Analytic expansions of characteristic functions and densities, and applications in finance

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Semi-elliptic Cauchy problems in finance (on the domain $[0,T] \times \mathbb{R}^n$):

(i) : Diffusion models of Hörmander type:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i=1}^{m} A_i^2 u + A_0 u \\ \\ u(0,x) = f(x), \end{cases} \tag{1}$$

(ii) CP with Integro-differential operator A

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = Av\,(t,x), \\ \\ v(0,x) = f(x). \end{cases} \tag{2}$$

More precisely:

(i) : vector fields with ($a_{ji} \in C_b^\infty(\mathbb{R}^n)$)

$$A_i = \sum_{j=1}^{n} a_{ji} \frac{\partial}{\partial x_j}, \qquad (3)$$

which satisfy a Hörmander type condition, i.e. for all x the sets

$$\left\{ A_i, \left[A_j, A_k \right], \left[\left[A_j, A_k \right], A_l \right], \cdots \mid 1 \le i \le m,\ 0 \le j, k, l \cdots \le m \right\} \qquad (4)$$

span some subspace of dimension $d \le n$.

(ii) CP with Integro-differential operator A

$$A[f](x) \equiv \frac{1}{2} \sum_{ij} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x) \frac{\partial}{\partial x_i} f(x) + \int \left[f(x+z) - f(x) - \frac{\partial f}{\partial x}(x) \cdot z 1_D(z) \right] \nu(x, dz). \qquad (5)$$

Densities and characteristic functions:

(i) Densities (for time-homogenous models) satisfy:

$$\begin{cases} \frac{\partial p}{\partial t} = \frac{1}{2}\sum_{i=1}^{m} A_i^2 p + A_0 p \\ \\ p(0,x,y) = \delta_y(x), \end{cases} \tag{6}$$

(ii) CP for characteristic function by Fourier-transform w.r.t. y

$$\begin{cases} \frac{\partial \widehat{p}}{\partial t}(t,x,u) = A\widehat{p}\,(t,x,u), \\ \\ \widehat{p}(0,x,u) = \exp(iux). \end{cases} \tag{7}$$

Probabilistic context for (i)

Theorem 1 *Let* $T > 0$ *and let*

$$b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \ \sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$$

be measurable functions; for $x \in \mathbb{R}^n, \ t \in [0, T]$

$$|b(t, x)| + |\sigma(t, x)| \leq C(t + |x|), \tag{8}$$

for some constant generic $C > 0$*, and wher* $|\sigma(t, x)| = \sqrt{\sum_{ij} |\sigma_{ij}|^2}$ *(*$|.|$ *denoting the Euclidean norm). Assume that for* $x \in \mathbb{R}^n, \ t \in [0, T]$

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq C|x - y|. \tag{9}$$

Then the stochastic differential equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t),$$

$$X(0) = x \tag{10}$$

has a global unique t*-continuous solution.*

However densities may exist only in a distributional sense for semi-ellitic problems even if

$$b_i, \sigma_{ij} \in C_b^\infty \left([0,T] \times \mathbb{R}^n \right).$$

Consider

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x_1^2} + \mu \frac{\partial u}{\partial x_2}, \\ \\ u(0,x) = f(x_1) + g(x_2). \end{cases} \tag{11}$$

The solution of this equation is

$$\begin{aligned} &\textstyle\int_{\mathbb{R}} f(y_1) \frac{1}{\sqrt{2\pi t^2}} \exp\left(-\frac{(x_1-y_1)^2}{2\sigma^2 t} \right) dy_1 \\ &+ g(x_2 + \mu t). \end{aligned} \tag{12}$$

This leads us to a ‘distributional density’ of the form

$$\begin{aligned} p(t,x,y) := \quad & \frac{1}{\sqrt{2\pi t^2}} \exp\left(-\frac{(x_1-y_1)^2}{2\sigma^2 t} \right) \times \\ & \delta(x_2 + \mu t - y_2). \end{aligned} \tag{13}$$

Indeed, formally we have (let us denote $dy = dy_1 dy_2$)

$$\begin{aligned} & \int \left(f(y_1) + g(y_2)\right) p(t,x,y) dy = \\ & \int f(y_1) \frac{1}{\sqrt{2\pi t^2}} \exp\left(-\frac{(x_1-y_1)^2}{2\sigma^2 t}\right) \delta(x_2 + \mu t - y_2) dy_1 dy_2 \\ + & \int g(y_2) \frac{1}{\sqrt{2\pi t^2}} \exp\left(-\frac{(x_1-y_1)^2}{2\sigma^2 t}\right) \delta(x_2 + \mu t - y_2) dy_1 dy_2 \\ = & \int f(y_1) \frac{1}{\sqrt{2\pi t^2}} \exp\left(-\frac{(x_1-y_1)^2}{2\sigma^2 t}\right) dy_1 + g(x_2 + \mu t - y_2) \\ = & \; u(t,x), \end{aligned} \tag{14}$$

If Hörmander's condition is satisfied, i.e. (4) holds for the whole space $\mathbb{R}^n$ for all $x \in \mathbb{R}^n$, then the operator A is microhypoelliptic, i.e.

$$\mathrm{WF}(u) \;=\; \mathrm{WF}(Au), \tag{15}$$

where WF(u) denotes the wave front set of a distribution u.

Especially it is hypoelliptic (recall that a differential operator L with C^∞-coefficients is called hypoelliptic on an open set $\Omega \subseteq \mathbb{R}^n$ if for any distribution u on Ω is in C^∞ if $Lu \in C^\infty$.)

Now, for u with

$$u(t,x) = \int_{\mathbb{R}} f(y_1)\frac{1}{\sqrt{2\pi t^2}}\exp\left(-\frac{(x_1-y_1)^2}{2\sigma^2 t}\right)dy_1$$

$$+g(x_2+\mu t). \tag{16}$$

we have

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x_1^2} - \mu\frac{\partial u}{\partial x_2} = 0, \tag{17}$$

where g may be C^1, and this shows that the semi-elliptic operators considered above are not hypoelliptic in general.

First motivation for semi-elliptic market models which are not micro-hypoelliptic:

Factor reduced market models:

Example: Libor market model.

- tenor structure $0 < T_1 \ldots < T_{n+1}$.

- dynamics of the forward Libors $L_i(t)$, in the interval $[0, T_i]$ for $1 \leq i \leq n$, is described by

$$dL_i = -\sum_{j=i+1}^{n} \frac{\delta_j L_i L_j \, \gamma_i^\top \gamma_j}{1+\delta_j L_j} \, dt + L_i \, \gamma_i^\top dW^{(n+1)}$$

$$=: \mu_i^L(t, L) L_i + L_i \, \gamma_i^\top dW^{(n+1)},$$

- here, $\delta_i = T_{i+1} - T_i$ and

$$t \to \gamma_i(t, L) = (\gamma_{i,1}(t, L), \ldots, \gamma_{i,d}(t, L)),$$

  with $0 \leq t \leq T_i$ volatility vector functions (in classical models dependence only on t).

Reduced Libor factor models

- Consider logarithmic coordinates $K = \ln(L)$.

- governing equation

$$dK = \mu^K(t, K)dt + FdW,$$

where F is a $n \times d$ matrix with $d \leq n$

(e.g. 40×5 matrix)

- Note that

$$FF^T$$

is a $n \times n$ matrix of rank d while $F^T F$ is a $d \times d$ matrix of rank d.

- let $G = (f_{d+1}, \cdots f_n)$ be the matrix consisting of the eigenvectors f_i of

$$\text{ker}\left(FF^T\right)$$

- equivalent system is

$$\begin{cases} d\left(F^T K\right) = F^T \mu^K(t,K)dt + F^T F dW^{(n+1)} \\ \\ d(G^T K) = G^T \mu(t,K)dt, \end{cases}$$

- note that $G^T F = 0$.

Reasons for reduced market models in practice:

- Computational parsimony $\rightarrow$ in practice, models with n factors but lower number $k < n$ of Brownian motions appear
$\Rightarrow$ semi-ellipticity, but not hypo-ellipticity.

- financial models may be high-dimensional; especially Libor market models may have 20-40 factors. Experience shows that more then $5-7$ factors induce numerical noise

- empirically $5-7$ factors are enough to calibrate the model

Second example for semi-elliptic market models where iterated subproblems are not micro-hypoelliptic:
American derivative on a hypoelliptic market model

Consider the Cauchy problem

$$\begin{cases} \max\left\{\frac{\partial u}{\partial t} + Lu, \phi - u\right\} = 0, \text{ in } [0,T) \times \mathbb{R}^n \\ \\ u(T,x) = \phi(T,x), \text{ in } \{T\} \times \mathbb{R}^n \end{cases} \tag{18}$$

on $[0,T] \times \mathbb{R}^n$ (where $T > 0$ is an arbitrary finite horizon)

$$L \equiv \frac{1}{2}\sum_{i=1}^{m} V_i^2 u + V_0 u \tag{19}$$

This may be rewritten in the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{n} v_{ij}^*(t,x)\frac{\partial^2}{\partial x_i \partial x_j}u + \sum_{j=1}^{n} v_{j0}(t,x)\frac{\partial u}{\partial x_j}, \tag{20}$$

where

$$\left(v_{ij}^{*}\right)(t,x) = \sum_{k=1}^{m} (V_i)^{\otimes^2}. \qquad (21)$$

Assume that the Hörmander condition holds in the continuation region $\mathcal{C}$, i.e., for all $(t,x) \in \mathcal{C} \subset [0,T] \times \mathbb{R}^n$

$$\begin{aligned} &\Big\{V_i, \left[V_j, V_k\right], \left[\left[V_j, V_k\right], V_l\right], \\ &\cdots | 1 \le i \le m,\ 0 \le j,k,l \cdots \le m\Big\} \end{aligned} \qquad (22)$$

spans $\mathbb{R}^n$.

