Discrete parametrix method and its applications

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Model description

The Mc Kean-Singer approach (JDG 67) for «jump-diffusion» processes. Let $(X_t)_{t\geq 0}$ be a Markov process with values in R^d with the following generator

$$L_t\varphi(x) = \langle b(t,x), \nabla\varphi(x)\rangle + \frac{1}{2} tr\left(a(t,x) D_x^2 \varphi(x)\right)$$

$$+ \int_{R^d} (\varphi(x+y) - \varphi(x) - \frac{\langle y, \nabla\varphi(x)\rangle}{1+|y|^2}) \nu(t,x,dy)$$

(1)

Model description

where

- $a(t,\cdot)$ is a positive semi definite diffusion matrix, $t \geq 0$.
- $b(t,\cdot)$ is a drift vector.
- $\nu(t,x,dy)$ is a Levy measure depending on a parameter $x \in R^d$.
- $\varphi \in C_0^\infty(R^d)$ is a test function.

The Mc Kean-Singer approach (JDG, 67)

Key ideas: freeze the terminal point x' and use the Kolmogorov equations satisfied by $\tilde{p}^{x'}, p$

$$\tilde{L}_t^{x'}\varphi(x) = \langle b(t,x'), \nabla\varphi(x)\rangle + \frac{1}{2} tr(a(t,x')D_x^2\varphi(x))$$

$$+\int_{R^d} (\varphi(x+y) - \varphi(x) - \frac{\langle y, \nabla\varphi(x)\rangle}{1+|y|^2})\nu(t,x',dy)$$

The Mc Kean-Singer approach (JDG,67)

- $b(t,x')$ is a «frozen» drift, $t \geq 0$.
- $\nu(t,x',dy)$ is a «frozen» Levy measure, $t \geq 0$.
- $a(t,x')$ is a «frozen» diffusion matrix, $t \geq 0$.

Proposition. (Mc Kean-Singer Parametrix expansion)

For all $0 \leq s < t \leq T$

$$p(s,t,x,x') = \sum_{r=0}^{\infty} \tilde{p} \otimes H^{(r)}(s,t,x,x')$$

Mc Kean-Singer Parametrix expansion

where

$$f \otimes g(s,t,x,y) = \int_s^t du \int_{R^d} f(s,u,x,z) g(u,t,z,x')dz$$

$$\tilde{p} \otimes H^{(0)} = \tilde{p}$$ and $$H^{(r)} = H \otimes H^{(r-1)}, r > 0,$$

denotes the r- fold convolution of the kernel H

Proof. $$(p - \tilde{p}^{x'})(s,t,x,x') =$$

$$\int_s^t du \partial_u \int_{R^d} dz p(s,u,x,z) \tilde{p}^{x'}(u,t,z,x')$$

Mc Kean-Singer Parametrix expansion

$$= \int_s^t du \int_{R^d} dz(\partial_u p(s,u,x,z)\tilde{p}^{x'}(u,t,z,x')$$

$$+ p(s,u,x,z)\partial_u \tilde{p}^{x'}(u,t,z,x'))$$

$$= \int_s^t du \int_{R^d} dz(L_u^* p(s,u,x,z)\tilde{p}^{x'}(u,t,z,x')$$

$$- p(s,u,x,z)\tilde{L}_u^{x'}\tilde{p}^{x'}(u,t,z,x'))$$

$$= \int_s^t du \int_{R^d} dz p(s,u,x,z)(L_u - \tilde{L}_u^{x'})\tilde{p}^{x'}(u,t,z,x')$$

Mc Kean - Singer Parametrix expansion

$$= p \otimes H(s,t,x,x'), \qquad \text{where}$$

$$H(u,t,z,x') = (L_u - \tilde{L}_u^{x'})\tilde{p}^{x'}(u,t,z,x')$$

A simple iteration completes the proof.

