An application of the implied copula model to the risk evaluation of a portfolio

Graduate School of Social Sciences
Tokyo Metropolitan University
Yukio Muromachi

Email: muromachi-yukio@tmu.ac.jp

Purposes (1)

- Before the financial crisis during these years, none of the risk evaluation models can show the possibility to suffer such enormous losses.
- The above is quite natural. Because …
  - ✓ The existing risk evaluation models are purely(?) statistical models, so that the estimated risk is based on the observed data, that is, past events.
  - ✓ Crash of the securitization market is the first event.

- We would like to obtain some forward-looking risk evaluation tools for financial risk management in future.

Purposes (2)

- By the way, before the financial crisis, in the derivative (securitization) markets, there existed some information implying that there would be a small probability of such a crisis.
- Those are the prices of the super-senior tranches in CDO market.

- We think about new risk evaluation models, which can show a probability of such a financial crisis and show the change of the probability day by day based on the market data.
- Such models will be useful as one of the complements for the existing statistical risk evaluation models.

Contents

1. Implied Copula Model

Market Standard Model for Pricing CDOs

is called "one-factor Gaussian Copula Model."

- Gaussian Copula is used as a joint distribution function of default times.

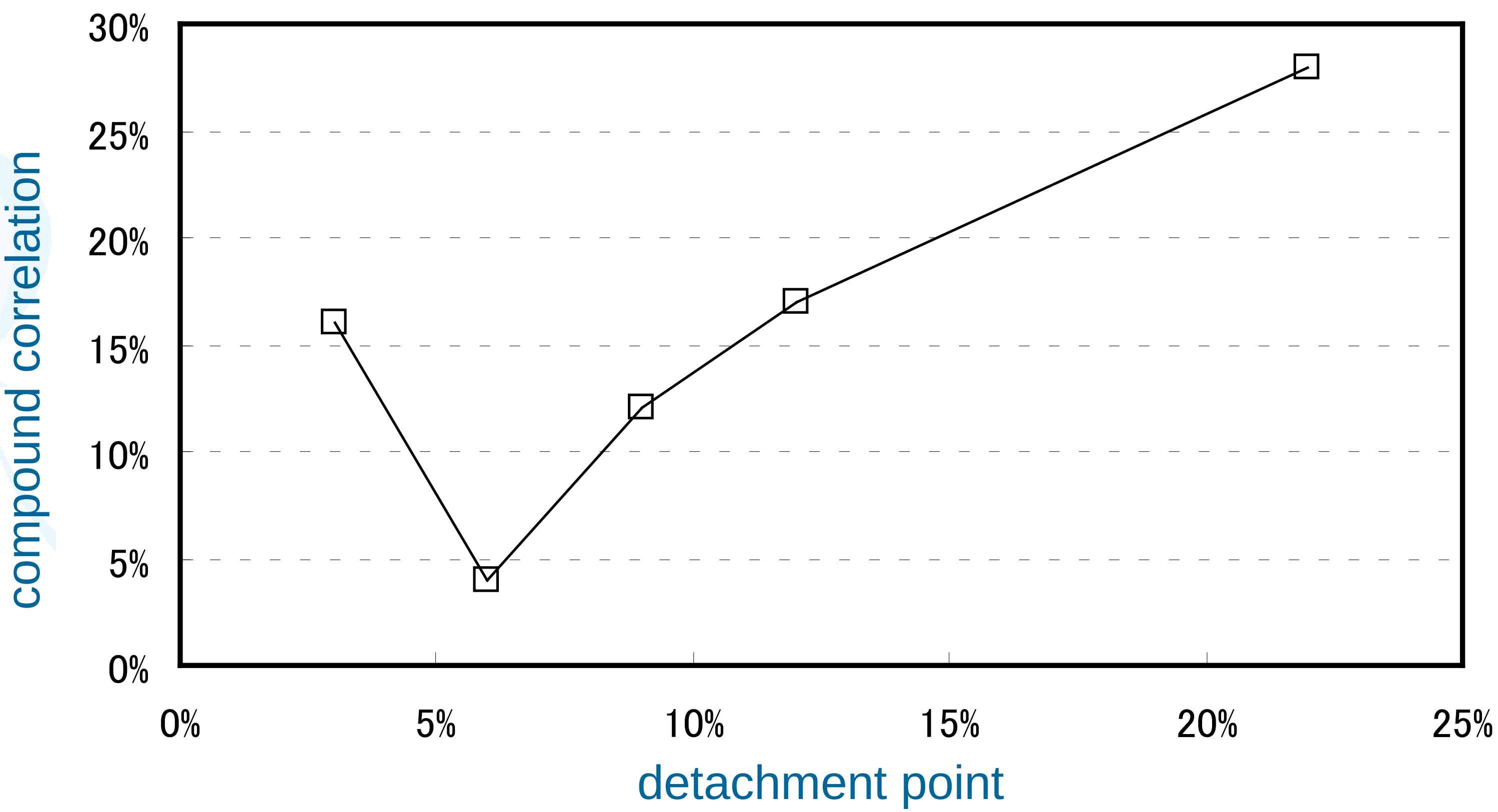
$$C^{Ga}(u_1,\cdots,u_n)=\Phi_n(\Phi^{-1}(u_1),\cdots,\Phi^{-1}(u_n);\Sigma)$$