Consider lognormal coordinates $S = (S_1, \cdots, S_n)$ and assume that a multivariate Amercan derivative is convex with respect to the spatial variables. Then the following assumption is satisfied.

(GG) for each $t \in [0, T]$ we assume that $0 \in \mathcal{E}_t$ and that $\mathcal{E}_t$ is star-shaped with respect to 0, i.e. for all $S \in \mathcal{E}_t$ and all $\lambda \in [0, 1]$ we assume that $\lambda S \in \mathcal{E}_t$. Note that this means that for a fixed "angle" at $S = (S_1, \cdots, S_n)$, i.e. at

$$\phi_S := \left(\frac{S_2}{\sum_{i=1}^n S_i}, \cdots, \frac{S_n}{\sum_{i=1}^n S_i} \right)$$

we have one intersection point of the free boundary of the section $\mathcal{E}_t$ and the ray through 0 which is determined by the angle ϕ_S.

Indeed, this often follows from arbitrage considerations. For example consider the American index Put

$$p_A(t, S; K) = \sup_{\tau \in \mathsf{Stop}_{[0,T]}} E_Q\left(K - \sum_{j=1}^{n} S_j\right). \tag{23}$$

Observe that

$$(t, S; K) \to p_A(t, S; K) \tag{24}$$

is homogenous of degree 1 with respect to (S, K), i.e. for any $\lambda > 0$ we have for all $t \in [0, T]$

$$p_A(t, \lambda S; \lambda K) = \lambda p_A(t, S; K). \tag{25}$$

Convexity with respect to K then implies convexity with respect to S.

Hence, the free boundary can be written in terms of the angles in form

$$(t, \phi_S) \rightarrow F(t, \phi_S). \tag{26}$$

We consider the transformation

$$\psi : (0, T) \times \mathbb{R}^n_+ \rightarrow (0, T) \times [1, \infty) \times (0, 1)^{n-1},$$

$$\psi(t, S_1, \cdots, S_n) = \left(t, \frac{\sum_{i=1}^n S_i}{F}, \frac{S_2}{\sum_{i=1}^n S_i}, \cdots, \frac{S_n}{\sum_{i=1}^n S_i}\right). \tag{27}$$

Continuation region is transformed to the half space $H_{\geq 1} = \{x \in \mathbb{R}^n | x_1 \geq 1\}$. We have

$$S_1 = x_1 F \left(1 - \sum_{j \geq 2} x_j\right), \; S_j = x_j x_1 F. \tag{28}$$

We get

$$\begin{cases} u_t = \frac{F_t}{F} x_1 \frac{\partial u}{\partial x_1} + \frac{1}{2}\sum_{ij} a^F_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_j b^F_j \frac{\partial u}{\partial x_j} \\ +r\left(x_1 \frac{\partial u}{\partial x_1} - u\right), \\ (BC1) \quad u(0,\infty,x_2,\cdots,x_n) = 0 \text{ on } x_1 = \infty \\ (BC2) \quad u_{x_1}(t,1,x_2,\cdots,x_n) - u(t,1,x_2,\cdots,x_n) \\ \qquad\qquad = -K \qquad\qquad\qquad\qquad \text{on } x_1 = 1 \\ (BC3) \quad F(t,x_2,\cdots,x_n) = K - u(t,1,x_2,\cdots,x_n) \\ (IC) \quad u(0,x) = \max\{K - x_1, 0\} \end{cases} \tag{29}$$

You compute

$$F\frac{\partial x_j}{\partial S_i} = \frac{\delta_{ij}-x_j}{x_1},$$

$$F\frac{\partial x_j}{\partial S_1} = 1 - \sum_{j\geq 2}(\delta_{ij} - x_j)\frac{F_j}{F}, \qquad (30)$$

$$\frac{\partial}{\partial S_i} = \sum_j \frac{\partial x_j}{\partial S_i}\frac{\partial}{\partial x_j},$$

and observe that

$$\sum_i S_i \frac{\partial x_j}{\partial S_i} = \sum_i S_i \frac{\delta_{ij} - x_j}{x_1 F} = 0. \qquad (31)$$

It follows that

$$r\left(\sum_i S_i \frac{\partial}{\partial S_i}\right) = r x_1 \frac{\partial}{\partial x_1}. \qquad (32)$$

Furthermore

$$a_{ij}^F = \sum_{kl} v_{kl} S_k S_l \frac{\partial x_i}{\partial S_l}\frac{\partial x_l}{\partial S_k},$$
$$b_j^F = \sum_{kl} v_{kl} S_k S_l \frac{\partial^2 x_j}{\partial S_k \partial S_l}. \tag{33}$$

In order to determine the latter coefficient functions we compute

$$\frac{\partial x_j}{\partial S_i} = \frac{1}{F}\frac{\delta_{ij} - x_j}{x_1}, j \geq 2, \quad \frac{\partial x_1}{\partial S_i} = \frac{1}{F}\left(1 - \sum_{j \geq 2} (\delta_{ij} - x_j)\frac{F_j}{F}\right) \tag{34}$$

Next, for $j \geq 2$ we have

$$\frac{\partial^2 x_j}{\partial S_i \partial S_k} = \sum_l \frac{\partial \left(\frac{\delta_{ij} - x_j}{F x_1}\right)}{\partial x_l}, \text{ and} \tag{35}$$

$$\frac{\partial \left(\frac{\delta_{ij} - x_j}{F x_1}\right)}{\partial x_l} = -\frac{\delta_{jl}}{x_1 F} + \frac{(x_j - \delta_{ij})(\delta_{1l} F + x_1 F_l (1 - \delta_{1l}))}{(x_1 F)^2}. \tag{36}$$

Finally,

$$\frac{\partial^2 x_1}{\partial S_i \partial S_k} = \sum_l \frac{\partial}{\partial x_l} \left(\frac{1}{F} - \sum_{j \geq 2} (\delta_{ij} - x_j) \frac{F_j}{F^2} \right) \frac{\partial x_l}{\partial S_k}, \tag{37}$$

where

$$\frac{\partial}{\partial x_l} \left(\frac{1}{F} - \sum_{j\geq 2} (\delta_{ij} - x_j) \frac{F_j}{F^2} \right)$$

$$= -\frac{F_l}{F^2}(1 - \delta_{1l})$$

$$- \sum_{j\geq 2} \frac{-\delta_{jl} F_j + (\delta_{ij} - x_j) F_{jl} (1 - \delta_{1l}) - 2 F_j F_l (1 - \delta_{1l})(\delta_{ij} - x_j)}{F^3}. \tag{38}$$

Special interest in densities and characteristic functions of semi-elliptic models in finance

- Computing Greeks for higher dimensional diffusion problems

- higher dimensional models $\rightarrow$ financial quantities are computed via probabilistic representations and MC methods

- Derivative values: $E(f(X^\lambda))$

  (X^λ a process depending on some parameters λ)

- of special interest: Sensitivities

$$\frac{\partial}{\partial \lambda} E(f(X^\lambda))$$

  (also with respect to underlyings)

- performance and accuracy may be enhanced by using weighted MC methods with

$$E(f(X^\lambda)\pi),$$

where the random variable π is called a *Greek weight.*

- in practice diffusion models (without jumps) are of particular interest
(issues of calibration)

- Computational parsimony $\to$ in practice, models with d factors but lower number $k < d$ of Brownian motions appear
$\Rightarrow$ semi-ellipticity!

Problem: for a reasonable class of models set up a weighted MC scheme for derivative values and sensitivities

- which is computationally parsimonious (hence include sem-ielliptic models)

- has bounded variance (hence sensitivities have to be treated carefully)

- uses optimal weights (we have to go beyond the standard Malliavin calculus based schemes)

Before we establish weak higher order probabilistic schemes we need analytic expansions. Density expansions and expansions of characteristic functions have different advantages /disadvantages.

- analytic expansions of the characteristic expansion may include jump-diffusion models

- they are (more or less) restricted to affine processes

- density expansions are possible for nonlinear coefficients

- expansions of jump-diffusion beyond affine processes converge only on cones

Let's start with expansions of characteristic functions:

Bochner told us to investigate Markov processes by the symbols of their associated infinitesimal generators.

Consider a regular Feller process

$$\left((X_t^x)_{t\geq 0}, P^x\right)_{x\in\mathbb{R}^d}$$

with values in $\mathbb{R}^d$.

The function

$$\sigma(x,\xi) := -\lim_{t\downarrow 0} \frac{E^x\left(e^{i(X_t^x - x)\cdot\xi}\right) - 1}{t} \tag{39}$$

is called the symbol of the process where

$$c_t(x,\xi) = E^x\left(e^{i(X_t^x - x)\cdot\xi}\right) \tag{40}$$

is called the characteristic function.

Now, if $(T_t)_{t\geq 0}$ is called the semigroup associated with $\left((X_t^x)_{t\geq 0}, P^x\right)_{x\in\mathbb{R}^n}$, then we have

$$T_t f(x) = E^x\left(f(X_t^x)\right) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\xi} c_t(x,\xi)\hat{f}(\xi)d\xi, \tag{41}$$

where $\hat{\cdot}$ denotes the Fourier transform.

It follows that the generator takes the form

$$Au(x) = \lim_{t\downarrow\infty}\frac{T_t u(x)-u(x)}{t} = -(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi \tag{42}$$

Hence the symbol completely characterizes the process.