Three basic partial cases of the model (1)

Three basic cases

- **Nondegenerate diffusions:** $\nu = 0, a(t,\cdot)$ is positively definite and UE, $b(t,\cdot)$ is bounded
- **Degenerated diffusions** (Kolmogorov type equations): $\nu = 0, a(t,\cdot)$ is positively semi definite, $b(t,\cdot)$ is **Not bounded!**
- **SDE driven by α-stable symmetric random processes**

Three basic cases

We consider SDE driven by α-stable symmetric process Z_s

$$X_t = x + \int_0^t m(X_{s\_})ds + \int_0^t \sigma(X_{s\_})dZ_s,$$

$$\mathsf{E}[\exp(i\langle u, Z_t\rangle)] =$$

$$\exp\{it\langle\gamma, u\rangle - t\int_{S^{d-1}} |\langle s, u\rangle|^{\alpha}\lambda(ds)\}$$

Three basic cases

For this case

$$\nu(t, x, A) = \nu(x, A) = \nu\{z \in R^d : \sigma(x) z \in A\},$$

$$b(t, x) = m(x) + \sigma(x)\gamma$$ for $1 < \alpha < 2$,

$$b(t, x) = 0$$ for $0 < \alpha \le 1,\quad a(t, x) = 0.$

Nondegenerate diffusions

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

(UE): $\lambda^{-1}\|z\|^2 \le \langle a(t,x)z, z\rangle \le \lambda\|z\|^2, \lambda > 0$

$$\forall (t,x) \in R^+ \times R^d, a = \sigma\sigma^*.$$

(R): $b(t,\cdot)$ is bounded and satisfies mild regularity conditions.

Nondegenerate diffusions

Mc Kean-Singer paramerix series has the following form

$$p(s,t,x,y)=\sum_{r=o}^{\infty}\tilde{p}\otimes H^{(r)}(s,t,x,y),$$

where

$$H(u,t,z,y)=(L_u-\tilde{L}_u^y)\tilde{p}^y(u,t,z,y),$$

$$L_u=\frac{1}{2}\sum_{i,j}a_{ij}(u,z)\partial_{z_i z_j}+\sum_i b_i(u,z)\partial_{z_i},$$

$$\tilde{L}_u^y=\frac{1}{2}\sum_{i,j}a_{ij}(u,y)\partial_{z_i z_j}+\sum_i b_i(u,y)\partial_{z_i}$$

Discrete Mc Kean Singer parametrix (KMol 84, KM 00,02,09)

Let $X_{t_i}^h, t_i = ih,$ be a Markov chain with values in R^d and with the following dynamics

$$X_{t_{i+1}}^h = X_{t_i}^h + b(t_i, X_{t_i}^h)h + h^{1/2}\xi_{t_{i+1}}^h, X_0^h = x.$$

$$\mathscr{L}(\xi_{t_{i+1}}^h \mid X_{t_i}^h = x_i, X_{t_{i-1}}^h = x_{i-1}, \dots) \sim q_{t_i, x_i}(\cdot)$$

depending only on the last values (t_i, x_i)

The two parameter family of densities $q_{t,x}(\cdot)$

Discrete parametrix

(B1) $$\int z q_{t,x}(z)dz = 0, \forall (t,x) \in R^{+} \times R^{d}$$

(B2) $$\int z_i z_j q_{t,x}(z)dz = a_{ij}(t,x), (t,x) \in R^{+} \times R^{d}$$

(B3) $\exists \psi : R^{d} \to R$ such that

$$\sup_{x \in R^{d}} |\psi(x)| < \infty, \quad \int_{R^{d}} \|x\|^{S} \psi(x)dx < \infty$$

Discrete parametrix

and $\forall (t,x,z) \in [0,T] \times R^d \times R^d$

$$\left| D_z^{\nu} q_{t,x}(z) \right| \le \psi(z), |\nu| = 0,1,2,3,4.$$

$$\left| D_x^{\nu} q_{t,x}(z) \right| \le \psi(z), |\nu| = 0,1,2.$$

(B4) mild regularity conditions on $b(t,x)$ and $a(t,x)$

Discrete parametrix

Example. The Euler scheme. For this case

$$X_{t_{i+1}}^h = X_{t_i}^h + b(t_i, X_{t_i}^h)h + \sigma(t_i, X_{t_i}^h)(W_{t_{i+1}} - W_{t_i}),$$

$$\xi_{t_{i+1}}^h = h^{-1/2}\sigma(t_i, X_{t_i}^h)(W_{t_{i+1}} - W_{t_i}),$$

and $q_{t,x}(z)$ is the family of Gaussian densities

$$q_{t,x}(z) = \phi_{0,a(t,x)}(z)$$

Discrete parametrix

Goal: handle more general random sources than $(W_{t_{i+1}} - W_{t_i})$

Error analysis: get a scheme that admits a «parametrix» representation comparable to the continuous case. **Frozen Markov chain**

$$\tilde{X}^{h,x'}_{t_{i+1}} = \tilde{X}^{h,x'}_{t_i} + b(t_i, x')h + h^{1/2}\tilde{\xi}^{h}_{t_{i+1}}, \tilde{X}^{h,x'}_{0} = x,$$

$$\mathscr{L}(\xi^h_{t_{i+1}} \mid X^h_{t_i} = x_i, X^h_{t_{i-1}} = x_{i-1}, \ldots) \sim q_{t_i, x'}(\cdot)$$

