1_{\{\tau\leq t\}}$. Assume that $H_t(\tau)$ is positive or bounded. Using the density $\alpha_t(\theta)$, we obtain

$$\begin{aligned}
\mathbb{E}[H_t(\tau)1\!\!1_{\{\tau\leq t\}}Y_T(\tau)] &= \int d\theta\,\mathbb{E}[H_t(\theta)1\!\!1_{\{\theta\leq t\}}Y_T(\theta)\alpha_T(\theta)] \\
&= \int d\theta\,\mathbb{E}\big[H_t(\theta)1\!\!1_{\{\theta\leq t\}}\mathbb{E}[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t]\big] \\
&= \int d\theta\,\mathbb{E}\Big[H_t(\theta)1\!\!1_{\{\theta\leq t\}}Y_t^{\mathrm{ad}}(T,\theta)\alpha_t(\theta)\Big] \\
&= \mathbb{E}\Big[H_t(\tau)1\!\!1_{\{\tau\leq t\}}Y_t^{\mathrm{ad}}(T,\tau)\Big],
\end{aligned}$$

Immersion property (H-hypothesis)

- A standard hypothesis in the credit risk modelling is the immersion property, or the H-hypothesis, which asserts that any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale.
- In the case where

$$\alpha_t(\theta) = \alpha_\theta(\theta), \quad \forall \theta \leq t$$

one has for any $T \geq t$,

$$S_t = 1 - \int_0^t \alpha_t(\theta)d\theta = 1 - \int_0^t \alpha_T(\theta)d\theta = \mathbb{P}(\tau > t|\mathcal{F}_T).$$

- This last equality implies

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty).$$

and is equivalent to the immersion property.

- Conversely, if immersion property holds, i.e. $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$, then the process S is decreasing. In addition, for $t > \theta$,

$$S_t(\theta) = S_\theta(\theta) \quad \text{and} \quad \alpha_t(\theta) = \alpha_\theta(\theta).$$

- Under immersion property, we have for any $T \geq t$,

$$\mathbb{E}[Y_T(\tau)|\mathcal{G}_t]1\!\!1_{\{\tau \leq t\}} = 1\!\!1_{\{\tau \leq t\}}\mathbb{E}[Y_T(\theta)|\mathcal{F}_t]\,\Big|_{\theta=\tau} \quad a.s.$$

- This implies that H-hypothesis, which is natural and often supposed for the before-default studies, becomes more restrictive in the after-default analysis.

Relationship with the intensity

Definition

Let τ be a $\mathbb{G}$-stopping time. The $\mathbb{G}$-compensator $\Lambda^{\mathbb{G}}$ of τ is the $\mathbb{G}$-predictable increasing process such that $(1\!\!1_{\{\tau\leq t\}} - \Lambda_t^{\mathbb{G}}, t \geq 0)$ is a $\mathbb{G}$-martingale.

- The $\mathbb{G}$-compensator is stopped at τ, i.e., $\Lambda_t^{\mathbb{G}} = \Lambda_{t\wedge\tau}^{\mathbb{G}}$.
- $\Lambda^{\mathbb{G}}$ coincides, on the set $\{\tau \geq t\}$, with an $\mathbb{F}$-predictable process $\Lambda^{\mathbb{F}}$, i.e. $\Lambda_t^{\mathbb{G}} 1\!\!1_{\{\tau\geq t\}} = \Lambda_t^{\mathbb{F}} 1\!\!1_{\{\tau\geq t\}}$.
- If $\Lambda^{\mathbb{G}}$ is absolutely continuous, i.e., $\Lambda_t^{\mathbb{G}} = \int_0^t \lambda_s^{\mathbb{G}} ds$, then $\lambda^{\mathbb{G}}$ is called the $\mathbb{G}$-intensity of τ. The $\mathbb{F}$-intensity $\lambda^{\mathbb{F}}$ is defined in a similar way.