Goal: recovering the characteristic function from its symbol function.

Remarks:

- goal is far from trivial, it is helpful that the wave functions $\exp(iux)$ from a set of analytic vectors for a consirable class of affine operators

- it is far from trivial to say under which condition the symbol of a given partial integro-differential operator corresponds to a Markov process. Hörmanders standard class of symbols of pseudo differential operators, namely functions $(x,\xi) \to \sigma(x,\xi)$ which satisfy
$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leq c_{\alpha\beta,K}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \tag{43}$$
where $0 \leq \delta < \rho < 1$ do not fit very well in order to analyze this question.

For processes X where for each $t > 0$ X_t has an infinitely divisible distribution (such processes correspond Levy processes in law), we have the Levy-Kintchine formula, i.e.

$$\mathbb{E}[e^{i\langle u, X_t^x - x\rangle}] :=$$

$$e^{-\frac{1}{2}\langle Au,u\rangle t + i\langle r,u\rangle t + t\int_{\mathbb{R}^d}\left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle 1_D(x)\nu(dx)\right)}, \tag{44}$$

where A is a symmetric nonnegative $d \times d$ matrix, $r \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$ satisfying

$$\begin{aligned} &\nu(\{0\}) = 0, \text{ and} \\ &\int_{\mathbb{R}^d \setminus \{0\}} \min\left\{|y|^2, 1\right\} \nu(dy) < \infty. \end{aligned} \tag{45}$$

Remarks:

- For affine processes explicit formulas for the characteristic function are known only in special cases (cf. the example in Duffie, Pan, Singleton (2000) for the case of affine processes without jumps).

- In the general case the computation of the (conditional) Fourier transform of an affine process is shown to be reducible to solving systems of general Riccati differential equations (Duffie, D., Filipović, D. and Schachermayer, W. (2003)). Analytical solutions of the generalized Riccati differential equation systems are not known, and even numerically they are often difficult to solve, especially due to quadratic terms involved.

- In Belomestny, Kampen Schoenmakers (2009) it is shown that for a class of affine equations a basis of analytic vectors can be found. In particular, this leads to recursive representations of characteristic functions in terms of constituents of the generator avoiding the generalized Riccati equations.

For affine process $X^{0,x}$ with $X_0^{0,x} = x$ we search for an explicit representation of $\widehat{p}(t, x, u) = E[e^{iuX_t^{0,x}}]$. Here we denote $e^{iux} := e^{\sum_{j=1}^d u_j x_j}$ for simplicity of notation.

We assume that the function $\widehat{p}$ is a global solution of the Cauchy problem

$$\begin{cases} \frac{\partial \widehat{p}}{\partial t}(t,x,u) = A\widehat{p}\,(t,x,u), \\ \\ \widehat{p}(0,x,u) = \exp(iux), \quad t \geq 0, \quad x \in \Omega \subset \mathbb{R}^d, \end{cases} \tag{46}$$

where Ω is some domain and

$$A[f](x) \equiv \frac{1}{2}\sum_{ij} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x)\frac{\partial}{\partial x_i} f(x) + \int_{\mathbb{R}^d \setminus \{0\}} \left[f(x+z) - f(x) - \frac{\partial f}{\partial x}(x)\cdot z 1_D(z)\right] \nu(x,dz). \tag{47}$$

Here we assume that the equation is semielliptic, i.e.

$$(a_{ij}) \geq 0,$$

and we assume that the operator is closed on some appropriate function space.

Furthermore 1_D is a indicator function of a bounded set D, $a_{ij}(x)$, $b_i(x)$ are affine functions, and $\nu(x,dz)$ is a Borel measure which is affine in x, i.e. we have the representations

$$a_{ij}(x) = a_{ij0} + \sum_{l=1}^d a_{ijl} x_l$$

$$b_i(x) = b_{i0} + \sum_{l=1}^d b_{il} x_l$$

$$\nu(x,dz) = 1_{\Omega_0}\nu_0(dz) + \sum_{l=1}^d x_l 1_{\Omega_l}\nu_l(dz), \quad (48)$$

where $\Omega_0, \Omega_l \subset \mathbb{R}^d$, $1 \le l \le d$. We define the symbol function

$$\sigma : \Omega \times i\mathbb{R}^d \to \mathbb{C},$$

$$\sigma(x, iu) := \exp(-iux)A[\exp iux] \equiv$$

$$-\frac{1}{2}\sum_{jk} a_{jk}(x)u_j u_k + \sum_j b_j(x)iu_j + \exp(-iux)\times$$

$$\int_{\mathbb{R}^d\setminus\{0\}}[\exp(iu(x+z)) - \exp(iux)$$

$$-iu\exp(iux)\cdot z1_D(z)]\nu(x, dz) =$$

$$-\frac{1}{2}\sum_{jk} a_{jk}(x)u_j u_k + \sum_j b_j(x)iu_j +$$

$$\int_{\mathbb{R}^d\setminus\{0\}}[\exp(iuz) - 1 - iu\cdot z1_D(z)]\,\nu(x, dz) \tag{49}$$

Since σ is affine with respect to x it is useful to define

$$\sigma_l(iu) := \partial_{x_l}\sigma(x,iu) = -\tfrac{1}{2}\sum_{jk} a_{jkl}u_ju_k + \sum_j b_{jl}iu_j$$
$$+\sum_{l=1}^{d}\int_{\mathbb{R}^d\setminus\{0\}}\left[\exp(iuz) - 1 - iu\cdot z1_D(z)\right]\nu_l(dz) \tag{50}$$

In the case $d = 1$ we define

$$\sigma_1(iu) := \frac{\partial}{\partial\xi}\sigma(x,iu). \tag{51}$$

In general we denote multivariate derivatives of order $\alpha = (\alpha_1, \cdots, \alpha_d)$ of the symbol function with respect to the symbol variable ξ by $\partial_\xi^\alpha\sigma$.

Plugging in

$$\widehat{p}\,(t,x,u) = \mathrm{exp}(iux)\left(1+\sum_k d_k t^k\right) \qquad (52)$$

we get

$$\frac{\partial \widehat{p}}{\partial t}(t,x,u) = \mathrm{exp}(iux)\left(\textstyle\sum_{k\geq 0} d_{k+1} t^k\right)$$

$$= \textstyle\sum_\alpha \beta_\alpha(x) D_x^\alpha \widehat{p}(t,x,y)$$

$$= \textstyle\sum_\alpha \beta_\alpha(x) D_x^\alpha \left(\mathrm{exp}(iux)\left(\sum_{k\geq 0} d_{k+1} t^k\right)\right) \qquad (53)$$

with $d_0 = 1$.

This leads to the recursion $d = 1$ and for $k \geq 0$

$$d_{k+1} = \frac{1}{k+1} \exp(-iux) \sum_{\alpha \geq 0} \beta_\alpha(x) D_x^\alpha \left(\exp(iux) d_k\right) \tag{54}$$

For example in the univariate case one computes

$$\begin{aligned}
d_0 &= 1 \\
d_1 &= \sigma(x, iu) \\
d_2 &= \tfrac{1}{2}\sigma(x, iu)^2 + \tfrac{1}{2}(\partial_\xi \sigma(x, iu))\sigma_1(iu) \\
d_3 &= \tfrac{1}{3!}\sigma(x, iu)^3 + \tfrac{3}{3!}\sigma(x, iu)(\partial_\xi \sigma(x, iu))\sigma_1(iu) \\
&\quad \tfrac{1}{3!}(\partial_\xi \sigma(x, iu))(\partial_\xi(\sigma_1(iu))\sigma_1(iu) + \tfrac{1}{3!}\left(\partial_\xi^2 \sigma(x, iu)\right)
\end{aligned} \tag{55}$$

Higher order terms are easy to compute as well.

The formula is coded by an affine triangle of rational numbers with kth row all numbers $c_{(\alpha^k,\beta^k)}$ where α^k, β^k are the k-tuples. We call that row the univariate affine triangle of the caracteristic function. The first row (terms of order 1) contains just one number

$$c_{(1,0)} = 1 \tag{56}$$

(note that the tuple $(0,1) \notin M_1$). The second row contains two numbers

$$c_{((2,0),(0,0))} = \frac{1}{2}, \quad c_{((0,1),(1,0))} = \frac{1}{2} \tag{57}$$

The third row contains four numbers

$$\begin{aligned} c_{((3,0,0),(0,0,0))} = \tfrac{1}{6} \quad c_{((1,1,0),(1,0,0))} = \tfrac{1}{2} \\ c_{((0,1,0),(1,1,0))} = \tfrac{1}{6} \quad c_{((0,0,2),(2,0,0))} = \tfrac{1}{6}, \end{aligned} \tag{58}$$

and so on.