Discrete parametrix

Consider for a test function $\varphi \in C_0^\infty(R^d)$

$$L_{t_j}^h \varphi(x) = h^{-1}\left\{E[\varphi(X_{t_j+h}^h \mid X_{t_j}^h = x)] - \varphi(x)\right\}$$

$$\widetilde{L}_{t_j}^{h,x'} \varphi(x) = h^{-1}\left\{E[\varphi(\widetilde{X}_{t_j+h}^{h,x'} \mid \widetilde{X}_{t_j}^{h,x'} = x] - \varphi(x)\right\}$$

$$H^h(t_i, t_j, x, x') = (L_{t_j}^h - \widetilde{L}_{t_j}^{h,x'})\widetilde{p}^h(t_{j+h}, t_{j'}, x, x')$$

Discrete parametrix

The following discrete Mc Kean-Singer type expansion holds

$$p^h(t_j,t_{j'},x,x') = \sum_{r=0}^{j'-j} \tilde{p}^h \otimes_h H^{h,(r)}(t_j,t_{j'},x,x'), \quad (2)$$

$$f \otimes_h g(t_j,t_{j'},x,x') =$$

$$\sum_{k=0}^{j'-j-1} h \int_{R^d} f(t_j,t_{j+k},x,z) g(t_{j+k},t_{j'},z,x') dz$$

Discrete parametrix

Main idea: comparison of two series (1) and (2)

Main tool: classical local limit theorems

$$p(0,T,x,y) = \sum_{r=0}^{\infty} \tilde{p} \otimes H^{(r)}(0,T,x,y) \qquad (1)$$

$$p^h(0,T,x,y) = \sum_{r=0}^{(T/h)-1} \tilde{p}^h \otimes_h H^{h,(r)}(0,T,x,y) \qquad (2)$$

Discrete parametrix

1. Control of the tails of these series
2. Control of the replacement $\otimes$ by $\otimes_h$ (replacement of integrals by Riemannian sums)
3. Control of the convergence of $\tilde{p}^h$ to $\tilde{p}$ (classical local limit multidimensional theorems)
4. Control of the difference of kernels H and H^h

Main result for the non-degenerate case

Theorem (KMam 00). Assume (UE), (B), (B1)-(B4). Then

$$\sup_{(x,y)\in R^{2d}} \left\{ T^{d/2}\left(1+\left(\frac{|y-x|}{\sqrt{T}}\right)^{S'}\right)\times \left|p^h(0,T,x,y)-p(0,T,x,y)\right| \right\} = O(n^{-1/2})$$

where

$$t\in[0,T], h=T/n, S'=(S-2d-4)/d$$

Main result for the non-degenerate case

Homogeneous case: $q_{t,x}(\cdot) = q_x(\cdot)$

(B5) For all $(x, y) \in R^d \times R^d$, $j \geq j_0$ with a bound j_0 that does not depend on x

$$\left| D_x^\nu q_x^{(j)}(y) \right| \leq C j^{-d/2} \psi(j^{-1/2} y), |\nu| = 0,1,2,3.$$

for a constant $C < \infty$. Here $q_x^{(j)}(y)$ denotes the j- fold convolution of q for fixed x as a function of y: $q_x^{(j)}(y) = \int q_x^{(j-1)}(u) q_x(y-u) du$

Main result for the non-degenerate case

Theorem (KMam 09). Assume (UE), (B), (B1)-(B5). Then there exists $\delta > 0$ such that

$$\sup_{(x,y)\in R^{2d}} T^{d/2}\left(1+\left(\frac{|y-x|}{\sqrt{T}}\right)^{S'}\right)\times\left|p^h(0,T,x,y)-p(0,T,x,y)\right.$$

$$\left.-h^{1/2}\pi_1(0,T,x,y)-h\pi_2(0,T,x,y)\right|=O(n^{-1-\delta})$$

Open question: is it possible to take δ=1/2 ?