- Variance-covariance matrix is constructed by one-factor Gaussian model.

$$Z_j=\rho_j X+\sqrt{1-\rho_j^2}\varepsilon_j,\quad j=1,\cdots n$$

- The model is tractable, but cannot explain the market data (price data) consistently.
  - ✓ Correlation Smile or Correlation Skew.

Correlation skew (smile)



Implied Copula Model

is proposed by Hull and White (2006) as a pricing model of CDOs (Collateralized Debt Obligations) in order to solve the problem.

- This model can explain the market prices of several tranches of CDOs better than previous models.

Essences of Implied Copula Model

are described as follows:

- The default times are **conditionally independent**.
- Hazard rates are stochastic.
  - ✓ When $h_j(t)$ denotes the hazard rate of j-th asset at time t, it can be written as

$$h_j(t) = \eta(T) \times h(t), \qquad 0 \le t \le T, \quad j = 1, \cdots n,$$

    and $\eta(T)$ is a stochastic variable dependent of maturity T.
  - ✓ For simplicity, hazard rates are assumed to be homogeneous, that is, $h_j(t), \quad j = 1, \cdots n,$ is the same value.
  - ✓ Distribution of $\eta(T)$ is calibrated non-parametrically based on the market price of CDOs.

Here we propose …

a credit risk evaluation model of a portfolio based on the implied copula model.

- Its essences are follows:
  - ✓ Two probability measures are used. One is the physical measure, and the other is the risk-adjusted measure for pricing.
  - ✓ Under each measure, hazard rates (default probabilities) are stochastic as they are in the implied copula model.
  - ✓ For a tractability, we construct our model by using default probabilities directly, in contrast to Hull and White (2006), in which they used hazard rates.

2. The Model

single-period model

Basic Settings (1)

- Consider N assets issued by different firms.
- P : the physical probability measure.
- An integer N, $N \in \{1, 2, \cdots, K\}$: the state of the future.
- There exists a unique risk-neutral probability measure $\tilde{P}$, and the default times are conditionally independent given N.
- τ_j : default time of j-th asset.
- $h_j(t)$: hazard rate of j-th asset under P.
- $\tilde{h}_j(t)$: hazard rate of j-th asset under $\tilde{P}$.
- $r(t)$: default-free instantaneous short rate.
  (Stochastic interest rate models can be used.)
- δ_j : recovery rate of j-th asset.