We introduce the following multiindex notation. For each positive integer k we denote

$$\alpha^k = (\alpha^k_0, \cdots, \alpha^k_{k-1}), \quad \beta^k = (\beta^k_0, \cdots, \beta^k_{k-1}). \tag{59}$$

where the entries α^k_j, β^k_j are nonnegative integers. For any positive integer k the projection of a k-tuples α^k (resp. β^k on a l-tuple for an nonnegative integer $l \leq k$ will be denoted by Π^k_l with

$$\Pi^k_l \alpha^k = (\alpha^k_0, \cdots, \alpha^k_{l-1}), \text{ (resp. } \Pi^k_l \beta^k = (\beta^k_0, \cdots, \beta^k_{l-1})). \tag{60}$$

Adding or subtracting integers to such multiindices is denoted as follows:

$$\alpha^k + i_j = (\alpha^k_0, \cdots, \alpha^k_j + i_j, \cdots, \alpha^k_{k-1}). \tag{61}$$

Similar for β^k and for subtraction, i.e. we have

$$\alpha^k - i_j = (\alpha^k_0, \cdots, \alpha^k_j - i_j, \cdots, \alpha^k_{k-1}). \tag{62}$$

We have

Theorem 2 *Locally, the following representation holds:*

$$\widehat{p}(t,x,u) = \mathsf{exp}(iux)\times$$

$$(1+\sum_{(\alpha^k,\beta^k)\in M^k} c_{(\alpha^k,\beta^k)} \Pi_{j=0}^{k-1} \left(\partial_\xi^j \sigma(x,iu)\right)^{\alpha_j^k}$$

$$\left(\partial_\xi^j \sigma_1(iu)\right)^{\beta_j^k} t^k), \tag{63}$$

where $\alpha^{)}$ *are* k*-tuples with*

$$M^k = \left\{ (\alpha^k,\beta^k) \Big| \sum_{j=0}^{k-1} \left(\alpha_j^k + \beta_j^k\right) = k \ \& \ \sum_{j=0}^{k-1} \beta_j^k \geq \sum_{j=1}^{k-1} \alpha_j^k \right\} \tag{64}$$

and

$$c_{(1,0)} = 1$$

and for $k \geq 2$

$$c_{(\alpha^{k+1},\beta^{k+1})} = \frac{1}{(k+1)!} \quad \textit{if} \quad \alpha_0^{k+1} = k+1$$

$$c_{(\alpha^{k+1},\beta^{k+1})} = \frac{1}{(k+1)!}\times \tag{65}$$

$$\sum_{j=0}^{k-1} \sum_{\sum_{i=0}^{k-1} \lambda_i^k = j} \binom{\sqcap_k \alpha^{k+1} - 1_j + \lambda^k}{\lambda_k} \times$$

$$c_{(\sqcap_k \alpha^{k+1} - 1_j + \lambda^k, \sqcap_k \beta^{k+1} - \lambda^k)}.$$

Here, the natural restriction holds that $\lambda^k = (\lambda_0^k, \cdots, \lambda_{k-1}^k)$ *are* k*-tuples such that*

$$\sqcap_k \alpha^{k+1} - 1_j + \lambda^k \in M_k. \tag{66}$$

(Alternatively we could define $c_{(\alpha^k,\beta^k)}$ *with pairs of* k*-tuples* (α^k, β^k) *not in* M_k *to be zero, of course).*

Such triangles can be derived also in the multivariate case.

But this looks rather general. Why can't we define integer triangles for general Cauchy problems?

Let us consider the one dimensional sphere $S^1 = [-\pi, \pi]$ and define function $p : [0, T) \times S^1 \times \mathbb{R} \to \mathbb{C}$ via

$$p(t, x; u) = e^{Lt} e^{ixu}, \tag{67}$$

where

$$L \equiv L^{(a,b,\lambda)} \equiv \frac{1}{2} a(x) \frac{\partial^2}{\partial x^2} - b(x) \frac{\partial}{\partial x} - \lambda(x) \sum_{l \geq 2} \frac{\gamma_l}{l!} \frac{\partial^l}{\partial x^l}. \tag{68}$$

with the moments $\gamma_l = \int_{\mathbb{R}} y^l \nu(dy)$.

We consider the standard topology induced by the scalar product (and its associated metric) on the set squared integrable functions $L^2(S^1)$ and consider the function space

$$\mathcal{F} = \Big\{(a, b, \lambda) \in L^2(S^1) \times L^2(S^1) \times L^2(S^1)\Big\} \tag{69}$$

equipped with the product topology $\mathcal{T}_p$. The following proposition holds

Theorem 3 *The set*

$$\mathcal{F}_{\textit{div}} = \Big\{(a, b, \lambda) \in \mathcal{F} | \quad e^{tL^{(a,b,\lambda)}} e^{ixu} \text{ is unbounded on any neighborhood of } 0\Big\} \tag{70}$$

is dense in $(\mathcal{F}, \mathcal{T}_p)$.

Proof. We consider the essential case where $b = 0$ and $\lambda = 0$ and prove denseness of the set

$$\mathcal{F}_{\mathsf{div}_0} = \Big\{ a \in L^2(S^1) \quad e^{tL^{(a,0,0)}} e^{ixu}$$

$$\text{is unbounded on any neighborhood of } 0 \Big\} \tag{71}$$

in $L^2(S^1)$ equipped with the topology induced by the L^2-norm. The general case can be proved along the same ideas and is only formally more complicated. Given any function $a \in L^2(S^1)$ it is well known that we can approximate it in L^2-sense by finite Fourier series

$$a(x) \approx \sum_{l=1}^{n} a_l e^{ilx}. \tag{72}$$

For given $\epsilon > 0$ we choose amplitudes a_l ($a_n \neq 0$) such that

$$\|a(x) - \sum_{l=1}^{n} a_l e^{ilx}\|_{L^2(S^1)} \leq \epsilon. \tag{73}$$

The corresponding approximating operator (of $a(x)\frac{\partial^2}{\partial x^2}$) is

$$A = \sum_{l=1}^{n} A_l = \sum_{l=1}^{n} a_l e^{ilx} \frac{\partial^2}{\partial x^2} \tag{74}$$

Write formally

$$\begin{aligned} e^{At}e^{iux} &= \textstyle\sum_{k=0}^{\infty} \frac{(\sum_{l=1}^{n} A_l)^k t^k}{k!} e^{iux} = \\ &\textstyle\sum_{k=0}^{\infty} \frac{(\sum_{1\le i_1\le\cdots i_k\le n} A_{i_1}\cdots A_{i_k})t^k}{k!} e^{iux}. \end{aligned} \tag{75}$$

Using the fact that for any $s_1, \ldots, s_k \in \mathbb{N}$

$$\begin{aligned} &A_{s_1}\cdots A_{s_k} e^{iux} = (-1)^k a_{s_1}\cdots a_{s_k} \\ &\textstyle\prod_{l=0}^{k-1}\left(u + \sum_{j=0}^{l} s_j\right)^2 e^{i(u+\sum_{j=1}^{k} s_j)x}, \end{aligned} \tag{76}$$

we can show that

$$S_N(x) = e^{-iux} \sum_{k=0}^{N} \frac{(\sum_{l=1}^{n} A_l)^k t^k}{k!} \qquad (77)$$

diverges for fixed n and $u > 0$. Indeed, set

$$F_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} S_N(x) dx, \quad k \in \mathbb{N} \qquad (78)$$

then $F_k = 0$ for $k > nN$ and

$$F_{nN} = \frac{(-1)^N a_n^N \prod_{l=0}^{N} (u + ln)^2 t^N}{N!} \to \infty, \quad N \to \infty. \qquad (79)$$

Due to Placherel Parseval identity

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |S_N(x)|^2 \, dx = \sum_{k=0}^{nN} |F_k|^2 \to \infty, \quad N \to \infty.$$

End Proof.

Remarks:

- Local convergence can be shown by counting terms

- for global formulas in space combine this with partitions of unity

- it is possible to derive globally recursively explicit formulas in time

Next let us look at densities:

The well-known expansion

$$p(t, x; 0, y) = \frac{1}{\sqrt{4\pi t}^n} \exp\left(-\frac{d_R^2}{4t}\right) \left(\sum_{k=0}^{\infty} d_k(t, x, y) t^k\right). \tag{80}$$

has some advantages compared to the WKB-expansion

$$p(t, x; 0, y) = \frac{1}{\sqrt{4\pi t}^n} \exp\left(-\frac{d_R^2}{4t} + \sum_{k=0}^{\infty} c_k(t, x, y) t^k\right). \tag{81}$$

Indeed the connection between the c_k and the d_k can be easily computed using Leibniz rule. We have

$$d_0 = \exp(c_0), \tag{82}$$

and

$$d_k = \sum_{i=1}^{k} \frac{i}{k} d_{k-i} c_i. \tag{83}$$

Since d_k recursions diificult to solve, compute first the c_k from WKB and then d_k recursively from c_k.

We have

Theorem 4 *Given coefficents with bounded analytic coefficients there exists a finite time horizon T_0 such that on the domain $\Omega \times (0, T_0]$ for any finite $T_0 > 0$ and any domain $\Omega \subseteq \mathbb{R}^n$ a constant β can be computed such that the fundamental solution of*

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} \tag{84}$$

has the pointwise valid representation

$$p(t,x,0,y) = \frac{1}{\sqrt{4\pi t}^n} \exp\left(-\frac{\sum_{i=1}^n \Delta x_i^2}{4t}\right)\left(\sum_{k=0}^{\infty} d_k(t,x,y)t^k\right), \tag{85}$$

for $j = 1, \cdots, n$, and for $(t,x) \in (0, \beta T_0) \times \Omega$. If the coefficients are given by finite Fourier representations a lower bound of the constant β is given by

$$\beta < \frac{1}{3(n(2|m_0|+1))\overline{e}R^2|m_0|^2}, \tag{86}$$

where

- $2|m_0|+1$ *is (an upper bound of) the number of terms in the finite Fourier representation of* b_i *along with* $|m_0| := \max_{j\in\{0,\cdots n\}} m_0^j$

- R *is a radius of a ball* $B_R(0)$ *such that the spatial part of the domain* Ω *is included*

- $\overline{e}$ *is an upper bound for the Fourier coefficients of the Fourier representation of the drift function* b_i *where* $i \in \{1,\cdots,n\}$.

