Pricing Asian Options under a General Jump Diffusion Model

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Joint work with Ning Cai at HKUST

Outline

Background on Asian Options

- Fixed strike Asian call option:

$$\left(\frac{S_0 A_t}{t} - K\right)^+, \quad A_t = \int_0^t e^{X(s)} ds, \quad X(t) = \log(S(t)/S(0)).$$

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- Insurance literature ...

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$$A_{T_\mu} \stackrel{d}{=} \frac{2}{\sigma^2} \frac{Z(1, -\gamma_1)}{Z(\beta_1)},$$

where $A_t = \int_0^t e_s^X \, ds$, $T_\mu \sim \mathrm{Exp}(\mu)$, and $\gamma_1 < 0 < \beta_1$ are two roots of the exponent equation $G(x) = \mu$.

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- The literature along this line includes Carmona, Petit and Yor (1994), Geman and Eydeland (1995), Fu, Madan and Wang (1999), Carr and Schroder (2004), Fusai (2004), Deywnne and Shaw (2008), $\cdots$

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- Moreover, various proofs for the representation were given; see, e.g., Dufresne (2001), Yor (2001), Matsumoto and Yor (2005), $\cdots$
- Two analytical approaches in the literature:
  - Advanced math tools: Lamperti's representation and Bessel process. See, e.g., Yor (1992, 2001).
  - Complicated computations: solve PDE or ODE using special functions such as Bessel and hypergeometric functions. See, e.g., Dufrense (2001).

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- It is difficult to generalize Lamperti's representation and Bessel processes for alternative models.

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- The error bounds for the Laplace inversion are given
- The inversion works even for low volatility, e.g. $\sigma = 0.05$.

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$$\mathcal{L}(\mu,\nu) = \frac{XE^*[A_{T_\mu}^{\nu+1}]}{\mu\nu(\nu+1)}\left(\frac{S_0}{X}\right)^{\nu+1}, \quad \mu > 0, \quad \nu > 0,$$

where $A_{T_\mu} = \int_0^{T_\mu} e^{X(s)}ds$ and T_μ is an exponential random variable with rate μ independent of $\{X(t) : t \geq 0\}$.

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- Need X for a rescaling for numerical Laplace inversion.
- Only need $E^*[A_{T_\mu}^{\nu+1}]$.

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- Consider the Laplace transform

$$y(s) = E[e^{-sA_{T_\mu}}]$$

Then by Feynman-Kac $y(s)$ satisfies a nonhomogeneous ODE

$$\mathcal{L}y(s) = (s+\mu)y(s) - \mu, \qquad \text{for } s \geq 0,$$

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- (1) How to solve this ODE?
- (2) The ODE has a regular singularity at 0 and irregular singularity at ∞, and is nonhomogeneous. The ODE has infinitely many solutions.

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- Theorem 1: There is at most one bounded solution to the ODE. More precisely, suppose $a(s)$ solves the ODE/OIDE and $\sup_{s\in[0,\infty)} |a(s)| \leq C < \infty$ for some constant $C > 0$. Then we must have
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Thus the bounded solution is unique.
- We start to look for a bounded solution.

Step 3. Find a particular bounded solution of the ODE via a recursion

- Consider a difference equation (or a recursion) for a function $H(\nu)$ defined on $(-1, \beta_1)$,

$$\begin{cases} h(\nu)H(\nu) = \nu H(\nu - 1) & \text{for any } \nu \in (0, \beta_1) \\ H(0) = 1 \end{cases} \tag{1}$$

where $h(\nu) = -\frac{\sigma^2}{2}(\nu - \beta_1)(\nu - \gamma_1)$.

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- Remark: In the case of Brownian motion we can show $E^*[A^{\nu}_{T_\mu}]$ satisfies the recursion (but not showing the uniqueness) via the Feynman-Kac formula (or alternatively a time reversal argument), although we do not need this result in this paper.

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- In general there is no unique solution to the recursion. In fact any two solutions must be of the form $h_1(v) = \theta(v)h_2(v)$, where $\theta(\nu)$ is any periodic function satisfies

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- Theorem 2: If there exists a nonnegative random variable X such that $H(\nu) = E[X^{\nu}]$ satisfies the difference equation, then the Laplace transform of X, i.e. $E[e^{-sX}]$, solves the nonhomogeneous ODE.

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- Theorem 2: If there exists a nonnegative random variable X such that $H(\nu) = E[X^{\nu}]$ satisfies the difference equation, then the Laplace transform of X, i.e. $E[e^{-sX}]$, solves the nonhomogeneous ODE.
- The proof uses a connection between fractional moments and the Laplace transform.