Proposition (density and intensity)

- Under the density hypothesis, the $\mathbb{G}$-compensator $\Lambda^{\mathbb{G}}$ of τ admits a density given by

$$\lambda_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}\lambda_t^{\mathbb{F}} = \mathbb{1}_{\{\tau>t\}}\frac{\alpha_t(t)dt}{S_{t-}} \quad a.s.$$

- Conversely, for any $T \geq t$,

$$\alpha_t(T) = \mathbb{E}[\lambda_T^{\mathbb{G}}|\mathcal{F}_t] \quad a.s.$$

Remark

- Given the density $\alpha_t(t)$, we obtain the intensity process since $S_t = \int_t^\infty \alpha_t(u)du = \int_t^\infty \mathbb{E}[\alpha_u(u)|\mathcal{F}_t]du$.
- From the intensity process, we deduce the density $\alpha_t(\theta)$ only for $\theta \geq t$. The intensity contains no information on $\theta < t$, which is important for the after-default case.

Density modelling

- We search for models of the default density $\alpha_t(\theta)$, which satisfies both the martingale property and the probability property, i.e. $\int_0^\infty \alpha_t(\theta)d\theta = 1$.
- Similar problems have been studied in the interest rate modelling by Brody-Hughston.
- The main point here is that in the interest rate models, the maturity θ is always larger than t, i.e., $\theta \geq t$, which corresponds to the before-default part of density.
- We here need the whole family, and in particular the after-default density where $\theta < t$.

The "after-default" density without H-hypothesis

- The change of probability plays an important role in constructing the density term structure, particularly for the "after-default" part.
- We begin from a probability measure $\mathbb{P}$ where the H-hypothesis holds.
- Suppose that $\mathbb{F}$ is generated by a Brownian motion W. Then W is also a $(\mathbb{G}, \mathbb{P})$-Brownian motion.
- Let $(N_t := 1\!\!1_{\{\tau \leq t\}} - \Lambda_t^{\mathbb{G}}, t \geq 0)$ be the $(\mathbb{G}, \mathbb{P})$-martingale of pure jump and assume the intensity hypothesis, i.e., $\Lambda_t^{\mathbb{G}} = \int_0^t \lambda_s^{\mathbb{G}} ds = \int_0^{t \wedge \tau} \lambda_s^{\mathbb{F}} ds$.
- Then the density of τ under $\mathbb{P}$ is

$$\alpha_t(\theta) = \mathbb{E}[\lambda_\theta^{\mathbb{F}} S_\theta | \mathcal{F}_t] = \mathbb{E}[\lambda_\theta^{\mathbb{F}} e^{-\Lambda_\theta^{\mathbb{F}}} | \mathcal{F}_t] \quad t, \theta \geq 0.$$

- Our aim is to find the "after-default" density under a probability $\mathbb{Q}$ where H-hypothesis is not satisfied.
- By the martingale representation theorem in $\mathbb{G}$ (Kusuoka), any positive martingale Q with expectation 1 can be written as the solution of a SDE

$$dQ_t = Q_{t-}(\Psi_t dW_t + \Phi_t dN_t), \quad Q_0 = 1$$

  where Ψ and Φ, $\Phi > -1$, are $\mathbb{G}$-predictable processes.
- Using the decomposed form

$$\Psi_t = \psi_t 1\!\!1_{\{t\leq\tau\}} + \psi_t(\tau) 1\!\!1_{\{t>\tau\}} \quad \text{and} \quad \Phi_t = \phi_t 1\!\!1_{\{t\leq\tau\}} + \phi_t(\tau) 1\!\!1_{\{t>\tau\}},$$

  it follows $Q_t = \overline{q}_t 1\!\!1_{\{t<\tau\}} + \overline{q}_t(\tau) 1\!\!1_{\{t\geq\tau\}}$ where

$$\begin{aligned} \overline{q}_t &= \exp\Big(\int_0^t \psi_u dW_u - \frac{1}{2}\int_0^t \psi_u^2 du - \int_0^t \lambda_u^{\mathbb{F}} \phi_u du\Big), t \geq 0 \\ \overline{q}_t(\theta) &= \overline{q}_\theta(\theta) \exp\Big(\int_\theta^t \psi_u(\theta) dW_u - \frac{1}{2}\int_\theta^t \psi_u(\theta)^2 du\Big), t \geq \theta. \end{aligned}$$

  with $\overline{q}_\theta(\theta) = \overline{q}_\theta(1 + \phi_\theta)$.