Main result for the non-degenerate case

where

$$\pi_1(0,T,x,y) = (p \otimes \mathcal{F}_2[p])(0,T,x,y),$$

$$\pi_2(0,T,x,y) = (p \otimes \mathcal{F}_2[p])(0,T,x,y)$$

$$+ (p \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p]])(0,T,x,y)$$

$$+ \frac{1}{2} p \otimes (L_*^2 - L^2) p(0,T,x,y),$$

Main result for the non-degenerate case

$$\mathcal{F}_i[f](s,t,x,y) = \sum_{|\nu|=i+2} \frac{\chi_\nu(y)}{\nu!} D_x^\nu f(s,t,x,y),$$

$i = 1,2.$

$\chi_\nu(y)$ is the ν-th cumulant of the density $q_y(\cdot)$,

L_* is the operator L with the coefficients frozen at x (not at y like in $\tilde{L}$!)

Degenerate diffusions. Kolmogorov example.

A.Kolmogorov (1934) example in R^2

$$L = \frac{1}{2}\partial_{xx}^2 + bx\partial_y, b \neq 0.$$

Solution (X_t, Y_t) is given by

$$X_t = x + W_t$$

$$Y_t = y + b(xt + \int_0^t W_s ds)$$

Kolmogorov example

Two important properties of this example:

- Transportation of the initial condition x to the second component
- Different time scales.

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} \sim \mathfrak{N}\left(\begin{pmatrix} x \\ y + bxt \end{pmatrix}, \begin{pmatrix} t & \frac{bt^2}{2} \\ \frac{bt^2}{2} & \frac{b^2t^2}{3} \end{pmatrix} \right)$$

Degenerate diffusion

Model description.

$$\begin{cases} X_t = x + \int_0^t b(X_s, Y_s)ds + \int_0^t \sigma(X_s, Y_s)dW_s \\ Y_t = y + \int_0^t F(X_s)ds \end{cases}$$

(A1) b, σ are C^1, bounded, resp. R^d and $R^d \otimes R^d$ valued Lipchitz continuous, $\sigma\sigma^*$ u.e.

Degenerate diffusion

(A2) F is R^d valued, $C^{2+\alpha}, \alpha > 0$, Lipchitz continuous mapping, **NOT bounded**, and $|\nabla F| \geq C > 0.$

- $(W_s)_{s \geq 0}$ standard d-dimensional BM.

How to chose correctly the frozen process?

Frozen diffusion process should compensate the additional singularity arising from the transportation of the initial condition.

Degenerate diffusion

Our assumptions and Ito formula imply that the process $\left(\bar{X}_s, \bar{Y}_s\right)_{s\geq 0} = \left(F(X_s), Y_s\right)_{s\geq 0}$ follows the dynamics of the same type with $F(x) = x$:

$$\begin{cases} X_t = x + \int_0^t b(X_s, Y_s)ds + \int_0^t \sigma(X_s, Y_s)dW_s \\ Y_t = y + \int_0^t X_s ds \end{cases}$$

Degenerate diffusion

Frozen diffusion process should be chosen to **compensate the additional singularity** arising from the transportation of the initial condition

$$\begin{cases} \tilde{X}_s^{1,x'} = x_1 + \int_0^s \sigma(x'_1, x'_2 - x'_1(t-u))dW_u + b(x'_1, x'_2)s \\ \tilde{X}_s^{2,x'} = x_2 + \int_0^s \tilde{X}_u^{1,x'} du \end{cases}$$

$$x' = (x'_1, x'_2), s \in [0, t].$$

Degenerate diffusion

Remark. Compensated value of the second component $\theta_u^2 = x'_2 - x'_1(t-u), u \in [0,s]$ is a solution of the system of ODE

$$d\theta_u^2 = x'_1\,du, \theta_t^2 = x'_2, u \in [0,s]$$

Degenerate diffusion

«Aronson type» bounds for the transition density. Assume (A1) and (A2). Then $\exists$ C, c such that $\forall$(t,x,x’) $\in (0,T]\times R^d \times R^d$

$$C^{-1} p_{c^{-1}}(t,x,x') \le p(t,x,x') \le C p_c(t,x,x')$$

where

Aronson type bounds

$$p_c(t,x,x') = \left(\frac{\sqrt{3}c}{2\pi(t-s)^2} \right)^{d/2} \times$$

$$\exp\left(-c\left\{ \frac{|x'_1 - x_1|^2}{4t} + 3\frac{\left| x'_2 - x_2 - \frac{x_1 + x'_1}{2} t \right|^2}{t^3} \right\} \right)$$

Aronson type bounds

Upper bound and partial lower bound were proved in (KMM, 2010), lower bound for more general model was proved in (DM, 2010).