Basic Settings (2)

- We use sub-filtration approach, which is often used in credit risk modeling.
- $H_j(t) = 1_{\{\tau_j \le t\}}, \quad j = 1, \cdots, n$
- $H_t^j = \sigma(H_j(s), 0 \le s \le t), \quad j = 1, \cdots, n$
- $H_t = H_t^1 \vee H_t^2 \vee \cdots \vee H_t^n$
- $F_t = G_t \vee H_t$
- H_t is the information of default times.
- G_t is other information.

- T : risk horizon. Hereafter, T = 1 year.

Default process (1)

- ✓ $\tilde{F}_j(t) = \tilde{P}\{\tau_j \le t\}$: default probability up to time t.
- ✓ $\tilde{F}_j(t,s) = \tilde{P}\{\tau_j \le s \mid \tau_j > t\}$: conditional default probability.

➢ Assumption 2.1

$$\tilde{F}_j(t \mid N=k) \;=\; \tilde{\kappa}(k)\,\tilde{F}_j(t), \qquad k = 1, \cdots, K$$

where $\tilde{\eta}(k) \;=\; \tilde{P}\{N = k\} \ge 0$ and

$$\sum_{k=1}^{K} \tilde{\kappa}(k)\tilde{\eta}(k) \;=\; 1, \qquad \sum_{k=1}^{K} \tilde{\eta}(k) \;=\; 1$$

➢ Assumption 2.2

Under $\tilde{P}$, default of each asset is N-conditionally independent.
Given N=k, each default occurs independently with prob. $\tilde{F}_j(t \mid N=k)$

Default process (2)

- We suppose $\tilde{F}_j(t) = E\left[\tilde{F}_j(t \mid N)\right]$.

  Therefore, we can also use the existing stochastic interest rate models.

- Assumption 2.3

  For simplicity, the conditional forward default probability is a deterministic function of time t. Therefore,

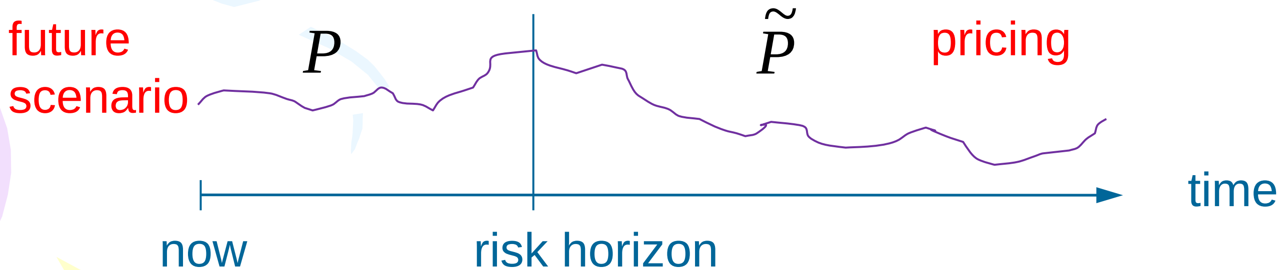
$$\tilde{F}_j(t,s \mid N=k) = \frac{\tilde{F}_j(s \mid N=k) - \tilde{F}_j(t \mid N=k)}{1-\tilde{F}_j(t \mid N=k)} = \tilde{\kappa}(k)\frac{\tilde{F}_j(s) - \tilde{F}_j(t)}{1-\tilde{F}_j(t \mid N=k)}$$

- Remark 2.1

  $0 \le \tilde{F}_j(t,s \mid N=k) \le 1$ must be satisfied.

Change of measure (1)

- Our model is future-value based model.
  - ✓ As the future value, we calculate the price under no-arbitrage condition.
  - ✓ Therefore, the cash flows after T (risk horizon) are evaluated under the risk-neutral probability measure $\tilde{P}$.
  - ✓ However, the scenarios up to T are under physical probability measure P .
- Therefore, we must use these two measures in our risk evaluation model.



Change of measure (2)

- Interest rate models:

  For example, Hull-White model.