- The restriction of Q on $\mathbb{F}$ is given by

$$Q_t^{\mathbb{F}} = \mathbb{E}[Q_t|\mathcal{F}_t] = \overline{q}_t S_t + \int_0^t \overline{q}_t(u)\lambda_u^{\mathbb{F}} S_u du.$$

- Let $\mathbb{Q}$ be the probability measure defined by $d\mathbb{Q} = Q_t d\mathbb{P}$ on $\mathcal{G}_t$. Then

$$\begin{aligned}
\alpha_t^{\mathbb{Q}}(\theta) &= \frac{1}{Q_t^{\mathbb{F}}} \overline{q}_t(\theta)\alpha_\theta(\theta), \ t \geq \theta \\
\alpha_t^{\mathbb{Q}}(\theta) &= \frac{1}{Q_t^{\mathbb{F}}} \mathbb{E}_{\mathbb{P}}[\overline{q}_\theta(1+\phi_\theta)\alpha_\theta(\theta)|\mathcal{F}_t], \ t < \theta.
\end{aligned}$$

Density modelling in the general case

- To apply the change of probability method in a general way, we need a characterization result of $\mathbb{G}$-martingales.
- A classical question in the enlargement of filtration is to give decomposition of $\mathbb{F}$-martingales in terms of $\mathbb{G}$-semimartingale.
- In the progressive enlargement with the density, if X is an $\mathbb{F}$-martingale, then

$$\widehat{X}_t = X_t - \int_0^{t\wedge\tau} \frac{d\langle X, S\rangle_u}{S_u} - \int_{t\wedge\tau}^{t} \frac{d\langle X, \alpha.(\theta)\rangle_u}{\alpha_u(\theta)}\bigg|_{\theta=\tau} \in \mathcal{M}(\mathbb{G})$$

- In the credit analysis, we shall study the problem in the converse sense, that is, we are interested in the $\mathbb{G}$-martingales and its relationship with the $\mathbb{F}$-martingales.

Characterization of $\mathbb{G}$-martingales

Proposition

A càdlàg process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale if and only if there exist a càdlàg $\mathbb{F}$-adapted process Y and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable process $Y_t(.)$ such that

$$Y_t^{\mathbb{G}} = Y_t 1\!\!1_{\{\tau > t\}} + Y_t(\tau) 1\!\!1_{\{\tau \leq t\}}$$

and that

- $(Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)ds, \ t \geq 0)$ is an $\mathbb{F}$-martingale;
- $(Y_t(\theta)\alpha_t(\theta), t \geq \theta)$ is an $\mathbb{F}$-martingale.

- Any $\mathbb{G}$-martingale may be splitted into two martingales, the first one stopped at time τ and the second one starting at time τ, that is $Y_t^{\mathbb{G}} = Y_t^{bd,\mathbb{G}} + Y_t^{ad,\mathbb{G}}$ where

$$Y_t^{bd,\mathbb{G}} = Y_{t\wedge\tau}^{\mathbb{G}} \text{ and } Y_t^{ad,\mathbb{G}} = (Y_t^{\mathbb{G}} - Y_\tau^{\mathbb{G}}) 1\!\!1_{\{\tau \leq t\}}.$$

- We study the two types of martingales respectively.