Markov chain approximation

Two scales. **«Micro» scale** with the step h_0, and **«macro» scale** with the step $h = nh_0$. The initial Markov chain «lives» in micro scale and the aggregated Markov chain «lives» in macro scale.

Degenerate Markov chain

The «frozen» model. Micro scale.

$$\widetilde{X}_{t_{i+1}}^{h_0,1,t,x'} = \widetilde{X}_{t_i}^{h_0,1,t,x'} + b(x')h_0$$

$$+\sigma(x'_1, x'_2 - x'_1 t)\sqrt{h_0}\,\widetilde{\xi}_{i+1},$$

$$\widetilde{X}_{t_{i+1}}^{h_0,2,t,x'} = \widetilde{X}_{t_i}^{h_0,2,t,x'} + \widetilde{X}_{t_{i+1}}^{h_0,1,t,x'} h_0 =$$

$$\widetilde{X}_{t_i}^{h_0,2,t,x'} + h_0 \widetilde{X}_{t_i}^{h_0,1,t,x'} + h_0^2 b(x') +$$

$$h_0^{3/2}\sigma\left(x'_1, x'_2 - x'_1 t\right)\widetilde{\xi}_{i+1}$$

Degenerate Markov chain

Where $(\xi_i)_{i \in N^*}$ are centered, i.i.d. with unit covariance matrix and with a density $q(\cdot)$. Note that $\begin{pmatrix} \tilde{\xi}_{i+1} \\ \tilde{\xi}_{i+1} \end{pmatrix}$ **does Not have a density in** R^{2d} !

To get a density **we have to iterate** n times and to consider our chain **in macro scale.**

Degenerate Markov chain

Aggregated frozen Markov chain. Marco scale.

$$\tilde{X}_{t_n}^{h_0,1,t,x'} = x_1 + (nh_0)b(x'_1, x'_2)$$

$$+\sigma(x'_1, x'_2 - x'_1 t)\sqrt{nh_0}\,\tilde{\xi}_n^{(1)},$$

$$\tilde{X}_{t_n}^{h_0,2,t,x'} = x_2 + (nh_0)x_1 + \frac{\gamma_n}{2}(nh_0)^2 b(x'_1, x'_2)$$

$$+(nh_0)^{3/2}\sigma(x'_1, x'_2 - x'_1 t)\tilde{\xi}_n^{(2)},$$

Degenerate Markov chain

where

$$\gamma_n = 1 + \frac{1}{n}, \tilde{\xi}_n^{(1)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{\xi}_i,$$

$$\tilde{\xi}_n^{(2)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{n-i+1}{n} \tilde{\xi}_i,$$

$$\begin{pmatrix} \tilde{\xi}_n^{(1)} \\ \tilde{\xi}_n^{(2)} \end{pmatrix} \to \begin{pmatrix} W_1 \\ \int_0^1 W_s ds \end{pmatrix}$$

Degenerate Markov chain

By Fourier methods $\Rightarrow \begin{pmatrix} \tilde{\xi}_n^{(1)} \\ \tilde{\xi}_n^{(2)} \end{pmatrix}$ admits a regular density $q_n(\cdot)$ in R^{2d}. This gives an idea how to construct the unfrozen Markov chain in macro scale.