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- It is easily verified that $H(\nu) = E[X^\nu]$ satisfies the recursion (1).

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- Combine this with Theorems 1 and 2 we have
- Theorem 3 (Originally proved by Geman and Yor (1993) in another way) Under the BSM, we have

$$A_{T_\mu} \stackrel{d}{=} \frac{2}{\sigma^2} \frac{Z(1, -\gamma_1)}{Z(\beta_1)}$$

and therefore

$$E[A^{\nu}_{T_\mu}] = \left(\frac{2}{\sigma^2}\right)^{\nu} \frac{\Gamma(\nu+1)\Gamma(\beta_1-\nu)\Gamma(1-\gamma_1)}{\Gamma(\beta_1)\Gamma(-\gamma_1+\nu+1)}, \quad \text{for any } \nu \in (-1, \beta_1$$

- Therefore, we have the double-Laplace transform $\mathcal{L}(\mu, \nu)$.

Outline

HEJD model (a flexible model)

- The model (proposed by many people independently)

$$X(t) = \left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right) t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \qquad X(0) = 0,$$

$$f_Y(y) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} I_{\{y \geq 0\}} + \sum_{j=1}^{n} q_j \theta_j e^{\theta_j y} I_{\{y < 0\}},$$

$$\zeta := E(e^{Y_1}) - 1 = \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - 1} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + 1} - 1.$$

- Motivation: It is hard to estimate the tail distribution
- Tractability, and hyper-exponential distribution can approximate any distributions with completely monotone density, which include both distributions with light and heavy tails

Properties of the Model

- $$G(x) := \frac{E\left[e^{xX(t)}\right]}{t}$$
  $$= \frac{1}{2}\sigma^2 x^2 + \left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right)x + \lambda\left(\sum_{i=1}^{m}\frac{p_i\eta_i}{\eta_i - x} + \sum_{j=1}^{n}\frac{q_j\theta_j}{\theta_j + x} - 1\right)$$
  The equation $G(x) = \mu$ has exactly $(m+n+2)$ roots $\beta_{1,\mu}, \cdots, \beta_{m+1,\mu}, \gamma_{1,\mu}, \cdots, \gamma_{n+1,\mu}$
- Additionally, the infinitesimal generator
  $$Lf(s) = \frac{\sigma^2}{2}s^2 f''(s) + (r - \lambda\zeta)sf'(s) + \lambda\int_{-\infty}^{+\infty}[f(se^u) - f(s)]f_Y(u)du,$$

Uniqueness of the OIDE

Theorem (Uniqueness): Suppose $a(s)$ solves the OIDE

$$Ly(s) = (s + \mu)y(s) - \mu,$$

and $\sup_{s \in [0,\infty)} |a(s)| \leq C < \infty$ for some constant $C > 0$. Then we must have

$$a(s) = E\left[\exp\left(-sA_{T_\mu}\right)\right] \quad \text{for any } s \geq 0.$$

A recursion and OIDE

Theorem: Consider a difference equation (or a recursion) for a function $H(\nu)$ defined on $(-1, \beta_1)$

$$h(\nu)H(\nu) = \nu H(\nu - 1) \quad \text{for any } \nu \in (0, \beta_1), \text{ and } H(0) = 1,$$

where

$$h(\nu) \equiv \mu - G(\nu) = \left(\frac{\sigma^2}{2}\right) \frac{\prod_{i=1}^{m+1}(\beta_i - \nu)\prod_{j=1}^{n+1}(-\gamma_j + \nu)}{\prod_{i=1}^{m}(\eta_i - \nu)\prod_{j=1}^{n}(\theta_j + \nu)}.$$

Here $\beta_1, \cdots, \beta_{m+1}, \gamma_1, \cdots, \gamma_{n+1}$ are $(m + n + 2)$ roots of the equation $G(x) = \mu$. If there is an nonnegative random variable X such that $H(\nu) = E[X^\nu]$ satisfies the difference equation, then the Laplace transform of X, i.e. $E[e^{-sX}]$, solves the nonhomogeneous OIDE

$$Ly(s) = (s + \mu)y(s) - \mu.$$

Asian options under HEJD model

- Distribution of A_{T_μ} under the HEM
  Theorem: Under the HEM, we have

$$A_{T_\mu} \overset{d}{=} \frac{2}{\sigma^2} \frac{Z(1,-\gamma_1)\prod_{j=1}^{n} Z(\theta_j+1,-\gamma_{j+1}-\theta_j)}{Z(\beta_{m+1})\prod_{i=1}^{m} Z(\beta_i,\eta_i-\beta_i)}$$

