Non-Gaussian quasi-likelihood estimation of jump processes

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Brief summary

"Inference for a class of Stochastic Differential Equations (SDE)"

- When observing a discrete-time but high-frequency sample

$$X_0, X_{h_n}, X_{2h_n}, \dots, X_{nh_n} \quad (h_n \to 0)$$

from the semi-parametric Lévy driven SDE

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dZ_t,$$

how can we estimate $\theta_0 = (\alpha_0, \gamma_0)$, the true value of $\theta := (\alpha, \gamma)$?

- We will provide an estimator $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\gamma}_n)$ s.t.

$$\left\{\left(\sqrt{n}h_n^{1-1/\beta}(\hat{\alpha}_n - \alpha_0), \sqrt{n}(\hat{\gamma}_n - \gamma_0)\right)\right\}_{n\in\mathbb{N}} \quad \text{is asymp. normal,}$$

with β denoting the Blumental-Getoor index of the Lévy process Z.

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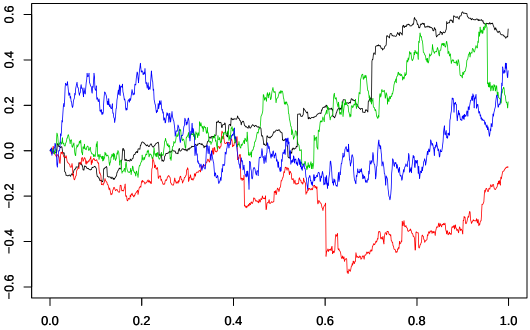
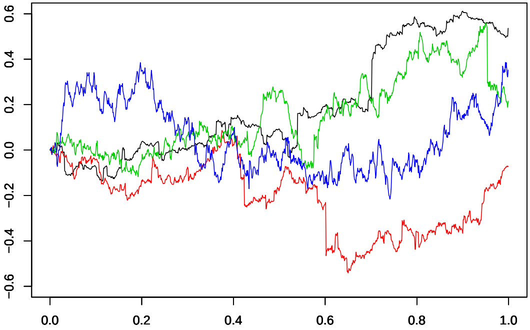
Statistics for SDE models

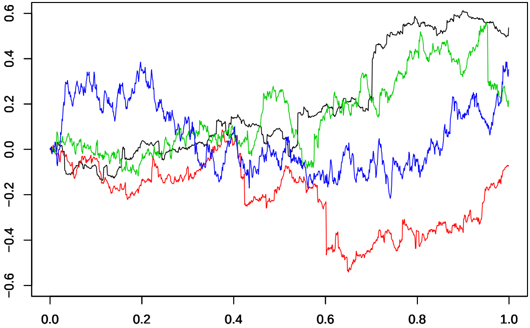


- **Time-varying phenomena $\longleftarrow$ "Stochastic process (SDE) models"**
  - **Mostly, data series exhibits dependence.**
  - **In real world, data is observed at discrete time instants.**



- **“Parameter estimation” is a standing problem in statistics.**
  - **We want a good estimation procedure for a model in question.**

⇒ “Estimation of continuous-time structure from discrete-time sample”.



- **A central issues in stochastic process modelling:**
  - **Continuous?**
  - **Including jumps?**
  - **...or, continuous with jumps?**

Why including jumps?

- Lévy process in finance (Cont and Tankov (2004)): e.g.,
  - Non-Gaussian stable... Mandelbrot (1963)
  - Normal inverse Gaussian... Barndorff-Nielsen (1995)
  - Hyperbolic... Eberlein and Keller (1995)
  - Generalized hyperbolic... Prause (1999), Raible (2000)
  - CGMY (tempered stable)... Carr et al. (2002)
  - Bilateral gamma... Küchler and Tappe (2008)
- Also, signal processing, turbulence, physical science, etc.
  - Non-Gaussian stable... e.g., Nikias and Shao (1995)
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$\Rightarrow$ Needs for statistics for jump processes

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$\Rightarrow$ **Needs for statistics for jump processes**

Non-Gaussian behavior in small time

- **In high-frequency data framework, jumps may be more conspicuous.**
- **Empirical evidence in financial returns, Grabchak and Samorodnitsky (2010):**
  - **Distribution tails appear to become:**
    - ⋆ **less heavy for less frequent (e.g. monthly) returns,**
    - ⋆ **than for more frequent (e.g. daily) returns.**
  - **Tempered heavy-tail models are reasonable.**