Degenerate Markov chain

Unfrozen Markov chain. Macro scale.

$$X_{t_{i+1}}^{h,1} = X_{t_i}^{h,1} + b(X_{t_i}^{h})h + \sigma\left(X_{t_i}^{h}\right)\sqrt{h}\eta_{i+1}^{1},$$

$$X_{t_{i+1}}^{h,2} = X_{t_i}^{h,2} + \left\{X_{t_i}^{h,1} + \frac{\gamma_n}{2} b(X_{t_i}^{h})h + \sigma\left(X_{t_i}^{h}\right)\sqrt{h}\eta_{i+1}^{2}\right\}h$$

$\begin{pmatrix} \eta_{i+1}^{1} \\ \eta_{i+1}^{2} \end{pmatrix}$ also has the same density $q_n(\cdot)$

LLT for the densities

(A3) $\int_{R^d} |u|^S |D_u^{\nu} q_n(u)| du < \infty, |\nu| \le 4, S > 4(d+1) + 2d^2.$

Theorem. Assume (A1) - (A3). Then $\exists$c > 0 such that

$$\sup_{x,x'\in R^{2d}} \left[(1+|x_1|+|x'_1|) \sup_{0\le\delta\le1} p_c(T(1+\delta), x, x') + \chi_{T^{1/2}}\left(x'_1 - x_1, x'_2 - x_2 - T\left(\frac{x_1 + x'_1}{2}\right)\right)\right] \times$$

LLT for the densities

$$\times \left| p_h(T,x,x') - p(T,x,x') \right| = O(h^{1/2}).$$

Where $p_c(t,x,x')$ is the Gaussian density defined above and

$$\chi_\rho(u_1,u_2) = \rho^{-4d} \frac{1}{1 + \left(\left| \frac{u_1}{\rho} \right|^2 + \left| \frac{u_2}{\rho^{3/2}} \right|^2 \right)^{\frac{S-4}{4d-1}}}$$

SDE's driven by α-stable symmetric processes

We consider

$$X_t = x + \int_0^t m(X_{s\_})ds + \int_0^t \sigma(X_{s\_})dZ_s,$$

where Z_s is α-stable symmetric process in R^d, $\alpha \in (0,2)$. The Brownian case α=2 was considered in [KMam 02]. **The Euler scheme:**

$$X_{t_{i+1}}^h = X_{t_i}^h + m(X_{t_i}^h)h + \sigma(X_{t_i}^h)(Z_{t_{i+1}} - Z_{t_i})$$

SDE's driven by α-stable symmetric processes

Assume:

(A1) For $d \geq 2$ the spherical measure λ has a surface density of class $C^q(S^{d-1})$ and for $d \geq 1$, there exist $0 < C_1 \leq C_2 < \infty, \forall p \in R^d$, such that

$$C_1 |p|^\alpha \leq \int_{S^{d-1}} |\langle p, s \rangle|^\alpha \lambda(ds) \leq C_2 |p|^\alpha$$

SDE's driven by α-stable symmetric processes

(A2) m, σ and their derivatives up to the order q are bounded. For $1 < \alpha < 2$, $b(x) = m(x) + \sigma(x)\gamma$ is uniformly bounded. For $0 < \alpha \leq 1$ we suppose that $b(x) = 0$ For all $x \in R^d$.

(A3) $\exists \Lambda \geq 1$ such that

$$\Lambda^{-1} |\xi|^2 \leq \langle \sigma(x)\xi, \xi \rangle \leq \Lambda |\xi|^2$$

SDE's driven by α-stable symmetric processes

Theorem. Suppose that $q > d + 4$ and (A1) – (A3). Let $0 < M \le q - (d+4)$. Then there exists a function $R_M(T, x, y)$,

$$|R_M(T,x,y)| \le C_M(T)\frac{1}{1+|y-x|^{d+\alpha}} = \rho_{\alpha,M}(T, y-x),$$

$C_M(T) < \infty$, such that

SDE's driven by α-stable symmetric processes

$$(p - p^h)(T, x, y) =$$

$$\sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} \pi_l^h(T, x, y) + h^M R_M(T, x, y),$$

where $$\sum_{l=1}^{M-1} \left| \pi_l^h(T, x, y) \right| \leq \rho_{\alpha,M}(T, y - x).$$

SDE's driven by α-stable symmetric processes

Example. For M=2

$$(p-p^h)(T,x,y)=$$

$$\frac{h}{2}(p\otimes(L^2-\tilde{L}_*^2)p)(T,x,y)+h^2R_2(T,x,y),$$

$$\left|R_2(T,x,y)\right|\le\rho_{\alpha,2}(T,y-x),\quad \tilde{L}_*^2p=\tilde{L}_*(\tilde{L}_*p),$$

$\tilde{L}_*$ is a generator (1) with a drift vector $b(x)$ and with a Levy measure $\nu(x,dy)$ frozen at x.

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