  and therefore for any $\nu \in (-1,\beta_1)$,

$$\begin{aligned} &E[A_{T_\mu}^{\nu}] \\ &= \left(\frac{2}{\sigma^2}\right)^{\nu} \frac{\Gamma(1+\nu)\Gamma(1-\gamma_1)}{\Gamma(1-\gamma_1+\nu)} \cdot \prod_{j=1}^{n}\left[\frac{\Gamma(\theta_j+1+\nu)\Gamma(1-\gamma_{j+1})}{\Gamma(1-\gamma_{j+1}+\nu)\Gamma(\theta_j+1)}\right] \\ &\cdot \prod_{i=1}^{m}\left[\frac{\Gamma(\beta_i-\nu)\Gamma(\eta_i)}{\Gamma(\eta_i-\nu)\Gamma(\beta_i)}\right] \cdot \frac{\Gamma(\beta_{m+1}-\nu)}{\Gamma(\beta_{m+1})}. \end{aligned}$$

Asian options under HEJD model

- A double Laplace transform of the Asian option price
- Theorem: Under the HEM, for every μ and ν such that $\mu > 0$ and $\nu \in (0, \beta_1 - 1)$, the double-Laplace transform of $XE(\frac{S_0}{X}A_t - e^{-k})^+$ is given by:

$$\mathcal{L}(\mu, \nu) = \frac{S_0^{\nu+1}}{\mu\nu(\nu+1)} E[A_{T_\mu}^{\nu+1}], \quad \mu > 0, \quad \nu > 0$$

$$= \frac{X}{\mu\nu(\nu+1)} \left(\frac{2S_0}{X\sigma^2}\right)^{\nu+1} \frac{\Gamma(2+\nu)\Gamma(1-\gamma_1)}{\Gamma(2-\gamma_1+\nu)}$$

$$\cdot \prod_{j=1}^{n} \left[\frac{\Gamma(\theta_j+2+\nu)\Gamma(1-\gamma_{j+1})}{\Gamma(2-\gamma_{j+1}+\nu)\Gamma(\theta_j+1)}\right]$$

$$\cdot \prod_{i=1}^{m} \left[\frac{\Gamma(\beta_i-\nu-1)\Gamma(\eta_i)}{\Gamma(\eta_i-\nu-1)\Gamma(\beta_i)}\right] \cdot \frac{\Gamma(\beta_{m+1}-\nu-1)}{\Gamma(\beta_{m+1})}.$$

- Two-sided, two dimensional Euler inversion algorithms apply; see Petrella (2004).

Outline

Numerical results under the BSM

- Comparison of accuracy with other existing methods.

Case	Cai-Kou	Linetsky	GY-Shaw	Vecer
1	0.0559860415	0.0559860415	0.0559860415	0.055986
2	0.2183875466	0.2183875466	0.2183875466	0.218388
3	0.1722687410	0.1722687410	0.1722687410	0.172269
4	0.1931737903	0.1931737903	0.1931737903	0.193174
5	0.2464156905	0.2464156905	0.2464156905	0.246416
6	0.3062203648	0.3062203648	0.3062203648	0.306220
7	0.3500952190	0.3500952190	0.3500952190	0.350095

Table: These seven cases are commonly used in the literature for testing the pricing algorithms of Asian options under the BSM; e.g., Fu et al. (1999), Craddock et al. (2000), Vecer (2001), Linetsky (2004), and Deywnne and Shaw (2008).

Numerical results: Low volatility in the BM Case

Extension of Case 1 when σ is extremely small.

Case	σ	DL Prices	GY-Shaw	MAE3	Zhang
1	0.1	0.0559860	0.0559860	0.0559860	0.0559860
1A	0.05	0.0339412	0.0339412	0.0339412	0.0339412
1B	0.01	NA	NA	0.0199278	0.0199278
1C	0.005	NA	NA	0.0197357	0.0197357
1D	0.001	NA	NA	0.0197353	0.0197353

Numerical results: BM Case

Comparison with single Laplace transform in Craddock, Heath, Platen (2000)

r	σ	t	K	S_0	DL	SL	MC
0.02	0.10	1.0	2	2	0.05599 (3.5 secs)	0.055 ($>$ 20 minutes)	0.05601
0.11	0.15	0.5	27	29	2.69787 (3.5 secs)	2.808 (570.56 secs)	2.69797
0.11	0.15	0.5	29	29	1.13474 (3.5 secs)	1.129 (470.72 secs)	1.13508
0.11	0.15	0.5	31	29	0.28532 (3.5 secs)	0.278 (408.59 secs)	0.28541