Maximum-Likelihood Estimation for Markov models

- **Maximum-Likelihood Estimator (MLE) is theoretically preferred.**
- **Data $Y_{t_1}, \dots, Y_{t_n}$ from a Markov process (Y_t)**
- **The MLE is defined to be the "argmax" of the log-likelihood function**

$$\theta \mapsto \log p_\theta(Y_{t_1}, \dots, Y_{t_n}) = \sum_{j=1}^{n} \log p_\theta(Y_{t_j} | Y_{t_{j-1}}).$$

- **For Y SDE, the transition density $p_\theta(y|x)$ is mostly unknown. What proxy can we make use of? How can we proceed in practice?**

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Gaussian Quasi-Likelihood Estimator (GQLE)

- Consists of fitting one-step conditional mean and variances:
  - Originally due to Wedderburn (1974);
  - A kind of generalized method of moments.

  To formulate the estimation procedure, it is enough to have

$$E[Y_{t_j}|Y_{t_{j-1}}] = m_{j-1}(\theta) \text{ and } Var[Y_{t_j}|Y_{t_{j-1}}] = v_{j-1}(\theta).$$

  explicitly.

- The GQLE is formally given by the argmax of

$$\theta \mapsto \sum_{j=1}^{n} \log \left\{ \frac{1}{\sqrt{v_{j-1}(\theta)}} \phi\left(Y_{t_j} - m_{j-1}(\theta) \right) \right\},$$

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  - **GARCH type model, Straumann and Mikosch (2006),**

$$Y_n = \sigma_n \epsilon_n, \quad n \in \mathbb{N},$$

$$\sigma_n = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2.$$

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$$Y_n = M_\theta(Y_{n-1}, Y_{n-2}, \dots)\epsilon_n + f_\theta(Y_{n-1}, Y_{n-2}, \dots).$$

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GQLE for discretely observed Lévy driven SDE *

- Based on $X_{h_n}, X_{2h_n}, \dots, X_{nh_n}$ stemming from the ergodic

$$dX_t = a(X_t, \alpha)dt + c(X_t, \gamma)dZ_t,$$

we want to estimate $\theta = (\alpha, \gamma)$, where Z is a Lévy process s.t. $E[Z_t] = 0$ and $E[Z_t^2] = t$.

- "Aggressive" approximation $\mathcal{L}(Z_{h_n}) \approx \mathcal{N}(0, h_n)$ for small h_n:

$$\begin{aligned} X_{jh_n} &\approx X_{(j-1)h_n} + a(X_{(j-1)h_n}, \alpha_0)h_n \\ &\qquad + c(X_{(j-1)h_n}, \gamma_0)(Z_{jh_n} - Z_{(j-1)h_n}) \\ &\sim \mathcal{N}\left(X_{(j-1)h_n} + a(X_{(j-1)h_n}, \alpha_0)h_n, c(X_{(j-1)h_n}, \gamma_0)^2 h_n\right), \end{aligned}$$

making the GQLE procedure explicit.

*M (2010, preprint) and the references therein.

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Resulting phenomenon and a practical caution

$$dX_t = a(X_t, \alpha)dt + c(X_t, \gamma)dZ_t$$

- **The GQLE $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\gamma}_n)$ are asymptotically normal:**

$$\left(\sqrt{nh_n}(\hat{\alpha}_n - \alpha_0), \sqrt{n}(\hat{\gamma}_n - \gamma_0)\right) \to^d \mathcal{N}(0, V') \quad \text{if } \nu(\mathbb{R}) = 0;$$

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where ν is the Lévy measure of Z.