For the coefficient functions d_k *we have*

$$d_0(t,x,y) =$$

$$\exp\left(\sum_m (y_m - x_m)\int_0^1 b_m(t,y+s(x-y))ds\right), \tag{87}$$

$$d_m(t,x,y) = \sum_{k=1}^{m} \frac{k}{m} d_{m-k} \int_0^1 R_{k-1}(t, y + s(x-y), y) s^k ds \tag{88}$$

with

$$R_{k-1}(t,x,y) = \frac{\partial}{\partial t} c_{k-1} + \Delta c_{k-1} + \sum_{l=1}^{n} \sum_{r=0}^{k-1} \left(\frac{\partial}{\partial x_l} c_r \frac{\partial}{\partial x_l} c_{k-1-r} \right) + \sum_i b_i(x) \frac{\partial}{\partial x_i} c_{k-1} \tag{89}$$

Note that the relation (86) contains also the relation of the time horizon of the convergence to the size of the domain, i.e. the time horizon is proportional to the inverse of the square of the domain.

Theorem 5 *More explicitly, we have*

$$c_0(t,x,y) = c_0(x,y) =$$

$$-\sum_i \sum_\gamma b_{i\gamma}(y)\Delta x^{\gamma+1_i}\frac{1}{1+|\gamma|} \tag{90}$$

$$\equiv \sum_\gamma c_{0\gamma}\Delta x^\gamma$$

and, given the power series representation

$$c_{k-1}(t,x,y) = \sum_{\gamma,l} c_{(k-1)\gamma l}(y)\Delta x^\gamma t^l \tag{91}$$

we have

$$c_k(t,x,y) = \sum_{\gamma,l} l c_{(k-1)\gamma l}(y)\Delta x^\gamma t^l +$$

$$\sum_\gamma \{ \sum_i \sum_{\rho+\alpha=\gamma}(\rho_i+1)(\alpha_i+1)c_{r(\beta+1_i)}c_{(k-1-r)(\alpha+1_i)}$$

$$+\sum_i(\gamma_i+2)(\gamma_i+1)c_{k(\gamma+2_i)} + \sum_{\rho+\alpha=\gamma}(\sum \frac{1}{\beta!}b_i(y)\times$$

$$(\alpha_i+1)c_{(k-1)(\alpha+1_i)}\}\left(\sum_{\delta=0}^{\gamma} p_{k\delta}^{y\gamma}\Delta x^\delta\right), \tag{92}$$

where with $\delta_\Sigma := \sum_{i=1}^{n} \delta_i$, *and*

$$\sum_{\delta=0}^{\gamma} p_{k\delta,\beta,\tau}^{y\gamma} \Delta x^{\delta} = \sum_{\delta=0}^{\gamma} \frac{\beta}{(1-\tau)\delta_\Sigma + k} \times \left[\prod_{i=1}^{n} \left(\frac{\gamma_i!}{\delta_i!(\gamma_i - \delta_i)!} \right) y^{(\gamma-\delta)} \right] \Delta x^{\delta}. \tag{93}$$

Similar theorems can be obtained in the case of variable corefficents *in terms of certain geodesics* determined by the line element

d_R is a Riemannian metric defined by the line element

$$ds^2 = \sum_{i,j=1}^{n} a^{*ij}(x) dx_i dx_j. \tag{94}$$

$$(a^{*ij}) \text{ inverse of diffusion matrix } (a^*_{ij}) \tag{95}$$

Consider the local expansion

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t^n}} \exp\left(-\frac{d^2(x,y)}{2t} + \sum_{k=0}^{\infty} c_k(x,y) t^k\right), \tag{96}$$

of the parabolic equation

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sum_{ij} a^*_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_i b_i(x)\frac{\partial u}{\partial x_i} = 0 := Lu \tag{97}$$

First we compute for $t > 0$

$$\frac{\partial p}{\partial t} = \left(-\frac{n}{2t} + \frac{d^2(x,y)}{2t^2} + \sum_{k=0}^{\infty}(k+1)c_{k+1}(x,y)t^k\right)p,$$

$$\frac{\partial p}{\partial x_i} = \left(-\frac{d^2_{x_i}}{2t} + \sum_{k=0}^{\infty}\frac{\partial c_k}{\partial x_i}(x,y)t^k\right)p,$$

$$\frac{\partial^2 p}{\partial x_i \partial x_j} = \Bigg(\left(-\frac{d^2_{x_i}}{2t} + \sum_{k=0}^{\infty}\frac{\partial c_k}{\partial x_i}(x,y)t^k\right)$$

$$\left(-\frac{d^2_{x_j}}{2t} + \sum_{k=0}^{\infty}\frac{\partial c_k}{\partial x_j}(x,y)t^k\right)$$

$$-\frac{d^2_{x_i x_j}}{2t} + \sum_{k=0}^{\infty}\frac{\partial^2 c_k}{\partial x_i \partial x_j}t^k\Bigg)p. \tag{98}$$

Plugging into 97 we get

$$\Big(-\tfrac{n}{2t} + \tfrac{d^2(x,y)}{2t^2} + \textstyle\sum_{k=0}^{\infty}(k+1)c_{k+1}(x,y)t^k$$

$$-\tfrac{1}{2}\textstyle\sum_{ij} a^*_{ij}(x)\Big(\Big(-\tfrac{d^2_{x_i}}{2t} + \textstyle\sum_{k=0}^{\infty}\tfrac{\partial c_k}{\partial x_i}(x,y)t^k\Big)$$

$$\left(-\tfrac{d^2_{x_j}}{2t} + \textstyle\sum_{k=0}^{\infty}\tfrac{\partial c_k}{\partial x_j}(x,y)t^k\right) - \tfrac{d^2_{x_i x_j}}{2t} + \textstyle\sum_{k=0}^{\infty}\tfrac{\partial^2 c_k}{\partial x_i \partial x_j}t^k\Big)\Big)$$

$$-\textstyle\sum_i b_i(x)\left(-\tfrac{d^2_{x_i}}{2t} + \textstyle\sum_{k=0}^{\infty}\tfrac{\partial c_k}{\partial x_i}(x,y)t^k\right)\Big)p \qquad (99)$$

Collecting terms of order t^{-2} we have

$$d^2 = \frac{1}{4}\sum_{ij} d^2_{x_i} a^*_{ij} d^2_{x_j}. \tag{100}$$

Note that here $d^2_{x_i}$ is the partial derivative of d^2 with respect to x_i. Equation (100) is closely connected to a Hamilton-Jacobi equation and admits a unique solution if the boundary condition, i.e. the condition $d(x,y) = 0$ if $x = y$, is satisfied. Collecting terms of order t^{-1} we get

$$\begin{aligned} &-\tfrac{n}{2} + \tfrac{1}{2}Ld^2 + \tfrac{1}{2}\textstyle\sum_{ij} a^*_{ij}(x)\left(\frac{d^2_{x_j}}{2}\frac{\partial c_0}{\partial x_i}(x,y)\right. \\ &\left.+\frac{d^2_{x_i}}{2}\frac{\partial c_0}{\partial x_j}(x,y)\right) = 0. \end{aligned} \tag{101}$$

Equation 101 is a linear first order equation which can be written as

$$\begin{aligned} &-\tfrac{n}{2} + \tfrac{1}{2}Ld^2 + \\ &\tfrac{1}{2}\textstyle\sum_i \left(\sum_j \left(a^*_{ij}(x) + a^*_{ji}(x)\right)\frac{d^2_{x_j}}{2}\right)\frac{\partial c_0}{\partial x_i}(x,y) = 0. \end{aligned} \tag{102}$$

together with some boundary condition

$$c_0(y,y) = -\frac{1}{2}\ln\sqrt{\det\left(a^{*ij}(y)\right)} \qquad (103)$$

determines c_0 uniquely for each $y \in \mathbb{R}^n$. For $k+1 \geq 1$ we get

$$(k+1)c_{k+1}(x,y) + \tfrac{1}{2}\sum_{ij} a^*_{ij}(x)\left(\frac{d^2_{x_i}}{2}\frac{\partial c_{k+1}}{\partial x_j} + \frac{d^2_{x_j}}{2}\frac{\partial c_{k+1}}{\partial x_i}\right)$$

$$= \tfrac{1}{2}\sum_{ij} a^*_{ij}(x)\sum_{l=0}^{k}\frac{\partial c_l}{\partial x_i}\frac{\partial c_{k-l}}{\partial x_j}$$

$$+\tfrac{1}{2}\sum_{ij} a^*_{ij}(x)\frac{\partial^2 c_k}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial c_k}{\partial x_i}, \qquad (104)$$

with the boundary conditions

$$c_{k+1}(x,y) = R_k(y,y) \text{ if } \quad x = y, \qquad (105)$$

R_k being the right side of 104.