$\mathbb{G}$-martingale stopped at time τ

- Any $\mathbb{G}$-adapted process $Y^{\mathbb{G}}$ stopped at time τ can be written in the form $Y_t^{\mathbb{G}} = Y_t 1\!\!1_{\{\tau > t\}} + Y_\tau(\tau) 1\!\!1_{\{\tau \leq t\}}$.
- It is a $\mathbb{G}$-martingale if and only if

$$(U_t := Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)ds,\ t \geq 0)$$

  is an $\mathbb{F}$-martingale.
- Idea of proof. Consider the $\mathbb{F}$-martingale defined by

$$Y_t^{\mathbb{F}} = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = Y_t S_t + \int_0^t Y_s(s)\alpha_t(s)ds$$

  Since $(\int_0^t Y_s(s)(\alpha_t(s) - \alpha_s(s))ds, t \geq 0)$ is an $\mathbb{F}$-martingale, and so is U. Conversely, if U is an $\mathbb{F}$-martingale, verify by the $\mathbb{G}$-conditional expectation computations that $\mathbb{E}[Y_T^{\mathbb{G}} - Y_t^{\mathbb{G}}|\mathcal{G}_t] = 0$.

$\mathbb{G}$-martingale starting at time τ

- Any $\mathbb{G}$-adapted process $Y^{\mathbb{G}}$ starting at τ can be written in the form $Y_t^{\mathbb{G}} = Y_t(\tau) 1\!\!1_{\{\tau \le t\}}$
- By direct computations, it is a $\mathbb{G}$-martingale if and only if $(Y_t(\theta)\alpha_t(\theta), t \ge \theta)$ is an $\mathbb{F}$-martingale.

Girsanov's theorem

- Let τ be given on $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ with density $\alpha_t(\theta), t, \theta \geq 0$.
- Let $(\beta_t(\theta), t \geq \theta)$ be a family of strictly positive $\mathbb{F}$-martingales and define $\beta_t(\theta) = \mathbb{E}[\beta_\theta(\theta)|\mathcal{F}_t]$ for $t < \theta$.
- Let

$$Q_t := \frac{q_t}{M_0^\beta} 1\!\!1_{\{\tau > t\}} + \frac{q_t(\tau)}{M_0^\beta} 1\!\!1_{\{\tau \leq t\}}.$$

where $M_t^\beta = \int_0^\infty \beta_t(\theta) d\theta$, $q_t(\theta) = \beta_t(\theta)/\alpha_t(\theta), t \geq \theta$ and $q_t S_t = \mathbb{E}\big[\int_t^\infty q_u(u)\alpha_u(u)du|\mathcal{F}_t\big]$.

- Then, Q is a positive $(\mathbb{G}, \mathbb{P})$-martingale with expectation 1 and defines a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{G}_\infty, \mathbb{G})$ equivalent to $\mathbb{P}$ and

$$\alpha_t^{\mathbb{Q}}(\theta) = \frac{\beta_t(\theta)}{M_t^\beta}, \quad t \geq 0, \ \theta \geq 0 \tag{2}$$

is the $(\mathbb{F}, \mathbb{Q})$-density process of τ.

Several remarks

- By adopting the density w.r.t the filtration $\mathbb{F}$, we start from the knowledge based on the default-free information.
- If we are only concerned with what happens up to the default time, it is natural to assume the H-hypothesis with a stochastic intensity process.
- We anticipate the default has a large impact on the market: with the non-constant after-default density, even the default-free market is “modified” after the default.
- This framework can be adopted to study counterparty risks and multiple default risks.

Application to modelling of successive defaults

- In the literature, there are two approaches to model the multiple default events: bottom-up and top-down.
- The "before default" and "after default" analysis adapt naturally to study the ordered default times in a recursive manner.
- There is a close link between the top-down models of the total loss process and the successive defaults.
- The correlation between default times is characterized by the joint density in a dynamic manner.