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Test statistics for the noise normality

$$\mathcal{T}_n := \frac{n}{6}\left\{\hat{\Phi}_n^{(3)} - \frac{3\sqrt{h_n}}{n}\sum_{j=1}^{n}\partial_x c(X_{(j-1)h_n}, \hat{\gamma}_n)\right\}^2 + \frac{n}{24}(\hat{\Phi}_n^{(4)} - 3)^2$$

$$\hat{\epsilon}_{nj} := \frac{X_{jh_n} - X_{(j-1)h_n} - a(X_{(j-1)h_n}, \hat{\alpha}_n)h_n}{c(X_{(j-1)h_n}, \hat{\gamma}_n)\sqrt{h_n}}, \quad \bar{\hat{\epsilon}}_n := \frac{1}{n}\sum_{j=1}^{n}\hat{\epsilon}_{nj},$$

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- Consistent and asymptotically distribution-free test:
  - $\mathcal{T}_n \to^d \chi^2(2)$ under $\mathcal{H}_0 : \nu(\mathbb{R}) = 0$;
  - $\mathcal{T}_n \to^p \infty$ under $\mathcal{H}_1 : \nu(\mathbb{R}) \in (0, \infty]$.
- We may proceed as follows: Using $\mathcal{T}_n$ with the GQLE,
  - $\mathcal{H}_0$ not rejected $\Rightarrow$ follow diffusion estimation procedures,
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Some important previous studies, some remarks

- **Jump detection filter may work well.**
  (Mancini, Shimizu and Yoshida, Shimizu, Ogihara and Yoshida.)
  - **Asymptotically efficient, may work well for compound Poisson jumps.**
  - **In principle, the coexistence of Wiener and Poisson parts makes estimation problem difficult when pursuing estimation efficiency.**
- **What will theoretically occur in general?**
  - **We do not known any general optimal behavior of estimators.**
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Our goal of this talk is to

- **Provide an estimator of the true value of $\theta = (\alpha, \gamma)$ in**

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dZ_t$$

  based on $X_0, X_{h_n}, X_{2h_n}, \dots, X_{nh_n}$ ($h_n \to 0$).
- **We want to deal with pure-jump Z with higher degree of activity; e.g. Generalized hyperbolic, Meixner, CGMY, etc.**
- **We here do not adopt:**
  - **the GQLE, unsatisfactory while usable, in the presence of any jump;**
  - **the jump detection filter approach, a nice device with a good choice of fine-tuning parameter**
    - ★ **under the presence of a Wiener part,**
    - ★ **when jump activity is finite (or moderate).**

Contents

Non-Gaussian Quasi-Likelihood Estimation (NGQLE)

Target:

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dZ_t, \quad \eta := \mathcal{L}(X_0)$$

- Z is a **pure-jump Lévy process of infinite activity.**
- **The parameter** $\theta := (\alpha, \gamma) \in \Theta_\alpha \times \Theta_\gamma = \Theta \subset \mathbb{R}^p$, **a bounded convex domain, the true value** $\theta_0 := (\alpha_0, \gamma_0) \in \Theta$.

Notation:

- $\Delta_j Y := Y_{jh_n} - Y_{(j-1)h_n}$ **for a process** Y;
- $f_{j-1}(\theta) := f(X_{(j-1)h_n}, \theta)$ **for any function of the form** $f(x, \theta)$.

A1. Regularity of the coefficients

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dZ_t$$

1. a and c are smooth in $\mathbb{R} \times \Theta$.
2. $a(\cdot, \alpha_0)$ and $c(\cdot, \gamma_0)$ are globally Lipschitz.
3. $\exists c \in (1, \infty)$ s.t. $\forall (x, \gamma)$: $0 < c^{-1} \leq c(x, \gamma) \leq c$.
4. If X is not a Lévy process, then $\exists c', M > 0$ s.t. $\forall |x| \geq M$: $xa(x, \alpha_0) \leq -c'|x|^2$.

* X is then ergodic under the true image measure P_0, the invariant measure denoted by $\pi_0(dx)$.

A2. Driving noise

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dZ_t$$

1. $\mathcal{L}(Z_1)$ is symmetric around 0, and the Lévy measure ν of Z fulfils

$$\nu(dz) = {}^{\exists}g_0(z)dz \quad \text{s.t.} \quad g_0(z) = \frac{c_0}{|z|^{1+\beta}}\{1 + O(|z|)\}, \quad |z| \to 0.$$

* $\mathcal{L}(h^{-1/\beta}Z_h) \underset{h\to 0}{\Rightarrow}$ β-stable law with the C.F. $u \mapsto \exp(-|u|^\beta)$ for some $\beta \in (0,2)$: ϕ_β denotes the density.