Let us look how such density expansions may be used in the context of semi-elliptic linear equations which are not hypoelliptic. We have

Theorem 6 *Let $0 < d \leq n$, $T > 0$ some real number, and $1 \leq p \leq \infty$. Consider a matrix function $x \rightarrow (a_{ji})^{d,m}(x),\ 1 \leq j \leq d$ on $\mathbb{R}^n$ with $n \geq d$, and m smooth vector fields of dimension d*

$$A_i = \sum_{j=1}^{d} a_{ji} \frac{\partial}{\partial x_j}, \tag{106}$$

where $0 \leq i \leq m$. Consider an additional vector field of dimension $n - d$

$$B := \sum_{j=d+1}^{n} a_{j0} \frac{\partial}{\partial x_j}, \tag{107}$$

Consider the Cauchy problem on $[0, T] \times \mathbb{R}^n$ (where $T > 0$ is an arbitrary finite horizon)

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i=1}^{m} A_i^2 u + (A_0 + B) u \\ \\ u(0, x) = f(x). \end{cases} \tag{108}$$

Assume that the Hörmander condition holds on the the subspace $\mathbb{R}^d$ of the first d coordinates. Assume that the initial data function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies

$$
\begin{array}{ll}
(i) & \text{for all } x_{d+1}, \cdots, x_n \text{ fixed the function} \\
& (x_1, \cdots, x_d) \rightarrow f(x_1, \cdots, x_d, x_{d+1}, \cdots, x_n) \\
& \text{is } L^p_{loc}, \quad 1 \leq p \leq \infty \text{ on } \mathbb{R}^d, \\
& \\
(ii) & \text{for all } x_1, \cdots, x_d \text{ fixed the function} \\
& (x_{d+1}, \cdots, x_n) \rightarrow f(x_1, \cdots, x_d, x_{d+1}, \cdots, x_n) \\
& \text{is } C^\infty\left(\mathbb{R}^{n-d}\right), \\
& \\
(iii) & \text{for all } x \in \mathbb{R}^n \\
& |f(x)| \leq C \exp(C|x|) \\
& \text{for some constant } C > 0.
\end{array}
\tag{109}
$$

Assume that the coefficients are smooth (i.e. C^∞) of linear growth with bounded derivatives, i.e.

$$
a_{ji} \in C^\infty_{l,b}\left(\mathbb{R}^n\right) \tag{110}
$$

for $i = 0$ and $1 \leq j \leq n$, or $1 \leq i \leq m$ and $1 \leq j \leq d$. Then the Cauchy problem above *has a global classical solution* u*, where*

$$u \in C^{\infty}\left((0,T] \times \mathbb{R}^n\right), \tag{111}$$

and where the singular behaviour in $t = 0$ *is determined by the Malliavin-type estimate in (cf. Kusuoka, S., Stroock, D. 1985) as follows: for given natural numbers* m *and* N *there is a number* q *such that the solution* u *and its time derivatives up to order* m *and its spatial derivatives up to order* N *are located in the space.*

$$C^q_{m,N}\left([0,T] \times \mathbb{R}^n\right) := \left\{v | t^q v \in C_{m,N}\left([0,T] \times \mathbb{R}^n\right)\right\}, \tag{112}$$

where

$$C_{m,N}\left([0,T] \times \mathbb{R}^n\right) :=$$

$$\left\{f | \|f\| + \sum_{l \leq m} \|D^l_t f\| + \sum_{|\alpha| \leq N} \|D^{\alpha}_x f\| < \infty\right\}, \tag{113}$$

with $\|.\|$ *denoting the supremum norm. Moreover,* $q = \max |\alpha| \leq N = n_{m,\alpha,\mathbf{0}} - n/2$ *where* $n_{m,\alpha,\mathbf{0}}$ *is determined by the estimate above of the singular behavior of the density.*

(Proof ideas)

- it is sufficient to prove the theorem under the stronger assumption of bounded payoff f, because we can transform the problem above for u to a problem for

$$\tilde{u} := e^{-d(x)} u := e^{-\sqrt{a+q|x|^2}} u \qquad (114)$$

for some $a > 0$, $q > C^2$, and where $|.|$ denotes the Euclidean norm. Then $\tilde{u}$ solves an equivalent problem with identical diffusion term but transformed drift vector $\tilde{\mathbf{b}} := \mathbf{b} - \frac{1}{2}\nabla d \cdot \sigma\sigma^T$ and an additional potential term $\tilde{c} := c + \mathbf{b} \cdot \nabla d - \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^T\right) D^2 d - \frac{1}{2}|\nabla d \sigma|^2$. Here $D^2 d$ denotes the Hessian of the function d and tr denotes the trace of a matrix.

Existence of the Vector Field

First we have

Proposition 1 *Fix $x^d \in \mathbb{R}^d$. Assume that the conditions of theorem 1 are satisfied. Assume that $g \in C^1([0,T] \times \mathbb{R}^n)$. Then there exists a smooth global flow $\mathcal{F}^t$ generated by the vector field on $[0,T] \times \mathbb{R}^{n-d}$ such that the first order equation problem*

$$\frac{\partial u}{\partial t} = \sum_{i=d+1}^{n} \mu_i(x^d, x^{n-d}) \frac{\partial}{\partial x_i} u + g(x^d, x^{n-d}),$$

$$u(0, x^d, x^{n-d}) = f(x^d, x^{n-d}), \tag{115}$$

has the solution

$$u(t, x^d, x^{n-d}) = f\left(x^d, \mathcal{F}^t x^{n-d}\right) + \int_0^t g(x^d, \mathcal{F}^{t-s} x^{n-d}) ds. \tag{116}$$

Construction of the solution via an AD-Scheme

Vector Field Step: $(l \geq 0)$

$$\frac{\partial u^{2l}}{\partial t} - \sum_{i=d+1}^{n} \mu_i(x) \frac{\partial u^{2l}}{\partial x_i}$$

$$= \begin{cases} 0 \text{ if } l = 0 \\ \\ \sum_{i,j=1}^{k} a^*_{ij}(x) \frac{\partial^2 u^{2l-1}}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \mu_i(x) \frac{\partial u^{2l-1}}{\partial x_i} \text{ if } l > 0 \end{cases} \tag{117}$$

and

Diffusion Step: $(l \geq 0)$

$$\frac{\partial u^{2l+1}}{\partial t} - \sum_{i,j=1}^{k} a^*_{ij}(x) \frac{\partial^2 u^{2l+1}}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \mu_i(x) \frac{\partial u^{2l+1}}{\partial x_i}$$

$$= \sum_{i=d+1}^{n} \mu_i(x) \frac{\partial u^{2l}}{\partial x_i}. \tag{118}$$

For each m we define $u^m(0,.) = f(.)$ and $u^{m+1}(0,.) = f(.)$. Here, in equation (118) we understand $(x_{d+1}, \cdots, x_n)$ to be fixed, and in (117) we understand $(x_1, \cdots, x_k)$ to be fixed. We construct the solution in the form

$$u^{\rho}(\tau,x)=u^{\rho,1}(\tau,x)+\sum_{l\geq 1}\delta u^{\rho,2l+1}(\tau,x), \quad (119)$$

where for $l\geq 1$

$$\delta u^{\rho,2l+1}=u^{\rho,2l+1}-u^{\rho,2l-1} \quad (120)$$

satisfies

$$\begin{array}{ll} & \frac{\partial \delta u^{\rho,2l+1}}{\partial\tau}-\rho\sum_{i,j=1}^{k}a^{*}_{ij}(x)\frac{\partial^2\delta u^{\rho,2l+1}}{\partial x_i\partial x_j}\\ & -\sum_{i=1}^{d}\rho\mu_i(x)\frac{\partial\delta u^{\rho,2l+1}}{\partial x_i} \quad (121)\\ = & \sum_{i=d+1}^{n}\rho\mu_i(x)\frac{\partial\delta u^{\rho,2l}}{\partial x_i},\end{array}$$

and in each substep where the right side in (121)

$$\delta u^{\rho,2l}=u^{\rho,2l}-u^{\rho,2l-2} \quad (122)$$

satisfies

$$\begin{aligned} & \frac{\partial \delta u^{\rho,2l}}{\partial \tau} - \sum_{i=d+1}^{n} \rho\mu_i(x) \frac{\partial \delta u^{\rho,2l}}{\partial x_i} \\ = \ & \sum_{i,j=1}^{k} \rho a^*_{ij}(x) \frac{\partial^2 \delta u^{\rho,2l-1}}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \rho\mu_i(x) \frac{\partial \delta u^{\rho,2l-1}}{\partial x_i}. \end{aligned} \tag{123}$$

Moreover, for $m \geq 1$, $\delta u^{\rho,m}$ has zero initial conditions, i.e. $\delta u^{\rho,m}(0,x) = 0$.

We claim that for small ρ the scheme just described is locally convergent with respect to time. Then iteration of the scheme in time using the semigroup property leads to a convergent scheme of a global solution.

Remarks

- Since the equations are only hypoelliptic on the subspace $\mathbb{R}^d$ we cannot apply Schauder estimates (at least not directly).

- However estimates in Kusuoka/Stroock based on the Malliavin calculus lead to estimates of densities at each iteration step on the subspace $\mathbb{R}^d$.