Numerical Results: HEJD model, $\lambda = 3$

σ	K	DL Prices	MC Prices	Std Err	Abs Err	Rel Err
0.1	90	13.48451	13.47574	0.00071	0.00877	0.0651%
0.1	95	9.20478	9.20559	0.00135	-0.00081	0.0088%
0.1	100	5.53662	5.53619	0.00207	0.00043	0.0078%
0.1	105	2.88896	2.88890	0.00249	0.00006	0.0021%
0.1	110	1.33809	1.33781	0.00238	0.00028	0.0210%
0.2	90	14.03280	14.03489	0.00193	-0.00289	0.0206%
0.2	95	10.32293	10.32461	0.00276	-0.00168	0.0163%
0.2	100	7.21244	7.21556	0.00343	-0.00312	0.0432%
0.2	105	4.78516	4.78822	0.00380	-0.00306	0.0638%
0.2	110	3.02270	3.02558	0.00380	-0.00288	0.0952%
0.3	90	15.19639	15.19689	0.00350	-0.00050	0.0033%
0.3	95	11.92926	11.93168	0.00431	-0.00242	0.0203%
0.3	100	9.14769	9.15063	0.00495	-0.00294	0.0321%
0.3	105	6.86049	6.86412	0.00533	-0.00363	0.0529%
0.3	110	5.04029	5.04400	0.00545	-0.00331	0.0656%

Numerical Results: HEJD model $\lambda = 5$

σ	K	DL Prices	MC Prices	Std Err	Abs Err
0.1	90	13.55964	13.56384	0.00102	-0.00420
0.1	95	9.41962	9.42350	0.00173	-0.00388
0.1	100	5.91537	5.91707	0.00246	-0.00170
0.1	105	3.35071	3.35124	0.00287	-0.00053
0.1	110	1.74896	1.74934	0.00281	-0.00038
0.2	90	14.17380	14.17586	0.00217	-0.00206
0.2	95	10.53795	10.53973	0.00300	-0.00178
0.2	100	7.48805	7.48864	0.00367	-0.00059
0.2	105	5.09001	5.09000	0.00405	0.00001
0.2	110	3.32061	3.31967	0.00409	0.00096
0.3	90	15.33688	15.33728	0.00367	-0.00040
0.3	95	12.10723	12.10732	0.00448	-0.00009
0.3	100	9.35336	9.35297	0.00511	0.00039
0.3	105	7.08059	7.07908	0.00551	0.00151
0.3	110	5.26109	5.25875	0.00565	0.00234

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- The error bounds for the Laplace inversion are given
- The inversion works even for low volatility, e.g. $\sigma = 0.05$.
- Future work: More complicated Levy processes, and insurance applications.

Proof of the Double Laplace Transform Representation

Applying Fubini's theorem yields

$$
\begin{aligned}
\mathcal{L}(\mu,\nu) &= X\int_0^\infty e^{-\mu t} E\left[\int_{-\ln(S_0 A_t/X)}^\infty e^{-\nu k}\left(\frac{S_0}{X}A_t - e^{-k}\right)dk\right]dt \\
&= X\int_0^\infty e^{-\mu t}\frac{E[A_t^{\nu+1}]}{\nu(\nu+1)}\left(\frac{S_0}{X}\right)^{\nu+1} dt \\
&= \frac{X}{\mu\nu(\nu+1)}\left(\frac{S_0}{X}\right)^{\nu+1}\cdot E[A_{T_\mu}^{\nu+1}],
\end{aligned}
$$

from which the proof is completed.

Proof of the Uniqueness of the ODE

In terms of $S(t)$, we can rewrite $E[\exp(-sA_{T_\mu})]$ as

$$E[\exp(-sA_{T_\mu})] = E_s\left[\int_0^\infty \mu \exp\left(-\int_0^t [\mu + S(u)]du\right) dt\right], \qquad (2)$$

where the notation E_s means that the process $\{S(t)\}$ starts from s, i.e. $S(0) = s$.
First, by Itô's formula, we have that

$$\begin{aligned} M_t \quad : \quad &= a(S(t)) \exp\left(-\int_0^t [\mu + S(u)]du\right) \\ &+ \int_0^t \mu \exp\left(-\int_0^v [\mu + S(u)]du\right) dv \end{aligned}$$

is a local martingale.