  - ✓ Under P , $dr(t) = (b(t) - ar(t))dt + \sigma\, dz(t)$
  - ✓ Under $\tilde{P}$, $dr(t) = (\phi(t) - ar(t))dt + \sigma\, d\tilde{z}(t)$

    where $\phi(t) = b(t) - \beta(t)\sigma,\ d\tilde{z}(t) = dz(t) + \beta(t)dt$
  - ✓ $\beta(t)$: market price of risk

- Hazard rate models:

  From Kijima and Muromachi (2000),

  - ✓ Under P , $h_j(t)$
  - ✓ Under $\tilde{P}$, $\tilde{h}_j(t) = h_j(t) + \ell_j(t)$
  - ✓ $\ell_j(t)$: risk-premia adjustments

Change of measure (3)

- **Assumption 2.4**

  We assume $1-\tilde{F}_j(t) = L_j(t)\,(1-F_j(t)), \quad 0 \le t \le T^*$

  ✓ This can be derived from

$$\tilde{P}\{\tau_j > t\} = E\left[\exp\left(-\int_0^t \tilde{h}_j(s)ds\right)\right] = E\left[\left[\exp\left(-\int_0^t \ell_j(s)ds\right)\right]\exp\left(-\int_0^t h_j(s)ds\right)\right]$$

  ✓ Here, we assume $\ell_j(t)$ are deterministic functions of time.

- **Assumption 2.5**

  Under P, default of each asset is N-conditionally independent.

  ✓ "Conditional independence" is not invariant w.r.t. change of measure.

Default process (3)

- Assumption 2.6

  Given N=k, the conditional default probability is given by

$$F_j(t \mid N=k) = \tilde{\kappa}(k)\, F_j(t), \qquad k = 1, \cdots, K$$

  - ✓ Distribution of $\tilde{\kappa}$ under P is the same as the one under $\tilde{P}$.
  - ✓ This assumption is ad hoc.

Interest rate process

- We can use existing interest rate processes in our model.

- For simplicity, we assume the default-free interest rates and the default probabilities are independent.
  - ✓ Even without this assumption, this independence is satisfied under our setting.
  - ✓ Additionally, in numerical examples, we assume a deterministic default-free interest rates.

Procedures for risk evaluation

1. Generate N with the probability distribution $\tilde{\eta}$.
2. Given N=k, calculate $\tilde{F}_j(t \mid N=k),\ \ j=1,\cdots,n$, and a sample-path of the short rates r(s) up to T.
3. Generate a default scenario. Namely, judge the states (default or survive) of all assets on T.
4. Given r(T), generate term structure of the interest rates at T.
5. Evaluate the prices of all assets at T.
6. Sum up the prices, and we obtain a sample of the future portfolio value.
7. Repeat from 1 to 6 until enough numbers of scenarios are obtained.

Valuation of Bond in future

- According to Jarrow and Turnbull (1995), we obtain present (t=0) value of the defaultable discount bond

$$v_j(0,\tau)=v_0(0,\tau)\left[\delta_j+(1-\delta_j)\tilde{P}\{\tau_j>\tau\}\right]$$

- Given N=k, its future value is obtained as

$$v_j(T,\tau)=v_0(T,\tau)\left[\delta_j+1_{\{\tau_j>T\}}(1-\delta_j)\tilde{P}_T\{\tau_j>\tau\mid N=k\}\right]$$

where $\tilde{P}_T\{\tau_j>\tau\mid N=k\}=1-F_j(T,\tau\mid N=k)$

Setting in “Jarrow and Turnbull” at maturity

If it defaults before maturity, the holder obtains $\$\delta_j$.

If others, he obtains \$1.

Valuation of CDO in future

- Present (t=0) value of CDO is calculated in a ordinary manner.

- In order to obtain a future value of CDO, consider
  - ✓ default legs, premium legs, and accrued interests before T.
  - ✓ future price of CDO without already defaulted assets.