The question now is in which Banach space we have a convergence of the functional series above. The guide is the density estimation in KS, 1985. Accordingly we define for each positiv real number q

$$C^q_{1,2}\left([0,T]\times\mathbb{R}^n\right) := \left\{v|t^q v \in C_{1,2}\left([0,T]\times\mathbb{R}^n\right)\right\}, \tag{124}$$

where

$$C_{1,2}\left([0,T]\times\mathbb{R}^n\right) := \left\{f|\|f\| + \|f_t\| + \sum_{|\alpha|\leq 2}\|D^\alpha_x f\| < \infty\right\} \tag{125}$$

along with $\|.\|$ denoting the supremum norm. In general (in order to prove higher regularity) we may consider the spaces

$$C^q_{m,N}\left([0,T]\times\mathbb{R}^n\right) := \left\{v|t^q v\in C_{m,N}\left([0,T]\times\mathbb{R}^n\right)\right\}, \tag{126}$$

where

$$C_{m,N}\left([0,T]\times\mathbb{R}^n\right) :=$$

$$\left\{f|\|f\|+\sum_{l\leq m}\|D^l_t f\|+\sum_{|\alpha|\leq N}\|D^\alpha_x f\|<\infty\right\}. \tag{127}$$

Theorem 7 *Consider a d-dimensional difffusion process of the form*

$$\mathrm{d}X_t \;=\; \sum_{i=1}^{d} b_i(X_t)\mathrm{d}t+\sum_{j=1}^{d}\sigma_{ij}(X_t)\mathrm{d}W^j_t \tag{128}$$

with $X(0)=x\in\mathbb{R}^d$ with values in $\mathbb{R}^d$ and on a time interval $[0,T]$. Assume that $b_i,\sigma_{ij}\in C^\infty_{lb}$. Then the law of the process X is absolutely continuous with respect to the Lebesgue measure, and the density p exists and is smooth, i.e.

$$p:(0,T]\times\mathbb{R}^d\times\mathbb{R}^d\rightarrow\mathbb{R}\in C^\infty\left((0,T]\times\mathbb{R}^d\times\mathbb{R}^d\right). \tag{129}$$

Moreover, for each nonnegative natural number j, and multiindices α, β there are increasing functions of time

$$A_{j,\alpha,\beta}, B_{j,\alpha,\beta} : [0, T] \rightarrow \mathbb{R}, \tag{130}$$

and functions

$$n_{j,\alpha,\beta}, m_{j,\alpha,\beta} : \mathbb{N} \times \mathbb{N}^d \times \mathbb{N}^d \rightarrow \mathbb{N}, \tag{131}$$

such that

$$\begin{array}{l} \left|\frac{\partial^j}{\partial t^j}\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}p(t,x,y)\right| \\ \\ \leq \frac{A_{j,\alpha,\beta}(t)(1+x)^{m_{j,\alpha,\beta}}}{t^{n_{j,\alpha,\beta}}}\exp\left(-B_{j,\alpha,\beta}(t)\frac{(x-y)^2}{t}\right) \end{array} \tag{132}$$

Moreover, all functions (130) and (131) depend on the level of iteration of Lie-bracket iteration at which the Hörmander condition becomes true.

We cannot apply the theorem directly, but it has a probabilistic side. We note

Corollary 1 *In the situation above, solution X_t^x starting at x is in the standard Malliavin space D^∞, and there are constants $C_{l,q}$ depending on the derivatives of the drift and dispersion coefficients such that for some constant $\gamma_{l,q}$*

$$|X_t^x|_{l,q} \leq C_{l,q}(1+|x|)^{\gamma_{l,q}}. \tag{133}$$

Here $|.|_{l,q}$ denotes the norm where derivatives up to order l are in L^q (in the Malliavin sense).

First we consider the Cauchy problem for $u^{\rho,1}$.

- In terms of the related Markov family $Y_t^{1,x}$ with $(1+|x|)^{\gamma_{l,q}} Y_t^x = X_t^{1,x}$, we have a probabilistic representation of the solution $u^{\rho,1}$

$$u^{\rho,1}(t,x) := E\left(f(X_t^x)\right) = E\left(g(Y_t^x)\right), \tag{134}$$

and where $g(.) := h((1+|x|)^{\gamma_{l,q}}.)$ is bounded and decays exponentially for $|x| \uparrow \infty$.

- $|\delta u^{\rho,2}|_0 \le \exp(-c|x|)$ as $|x| \uparrow \infty$ follows from proposition 1. Similarly, for $l \geq 1$ we use the probabilistic representation

$$\begin{aligned}\delta u^{\rho,2l+1}(t,x) &:= \int_0^t E\left(\delta u^{\rho,2l}(s, X_s^x)\right) ds \\ &= \int_0^t E\left(\delta u^{\rho,2l}(s,(1+|x|)^{\gamma_{l,q}} Y_s^x)\right) ds,\end{aligned} \tag{135}$$

and proposition 1 and get

$|\delta u^{\rho,2l}|_0,\ |\delta u^{\rho,2l+1}|_0 \le \exp(-c|x|)$ as $|x| \uparrow \infty$

inductively.

In terms of the densities $(t,x,y) \rightarrow p_Y(t,x,y)$ via

$$P^x(Y_t^x \in dy) = p_Y(t,x,y)dy, \tag{136}$$

we get the representations

$$\begin{aligned} u^{\rho,1}(t,x) &:= E\left(f(X_t^x)\right) = E\left(g(Y_t^x)\right) \\ &= \int g(y)p_Y(t,x,y)dy, \end{aligned} \tag{137}$$

and

$$\begin{aligned} \delta u^{\rho,2l+1}(t,x) &:= \int_0^t E\left(\delta u^{\rho,2l}(s,X_s^x)\right)ds \\ &= \int_0^t E\left(\delta u^{\rho,2l}(s,(1+|x|)^{\gamma_{l,q}}Y_s^x)\right)ds \end{aligned} \tag{138}$$

This leads to a absolutely convergent scheme.

Generalisation: Consider a matrix-valued function $x \to (v_{ji})^{n,m}(x)$, $1 \leq j \leq n$, $0 \leq i \leq m$ on $\mathbb{R}^n$, and m smooth vector fields

$$V_i = \sum_{j=1}^{n} v_{ji}(x) \frac{\partial}{\partial x_j}, \tag{139}$$

where $0 \leq i \leq m$. Cauchy problem on $[0,T] \times \mathbb{R}^n$ (where $T > 0$ is an arbitrary finite horizon)

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i=1}^{m} V_i^2 u + V_0 u \\ \\ u(0,x) = f(x). \end{cases} \tag{140}$$

Define for all $x \in \mathbb{R}^n$

$$W_x := \text{span}\{V_i(x), \left[V_j, V_k\right](x), \left[\left[V_j, V_k\right], V_l\right](x),$$

$$\cdots | 1 \leq i \leq m,\ 0 \leq j,k,l \cdots \leq m\}. \tag{141}$$

Furthermore let $I_H = \cap_{x \in \mathbb{R}^n} W_x$.

We have

Theorem 8 *Let $1 \leq p \leq \infty$. Consider the Cauchy problem (140) on $[0,T] \times \mathbb{R}^n$. Assume that the initial data function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$(i) \qquad \text{the function } f \text{ is } L^p_{loc}, 1 \leq p \leq \infty \text{ on } I_H,$$

$$(ii) \qquad \text{the function } f \text{ is } C^\infty \text{ on } \mathbb{R}^n \setminus I_H,$$

$$(iii) \quad \text{for all } x \in \mathbb{R}^n$$
$$|f(x)| \leq C\exp(C|x|) \quad \text{for some constant } C > 0 \tag{142}$$

Assume that the coefficients are smooth (i.e. C^∞) and of linear growth with bounded derivatives, i.e.

$$v_{ji} \in C^\infty_{l,b}(\mathbb{R}^n) \tag{143}$$

for $i = 0$ and $1 \leq j \leq n$, or $1 \leq i \leq m$ and $1 \leq j \leq n$. Then the Cauchy problem above has a global classical solution u, where

$$u \in C^\infty((0,T] \times \mathbb{R}^n). \tag{144}$$

Higher order probabilistic weighted Monte Carlo schemes and comparison with proxy schemes and partial proxy schemes.

Let $k \leq d$ be fixed and fix the coordinates $x^{n-k} = (x_{k+1}, \cdots, x_n)$. Let $p_{x^{n-k}}$ be the fundamental solution of

$$\frac{\partial u^{\rho,1}}{\partial \tau} - \rho \sum_{i,j=1}^{k} a_{ij}(x^k, x^{n-k}) \frac{\partial^2 u^{\rho,1}}{\partial x_i \partial x_j}$$
$$- \sum_{i=1}^{k} \rho \mu_i(x^k, x^{n-k}) \frac{\partial u^{\rho,1}}{\partial x_i} = 0. \tag{145}$$

Then according to our results above we get the probabilistic representation

$$u^{\rho,1}(\tau, .) = \int_{\mathbb{R}^k} f(y^k, x^{n-k}) p_{x^{n-k}}(\tau, x^k; 0, y^k) dy^k$$
$$+ \int_0^\tau \int_{\mathbb{R}^k} \sum_{i=k+1}^{n} \mu_i(t(s), y^k, x^{n-k}) \times$$
$$\frac{\partial f}{\partial x_i}(y^k, \mathcal{F}^\tau x^{n-k}) p_{x^{n-k}}(\tau, x^k; s, y^k) dy^k ds. \tag{146}$$

Similarly for the higher order terms we get the recursive probabilistic representation

$$\delta u^{\rho,2l+1}(\tau,.) = \int_0^\tau \int_{\mathbb{R}^k} \sum_{i=k+1}^{n} \rho\mu_i(y^k, x^{n-k})\times$$

$$\left(\frac{\partial}{\partial x_i}\int_0^\tau \sum_{i,j=1}^{k} \rho a_{ij}(y^k, x^{n-k})\frac{\partial^2 \delta u^{\rho,2l-1}}{\partial x_i \partial x_j}(y^k, \mathcal{F}^{\tau-s}x^{n-k})ds\right.$$

$$\left.+\frac{\partial}{\partial x_i}\int_0^\tau \sum_{i=1}^{k} \rho\mu_i(y^k, x^{n-k})\frac{\partial \delta u^{\rho,2l-1}}{\partial x_i}(s, y^k, \mathcal{F}^{\tau-s}x^{n-k})\right)$$

$$\times p_{x^{n-k}}(\tau, x^k; s, y^k)dy^k ds. \tag{147}$$

The representation (147) together with the initial representation (146) gives the following 'naive' weighted Monte-Carlo scheme. Let ζ^{x^k} be a random variable with normal density $\phi(t, x^k, \cdot)$.*

*Note that in general practice ϕ will be time-homogenous.