2. $\mathcal{L}(h^{-1/\beta}Z_h)$ admits a positive density $f_h(y)$ s.t.:
There exist constant $\epsilon_n \to 0$ and Lebesgue-integrable λ s.t.

$$\sqrt{n}\int |f_h(y) - \phi_\beta(y)|dy \to 0.$$

* This holds for, e.g., the NIG Z if $nh_n^{2-\kappa} \to 0$ for some $\kappa > 0$.

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* This holds for, e.g., the NIG Z if $nh_n^{2-\kappa} \to 0$ for some $\kappa > 0$.

A3. Sampling rate

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dZ_t$$

1. $\beta \geq 1$ if X is a Lévy process (we do not need $nh_n \to \infty$).
2. Otherwise, $\beta > 1$, $nh_n \to \infty$, and
   $\exists \epsilon_0 > 0$ s.t. $\limsup_{n\to\infty} nh_n^{3-2/\beta-\epsilon_0} < \infty$.

A4. Weight function; for heavy-tailed cases

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dZ_t$$

1. $W : \mathbb{R} \to \mathbb{R}_+$ **is bounded.**
2. **There exists a function** $K : \mathbb{R} \to \mathbb{R}_+$ **s.t.**
   1. $\sup_{\theta\in\Theta} W(x)\{|\partial_\alpha a(x,\alpha)| + |\partial_\alpha a(x,\alpha)|^2 + |\partial_\alpha^2 a(x,\alpha)| + |\partial_\gamma c(x,\gamma)| + |\partial_\gamma c(x,\gamma)|^2 + |\partial_\gamma^2 c(x,\gamma)|\} \le K(x)$,
   2. $\sup_{t\in\mathbb{R}_+} E_0[K(X_t)] < \infty$.

A5. Nonsingularity and identifiability

For $g(y) := \frac{\partial\phi_\beta}{\phi_\beta}(y)$,

1. $\det\left\{ \int W(x)\frac{[\partial_\alpha a(x,\alpha_0)]^{\otimes 2}}{c(x,\gamma_0)^2}\pi_0(dx)\right\}\cdot\det\left\{ \int W(x)\frac{[\partial_\gamma c(x,\gamma_0)]^{\otimes 2}}{c(x,\gamma_0)^2}\pi_0(dx)\right\} \neq 0.$
2. $\iint W(x)\frac{\partial_\alpha a(x,\alpha)}{c(x,\gamma)^2}\{a(x,\alpha_0)-a(x,\alpha)\}\partial g(\frac{c(x,\gamma_0)}{c(x,\gamma)}y)\phi_\beta(y)dy\pi_0(dx)=0$ **iff** $\theta=\theta_0$.
3. $\iint W(x)\frac{\partial_\gamma c(x,\gamma)}{c(x,\gamma)}\left\{1+\frac{c(x,\gamma_0)}{c(x,\gamma)}yg(\frac{c(x,\gamma_0)}{c(x,\gamma)}y)\right\}\phi_\beta(y)dy\pi_0(dx)=0$ **iff** $\theta=\theta_0$.

Construction of our estimator

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dZ_t$$

- **Again, the naive Euler type approximation:**

$$\begin{aligned} X_{jh_n} &\approx^{P_0} X_{(j-1)h_n} + a_{j-1}(\alpha_0)h_n + c_{j-1}(\gamma_0)\Delta_j Z \\ &= X_{(j-1)h_n} + a_{j-1}(\alpha_0)h_n + c_{j-1}(\gamma_0)h_n^{1/\beta} \cdot \frac{\Delta_j Z}{h_n^{1/\beta}} \end{aligned}$$

$$\therefore \epsilon_{nj}(\theta_0) := \frac{\Delta_j X - a_{j-1}(\alpha_0)h_n}{h_n^{1/\beta} c_{j-1}(\gamma_0)} \approx \beta\text{-stable, in law (density } \phi_\beta\text{).}$$

- **We define our estimator $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\gamma}_n)$ through the quasi-likelihood:**

$$\hat{\theta}_n \in \underset{\theta \in \Theta}{\operatorname{argmax}} \sum_{j=1}^{n} W_{j-1} \log \left\{ \frac{1}{h_n^{1/\beta} c_{j-1}(\gamma)} \phi_\beta\left(\epsilon_{nj}(\theta)\right) \right\}.$$