Density hypothesis for ordered defaults

- Let us consider a family of random times $(\tau_1, \cdots, \tau_n)$ on $(\Omega, \mathcal{A}, \mathbb{P})$ taking values on $\mathbb{R}_+^n$, whose increasing-ordered permutation is denoted by

$$\sigma_1 \leq \sigma_2 \cdots \leq \sigma_n.$$

- Joint density hypothesis for $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$: there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable functions $(\omega, \boldsymbol{u}) \to \alpha_t(\boldsymbol{u})$ with $\boldsymbol{u} = (u_1, \cdots, u_n) \in \mathbb{R}_+^n$, such that for any bounded Borel function $f : \mathbb{R}_+^n \to \mathbb{R}$,

$$\mathbb{E}[f(\boldsymbol{\sigma})|\mathcal{F}_t] = \int_{\mathbb{R}_+^n} f(\boldsymbol{u})\alpha_t(\boldsymbol{u})d\boldsymbol{u}, \quad t \geq 0. \tag{3}$$

- The density satisfies

$$\alpha_t(\boldsymbol{u}) = 0$$

for any $\boldsymbol{u}$ outside the set $\{u_1 \leq u_2 \leq \cdots \leq u_n\}$.

Default information

- The default information arrives progressively with successive defaults.
- For any $i = 1, \cdots, n$, let $\mathbb{D}^i = (\mathcal{D}^i_t)_{t \geq 0}$ be the filtration associated with σ_i, i.e. $\mathcal{D}^i_t = \sigma(\sigma_i \wedge t)$ and by

$$\mathbb{D}^{(i)} = (\mathcal{D}^{(i)}_t)_{t \geq 0} := \mathbb{D}^1 \vee \cdots \vee \mathbb{D}^i.$$

- The global information also contains the default-free information:

$$\mathbb{G}^{(i)} = (\mathcal{G}^{(i)}_t)_{t \geq 0} := \mathbb{F} \vee \mathbb{D}^{(i)}$$

and define for convenience $\mathbb{G}^{(0)} = \mathbb{F}$.
- Market full information : $\mathcal{G}^{(n)}_t$ and $\mathcal{G}^{(n)}_{t \wedge \sigma_i} = \mathcal{G}^{(i)}_t$.
- Any $\mathcal{G}^{(n)}_t$-measurable random variable X can be written in the form $X_t = \sum_{j=0}^n 1\!\!1_{\{\sigma_j \leq t < \sigma_{j+1}\}} X^j_t(\boldsymbol{\sigma}_{(j)})$ where $X^j_t(\boldsymbol{u}_{(j)})$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^j_+)$-measurable and $\boldsymbol{\sigma}_{(j)} = (\sigma_1, \ldots, \sigma_j)$.

A useful computation result

- Let $Y_T(\boldsymbol{u})$ be positive and $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^n_+)$-measurable, then

$$\mathbb{E}[Y_T(\boldsymbol{\sigma})|\mathcal{G}^{(n)}_t] = \int_{\mathbb{R}^n_+} \mathbb{E}\Big[\frac{Y_T(\boldsymbol{u})\alpha_T(\boldsymbol{u})}{\alpha_t(\boldsymbol{u})}\Big|\mathcal{F}_t\Big]\mu^{(n)}_t(d\boldsymbol{u})$$

  where

$$\mu^{(n)}_t(d\boldsymbol{u}) = \sum_{i=0}^{n} \mathbb{1}_{\{\sigma_i \le t < \sigma_{i+1}\}} \frac{\alpha_t(\boldsymbol{u})d\boldsymbol{u}_{(i+1:n)}}{\int_{]t,\infty[}\alpha_t(\boldsymbol{u})d\boldsymbol{u}_{(i+1:n)}} \delta_{\boldsymbol{\sigma}_{(i)}}(d\boldsymbol{u}_{(i)})$$

  with δ denoting the Dirac measure.
- k^{th}-to-default swap: $Y_T(\boldsymbol{\sigma}) = \mathbb{1}_{\{\sigma_k > T\}}$
- CDO tranche: $Y_T(\boldsymbol{\sigma}) = (\sum_{i=1}^n \mathbb{1}_{\{\sigma_i \le T\}} - k)_+$
- The prices depend on both the number and the occurrence timing of defaults.