Then we have the probabilistic representations

$$u^{\rho,1}(\tau,.) = E\Big(\frac{f(\zeta^{x^k},x^{n-k})p_{x^{n-k}}(\tau,x^k;0,\zeta^{x^k})}{\phi(\tau,x^k,\zeta^{x^k})}\Big)$$

$$+E\Big(\int_0^\tau \sum_{i=k+1}^n \rho\mu_i(\zeta^{x^k},x^{n-k})\times$$

$$\frac{\partial f}{\partial x_i}(\zeta^{x^k},\mathcal{F}^\tau x^{n-k})\frac{p_{x^{n-k}}(\tau,x^k;s,\zeta^{x^k})}{\phi(s,x^k,\zeta^{x^k})}\Big), \tag{148}$$

and

$$\delta u^{\rho,2l+1}(\tau,x) = E\Big(\int_0^\tau \sum_{i=k+1}^n \rho\mu_i(\zeta^{x^k},x^{n-k})\times$$

$$\Big(\frac{\partial}{\partial x_i}\int_0^\tau \sum_{i,j=1}^k \rho a_{ij}(\zeta^{x^k},x^{n-k})\frac{\partial^2 \delta u^{\rho,2l-1}}{\partial x_i \partial x_j}(s,\zeta^{x^k},\mathcal{F}^{\tau-s}x^{n-k})$$

$$+\frac{\partial}{\partial x_i}\int_0^\tau \sum_{i=1}^k \rho\mu_i(\zeta^{x^k},x^{n-k})\frac{\partial \delta u^{\rho,2l-1}}{\partial x_i}(s,\zeta^{x^k},\mathcal{F}^{\tau-s}x^{n-k})\Big)$$

$$\times\frac{p_{x^{n-k}}(\tau,x^k;s,\zeta^{x^k})}{\phi(s,x^k,\zeta^{x^k})}ds\Big). \tag{149}$$

Substitution of expectation E by the sum $\frac{1}{M}\sum_{m=1}^{M}$ and writing for each m the m-th realization of ζ^{x^k} as ${}_m\zeta^{x^k}$ leads to the following 'naive' Monte-Carlo scheme for the price

$$u^{\rho,1}(\tau,.) \cong \frac{1}{M}\sum_{m=1}^{M}\Big(\frac{f(\zeta^{x^k},x^{n-k})p_{x^{n-k}}(\tau,x^k;0,{}_m\zeta^{x^k})}{\phi(\tau,x^k,{}_m\zeta^{x^k})}\Big)$$

$$+\frac{1}{M}\sum_{m=1}^{M}\Big(\int_0^\tau\sum_{i=k+1}^{n}\rho\mu_i({}_m\zeta^{x^k},x^{n-k})\times$$

$$\frac{\partial f}{\partial x_i}(s,{}_m\zeta^{x^k},\mathcal{F}^\tau x^{n-k})\frac{p_{x^{n-k}}(\tau,x^k;s,{}_m\zeta^{x^k})}{\phi(s,x^k,{}_m\zeta^{x^k})}\Big), \tag{150}$$

and

$$\delta u^{\rho,2l+1}(\tau,x) \cong$$

$$\frac{1}{M}\sum_{m=1}^{M}\Big(\int_0^\tau \sum_{i=k+1}^{n} \rho\mu_i(t(s),_m\zeta^{x^k},x^{n-k})\times$$

$$(\frac{\partial}{\partial x_i}\int_0^\tau \sum_{i,j=1}^{k} \rho a_{ij}(t(s),_m\zeta^{x^k},x^{n-k})\times$$

$$\frac{\partial^2 \delta u^{\rho,2l-1}}{\partial x_i \partial x_j}(s,_m\zeta^{x^k},\mathcal{F}^{\tau-s}x^{n-k})ds$$

$$+\frac{\partial}{\partial x_i}\int_0^\tau \sum_{i=1}^{k} \rho\mu_i(_m\zeta^{x^k},x^{n-k})\times$$

$$\frac{\partial \delta u^{\rho,2l-1}}{\partial x_i}(s,_m\zeta^{x^k},\mathcal{F}^{\tau-s}x^{n-k}))\times$$

$$\frac{p_{x^{n-k}}(\tau,x^k;s,_m\zeta^{x^k})}{\phi(s,x^k,_m\zeta^{x^k})}ds\Big). \tag{151}$$

Note that we have the same density for both the lower and the higher order terms. The estimators for the Greeks are easily obtained by differentiating (148) and (147) and then writing the MC approximations for the corresponding differentiated equations. We call the estimate for derivatives 'naive' because it may have unbounded variance for small time and/or small variance.

For simplicity let us consider the Δs'. The 'naive' estimator is constructed from

$$\frac{\partial}{\partial x_l} u^{\rho,1}(\tau,.) = E\Big(\frac{\partial}{\partial x_l}\frac{f(\zeta^{x^k},x^{n-k})p_{x^{n-k}}(\tau,x^k;0,\zeta^{x^k})}{\phi(\tau,x^k,\zeta^{x^k})}\Big)$$

$$+E\Big(\frac{\partial}{\partial x_l}\int_0^\tau \sum_{i=k+1}^n \rho\mu_i(\zeta^{x^k},x^{n-k})\times$$

$$\frac{\partial f}{\partial x_i}(\zeta^{x^k},\mathcal{F}^\tau x^{n-k})\frac{p_{x^{n-k}}(\tau,x^k;s,\zeta^{x^k})}{\phi(s,x^k,\zeta^{x^k})}ds\Big), \tag{152}$$

and from

$$\frac{\partial}{\partial x_l}\delta u^{\rho,2l+1}(\tau,x) =$$

$$E\Big(\frac{\partial}{\partial x_l}\int_0^\tau \sum_{i=k+1}^n \rho\mu_i(\zeta^{x^k},x^{n-k})\times$$

$$(\frac{\partial}{\partial x_i}\int_0^\tau \sum_{i,j=1}^k \rho a_{ij}(\zeta^{x^k},x^{n-k})\times$$

$$\frac{\partial^2\delta u^{\rho,2l-1}}{\partial x_i\partial x_j}(s,\zeta^{x^k},\mathcal{F}^{\tau-s}x^{n-k})ds \qquad (153)$$

$$+\frac{\partial}{\partial x_i}\int_0^\tau \sum_{i=1}^k \rho\mu_i(\zeta^{x^k},x^{n-k})\times$$

$$\frac{\partial\delta u^{\rho,2l-1}}{\partial x_i}(s,\zeta^{x^k},\mathcal{F}^{\tau-s}x^{n-k}))\times$$

$$\frac{p_{x^{n-k}}(\tau,x^k;s,\zeta^{x^k})}{\phi(s,x^k,\zeta^{x^k})}ds\Big).$$

Consider $g : \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ with

$$|\partial g(t, x, z)/\partial z| \neq 0,$$

and an $\mathbb{R}^k$-valued random variable ξ on some probability space with density λ_t where $\lambda_t(z) \neq 0$ for all z and t (for example, the standard normal density).

Then we write $\zeta^{x^k} := g(\tau, x^k, \xi)$ where we assume that it has a density $\phi(\tau, x^k, \cdot)$.

Hence the first order term of our constructive scheme of the estimator becomes (analytic form)

$$\frac{\partial}{\partial x_l}u^{\rho,1}(\tau,.) =$$

$$E\Big(\frac{\partial}{\partial x_l}\frac{f(g(\tau,x^k,\xi),x^{n-k})p_{x^{n-k}}(\tau,x^k;0,g(\tau,x^k,\xi))}{\phi(\tau,x^k,g(\tau,x^k,\xi))}\Big)$$

$$+E\Big(\frac{\partial}{\partial x_l}\int_0^\tau \sum_{i=k+1}^n \rho\mu_i(g(\tau,x^k,\xi),x^{n-k})\times$$

$$\frac{\partial f}{\partial x_i}(s,g(\tau,x^k,\xi),\mathcal{F}^\tau x^{n-k})\frac{p_{x^{n-k}}(\tau,x^k;s,g(t,x^k,\xi))}{\phi(s,x^k,g(\tau,x^k,\xi))}ds\Big). \tag{154}$$

Again, substitution of expectation E by the sum $\frac{1}{M}\sum_{m=1}^{M}$ and for each m writing the m-th realization of ξ as ${}_m\xi$ gives the corresponding Monte Carlo estimator

$$\frac{\partial}{\partial x_l}u^{\rho,1}(\tau,.)=$$

$$\frac{1}{M}\sum_{m=1}^{M}\left(\frac{\partial}{\partial x_l}\frac{f(g(\tau,x^k,{}_m\xi),x^{n-k})p_{x^{n-k}}(\tau,x^k;0,g(\tau,x^k,{}_m\xi))}{\phi(\tau,x^k,g(t,x^k,{}_m\tilde{\xi}))}\right)$$

$$+\frac{1}{M}\sum_{m=1}^{M}\left(\frac{\partial}{\partial x_l}\int_0^{\tau}\sum_{i=k+1}^{n}\rho\mu_i(g(s,x^k,{}_m\xi),x^{n-k})\times\right.$$

$$\left.\frac{\partial f}{\partial x_i}(s,g(s,x^k,{}_m\tilde{\xi}),\mathcal{F}^{\tau}x^{n-k})\frac{p_{x^{n-k}}(\tau,x^k;s,g(s,x^k,{}_m\xi))}{\phi(s,x^k,g(s,x^k,{}_m\xi))}\right),\qquad(155)$$

and similar for the correction terms.

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