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Main claim: Asymptotic Normality

Under the aforementioned assumptions, the estimator is A.N.:

$$\left(\sqrt{n}h_n^{1-1/\beta}(\hat{\alpha}_n-\alpha_0), \sqrt{n}(\hat{\gamma}_n-\gamma_0)\right) \Rightarrow \mathcal{N}\left(0, \mathrm{diag}[U(\theta_0)^{-1}, V(\theta_0)^{-1}]\right),$$

where

$$U(\theta_0)=\int W(x)\frac{\{\partial_\alpha a(x,\alpha_0)\}^{\otimes 2}}{c(x,\gamma_0)^2}\pi_0(dx)\cdot\int\frac{\partial\phi_\beta(y)^2}{\phi_\beta(y)}dy,$$

$$V(\theta_0)=\int W(x)\frac{\{\partial_\gamma c(x,\gamma_0)\}^{\otimes 2}}{c(x,\gamma_0)^2}\pi_0(dx)\cdot\int\frac{\{\phi_\beta(y)+y\partial\phi_\beta(y)\}^2}{\phi_\beta(y)}dy$$

Remarks

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dZ_t.$$

Contrast	Rates	
	α	γ
Gaussian QL	$\sqrt{nh_n}$	$\sqrt{nh_n}$
Non-Gaussian (Stable) QL	$\sqrt{n}h_n^{1-1/\beta}$	$\sqrt{n}$

- GQLE is easier to use, but NGQLE has better performance.
- Both are somewhat robust for the specification of the Lévy measure.
- The technical conditions imposed are, unfortunately, not so mild.
- However, we conjecture that the NGQLE is asymptotically optimal.

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A small numerical example: NIG Lévy process

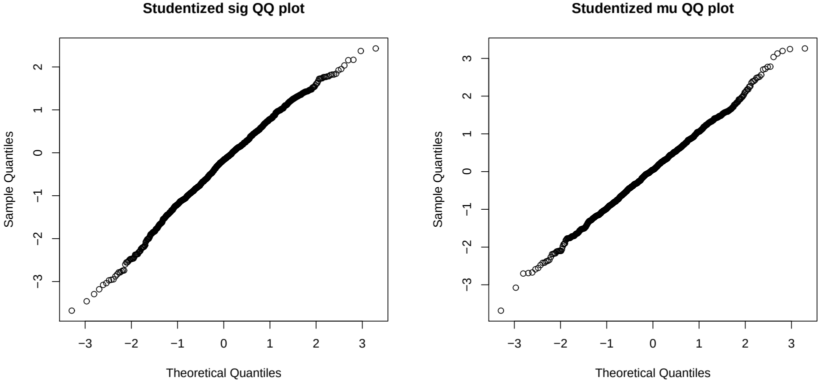
- We set $X_t = \alpha t + \gamma Z_t$ with $\mathcal{L}(Z_t) = NIG(a, 0, t, 0)$ for some (unknown) $a > 0$, hence

$$\frac{X_t - \alpha t}{\gamma t} \sim NIG(at, 0, 1, 0) \to^d \text{standard Cauchy.}$$

- $\theta_0 = (\alpha_0, \gamma_0) \leftarrow (-3, 2)$, $\beta = 1$, and $a = 2$.
- 1000 iterations with $n = 500$ and $h_n = 1/n$.
- Results.

	Sample median	Stable QLE α	Stable QLE γ
Mean	-2.9961	-2.9942	1.9781
S.D.	0.1430	0.1272	0.1237
Max	-2.5186	-2.5852	2.3635
Min	-3.4808	-3.4704	1.6225

Achieving the normality of the NGQLE



Summary and concluding remarks

- **The essential assumption: $\mathcal{L}(h^{-1/\beta}Z_h)$ is approximately β-stable.**
- Without imposing $nh_n \to \infty$ for all cases?: A suitable weak limit theorem is necessary for identifying possible limit distribution.
- Want to utilize the Cauchy quasi-likelihood ($\beta = 1$) for SDE.
- Estimation of the Blumental-Getoor index β: For Lévy driven OUP, we can apply LAD type estimate (M, 2010).
- Large deviation for the random fields, giving convergence of moments?
- Adaptive estimation for jump SDEs? (Uchida and Yoshida (2010) for diffusions)

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