Recursive point of view

- The above result can also be obtained from a recursive point of view.
- On the set $\{\sigma_{i+1} > t\}$, $\mathcal{G}_t^{(n)}$ and $\mathcal{G}_t^{(i)}$ coincide, so

$$1\!\!1_{\{\sigma_i \leq t < \sigma_{i+1}\}}\mathbb{E}[Y_T(\boldsymbol{\sigma})|\mathcal{G}_t^{(n)}] = 1\!\!1_{\{\sigma_i \leq t < \sigma_{i+1}\}}\mathbb{E}[Y_T(\boldsymbol{\sigma})|\mathcal{G}_t^{(i)}]$$

- Let $\boldsymbol{\sigma}_{(i+1:n)} = (\sigma_{i+1}, \cdots, \sigma_n)$. Its density $\alpha^{(i+1:n)|i}$ w.r.t. $\mathbb{G}^{(i)}$ can be given explicitly using the $\mathbb{F}$ density of $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$,
- Generalizing the before-default formula to $\boldsymbol{\sigma}_{(i+1:n)}$ and the corresponding reference filtration $\mathcal{G}_t^{(i)}$ implies the result.

Total loss process

- In the top-down models, one works with the cumulative loss of the underlying portfolio defined by

$$L_t := \sum_{i=1}^{n} 1\!\!1_{\{\tau_i \le t\}}.$$

- Loss information: $\mathbb{D}^L = (\mathcal{D}_t^L)_{t \ge 0}$ where $\mathcal{D}_t^L = \sigma(L_s, s \le t)$ including the number of defaults and the timing of past default events.
- It holds $L_t = \sum_{i=1}^{n} 1\!\!1_{\{\sigma_i \le t\}}$ and $\{L_t < k\} = \{\sigma_k > t\}$.
- The same information flow as the ordered defaults:

$$\mathcal{D}_t^{(n)} = \mathcal{D}_t^L, \quad t \ge 0.$$

Successive default intensities and loss intensity

- The $\mathbb{G}^{(k)}$-intensity of σ_k is the $\mathbb{G}^{(k)}$-adapted process λ^k such that $(1\!\!1_{\{\sigma_k \le t\}} - \int_0^t \lambda_s^k ds, t \ge 0)$ is a $\mathbb{G}^{(k)}$-martingale.
- The $\mathbb{G}^{(k)}$-intensity of σ_k coincides with its $\mathbb{G}^{(n)}$-intensity. It is null outside the set $\{\sigma_{k-1} \le t < \sigma_k\}$ and is given as

$$\lambda_t^k = 1\!\!1_{\{\sigma_{k-1} \le t < \sigma_k\}} \lambda_t^{k,\mathbb{F}}(\boldsymbol{\sigma}_{(k-1)})$$

where $\lambda_t^{k,\mathbb{F}}(\boldsymbol{s}_{(k-1)})$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^{k-1})$-measurable.

- The loss intensity, is the $\mathbb{G}^L$-adapted process λ^L such that $(L_t - \int_0^t \lambda_s^L ds, t \ge 0)$ is a $\mathbb{G}^L$-martingale.
- The loss intensity is the sum of the intensities of σ_k, i.e.

$$\lambda_t^L = \sum_{k=1}^n \lambda_t^k, \quad a.s..$$

- Some explicit models: Frey-Backhaus, Arnsdorf-Halperin, Herbertsson, Giesecke and al., etc.