Proof of the Uniqueness of the ODE

Indeed, we obtain by some algebra that

$$dM_t = \exp\left(-\int_0^t [\mu + S(u)]du\right) \cdot a'(S(t))\sigma S(t) dW(t),$$

which implies that $\{M_t\}$ is a local martingale.
Actually, $\{M_t\}$ is a true martingale as M_t is uniformly bounded, $\sup_{t\geq 0}|M_t| \leq \sup_{t\geq 0}\left\{Ce^{-\mu t} + \int_0^t \mu e^{-\mu v} dv\right\} = C + 1 < \infty$, because $S(u) \geq 0$. Thus, $a(s) = a(S(0)) = E_s[M_0] = E_s[M_t]$.

Proof of the Uniqueness of the ODE

Letting $t \to +\infty$, the first term in M_t goes to zero almost surely because $a(\cdot)$ is bounded, and therefore

$$M_t \to \int_0^\infty \mu \exp\left(-\int_0^v \{\mu + S(u)\} du\right) dv,$$

almost surely.
Accordingly, by the dominated convergence theorem,

$$\begin{aligned} a(s) &= E_s[\lim_{t\to\infty} M_t] \\ &= E_s\left[\int_0^\infty \mu \exp\left(-\int_0^v \{\mu + S(u)\} du\right) dv\right] = E[\exp\left(-sA_{T_\mu}\right)]. \end{aligned}$$

The theorem is proved.

A particular bounded solution via a recursion

Denote the Laplace transform of X by $y(s) = E[e^{-sX}]$, for $s \geq 0$.Note that for any $a \in (0, \min(\alpha_1, 1))$, we have

$$\int_0^{+\infty} s^{-a} e^{-sX} ds = \Gamma(1-a) X^{a-1}$$

$$\int_0^{+\infty} s^{-a-1} \left(e^{-sX} - 1 \right) ds = -\frac{\Gamma(1-a)}{a} X^a,$$

where the second equality holds due to integration by parts. Taking expectations on both sides of the two equations above and applying Fubini's theorem yields

$$E[X^{a-1}] = \frac{1}{\Gamma(1-a)} \int_0^{\infty} s^{-a} y(s) ds$$

$$E[X^a] = -\frac{a}{\Gamma(1-a)} \int_0^{\infty} s^{-a-1} (y(s) - 1) ds.$$

A particular bounded solution via a recursion

Thus, by the difference equation, we have

$$-\frac{ah(a)}{\Gamma(1-a)}\int_0^\infty s^{-a-1}\left(y(s)-1\right)ds = \frac{a}{\Gamma(1-a)}\int_0^\infty s^{-a}y(s)ds,$$

i.e.

$$0 = \int_0^\infty s^{-a-1}\left[sy(s) + h(a)(y(s)-1)\right]ds.$$

Setting $s = e^{-x}$, and $z(x) = y(s) - 1$, we have

$$0 = \int_{-\infty}^\infty e^{ax}\left\{e^{-x}(z(x)+1) + h(a)z(x)\right\}dx, \quad \text{for any } a \in (0, \min(\alpha_1, 1)).$$

For simplicity of notations, rewrite $h(a)$ as $h(a) = h_0 a^2 + h_1 a + h_2$, with $h_0 = -\frac{\sigma^2}{2}$, $h_1 = -r + \frac{\sigma^2}{2}$, and $h_2 = \mu$. Note that integration by parts yields

$$\int_{-\infty}^\infty e^{ax} a z(x) dx = -\int_{-\infty}^\infty e^{ax} z'(x) dx$$

$$\int_{-\infty}^\infty e^{ax} a^2 z(x) dx = \int_{-\infty}^\infty e^{ax} z''(x) dx$$

A particular bounded solution via a recursion

Then for any $a \in (0, \min(\alpha_1, 1))$,

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} e^{ax} \left\{ e^{-x}(z(x)+1) + \left(h_0 a^2 + h_1 a + h_2\right) z(x) \right\} dx \\ &= \int_{-\infty}^{\infty} e^{ax} \left\{ e^{-x}(z(x)+1) + h_0 z''(x) - h_1 z'(x) + h_2 z(x) \right\} dx. \end{aligned}$$

By the uniqueness of the moment generating function, we have an ODE

$$h_0 z''(x) - h_1 z'(x) + h_2 z(x) + e^{-x}(z(x)+1) = 0.$$

Now transferring the ODE for $z(x)$ back to one for $y(s)$, with $s = e^{-x}$ we have the required ODE

$$\frac{\sigma^2}{2} s^2 y''(s) + rsy'(s) - (s+\mu)y(s) = -\mu.$$

Thank you!