Immersion property

- The immersion holds between $\mathbb{G}^{(i)}$ and $\mathbb{G}^{(i+1)}$ for any $i = 0, \cdots, n-1$ if and only if $\mathbb{F}$ is immersed in $\mathbb{G}^L$ (Ehlers-Schönbucher).
- This is equivalent to

$$\sigma_i = \inf\{t \geq \sigma_{i-1} \ : \int_{\sigma_{i-1}}^{t} \lambda_s^{i,\mathbb{F}}(\boldsymbol{\sigma}_{(i-1)})ds \geq \eta_i\}.$$

where η_i is independent of $\mathbb{G}^{(i-1)}$, hence of $\eta_1, \ldots, \eta_{i-1}$.

- Assuming immersion between $\mathbb{F}$ and $\mathbb{G}^{(n)}$, then the loss distribution is given for $k = 1, \ldots, n-1$ by

$$\mathbb{P}(L_T \leq k|\mathcal{G}_t^L) = 1\!\!1_{\{t<\sigma_{k+1}\}}\mathbb{E}\Big[\exp\{-\int_t^T \lambda_s^{k+1,\mathbb{F}}(\boldsymbol{\sigma}_{(k)})ds\}|\mathcal{G}_t^{(k)}\Big].$$

- This is not true in general.

Joint density and successive intensity

- Under a probability $\mathbb{P}$ where H-hypothesis holds between $\mathbb{F}$ and $\mathbb{G}^L$, then for any $\boldsymbol{\theta} \in \mathbb{R}_+^n$ such that $\theta_1 \leq \cdots \leq \theta_n$ and any $0 \leq t \leq \theta_n$,

$$\alpha_t(\boldsymbol{\theta}) = \mathbb{E}\Big[\prod_{i=1}^{n} \lambda_{\theta_i}^{i,\mathbb{F}}(\boldsymbol{\theta}_{(i-1)}) \exp\Big\{-\int_{\theta_{i-1}}^{\theta_i} \lambda_u^{i,\mathbb{F}}(\boldsymbol{\theta}_{(i-1)})du\Big\}|\mathcal{F}_t\Big] \quad a.s.. \tag{4}$$

  If $t > \theta_n$, then $\alpha_t(\boldsymbol{\theta}) = \alpha_{\theta_n}(\boldsymbol{\theta})$.

- To obtain the whole term structure of the joint density in the general case, it needs a change of probability measure on $\mathbb{G}^L$ and more information.

Bottom-up vs top-down

- The ordered joint density of $\boldsymbol{\sigma}$ can be deduced from the non-ordered one of $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$. .
- Denote by $\beta_t(\boldsymbol{u})$, $t \geq 0$, $\boldsymbol{u} = (u_1, \cdots, u_n) \in \boldsymbol{R}_+^n$ the joint density of $\boldsymbol{\tau}$.
- For any $\boldsymbol{\theta} \in \mathbb{R}_+^n$ such that $\theta_1 \leq \cdots \leq \theta_n$,

$$\alpha_t(\theta_1, \cdots, \theta_n) = 1\!\!1_{\{\theta_1 \leq \cdots \leq \theta_n\}} \sum_{\Pi} \beta_t\big(\theta_{\Pi(1)}, \cdots, \theta_{\Pi(n)}\big) \tag{5}$$

  where $(\Pi(1), \cdots, \Pi(n))$ is a permutation of $(1, \cdots, n)$.
- In particular, if $\boldsymbol{\tau}$ is exchangeable, then

$$\alpha_t(\theta_1, \cdots, \theta_n) = 1\!\!1_{\{\theta_1 \leq \cdots \leq \theta_n\}}\, n!\ \beta_t(\theta_1, \cdots, \theta_n).$$

Concluding remarks

- The density approach can also be applied to non-ordered defaults $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ directly. However, the computation burden is heavy with 2^n default scenarios instead of $n + 1$.
- Several important points for giving explicit models of joint survival probability w.r.t. $\mathbb{F}$:
  - compatibility between the joint probability property and the martingale property
  - describe the correlation structure in a dynamic manner
  - methods: change of probability, diffusing a joint probability function as a martingale, backward construction...
- The density approach allows to give an analysis depending on both the number and the timing of past default events.

Thanks for your attention !