3. Numerical Example: a synthetic CDO

risk horizon = 1 year

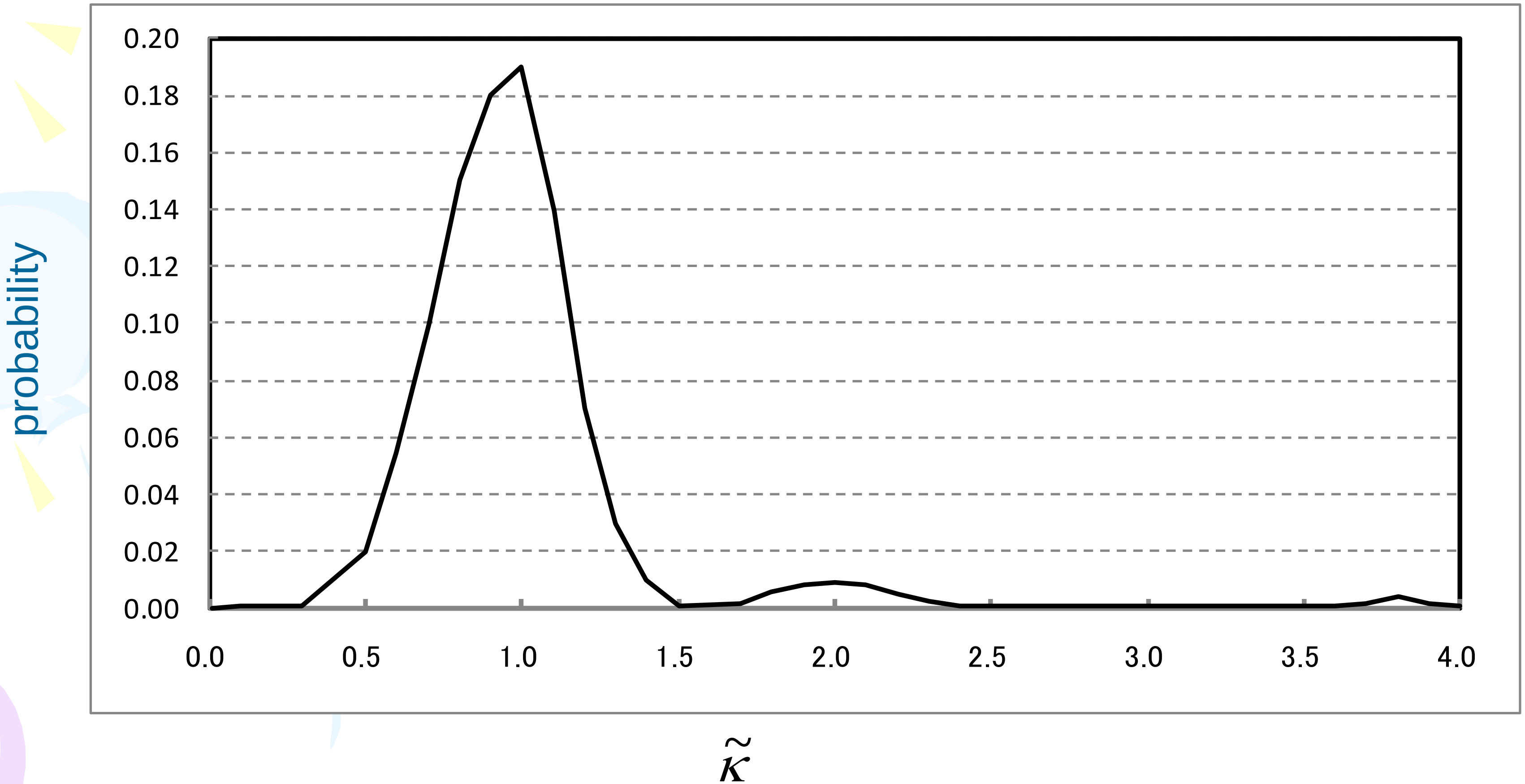
Monte Carlo (100,000 scenarios)

CDO setting

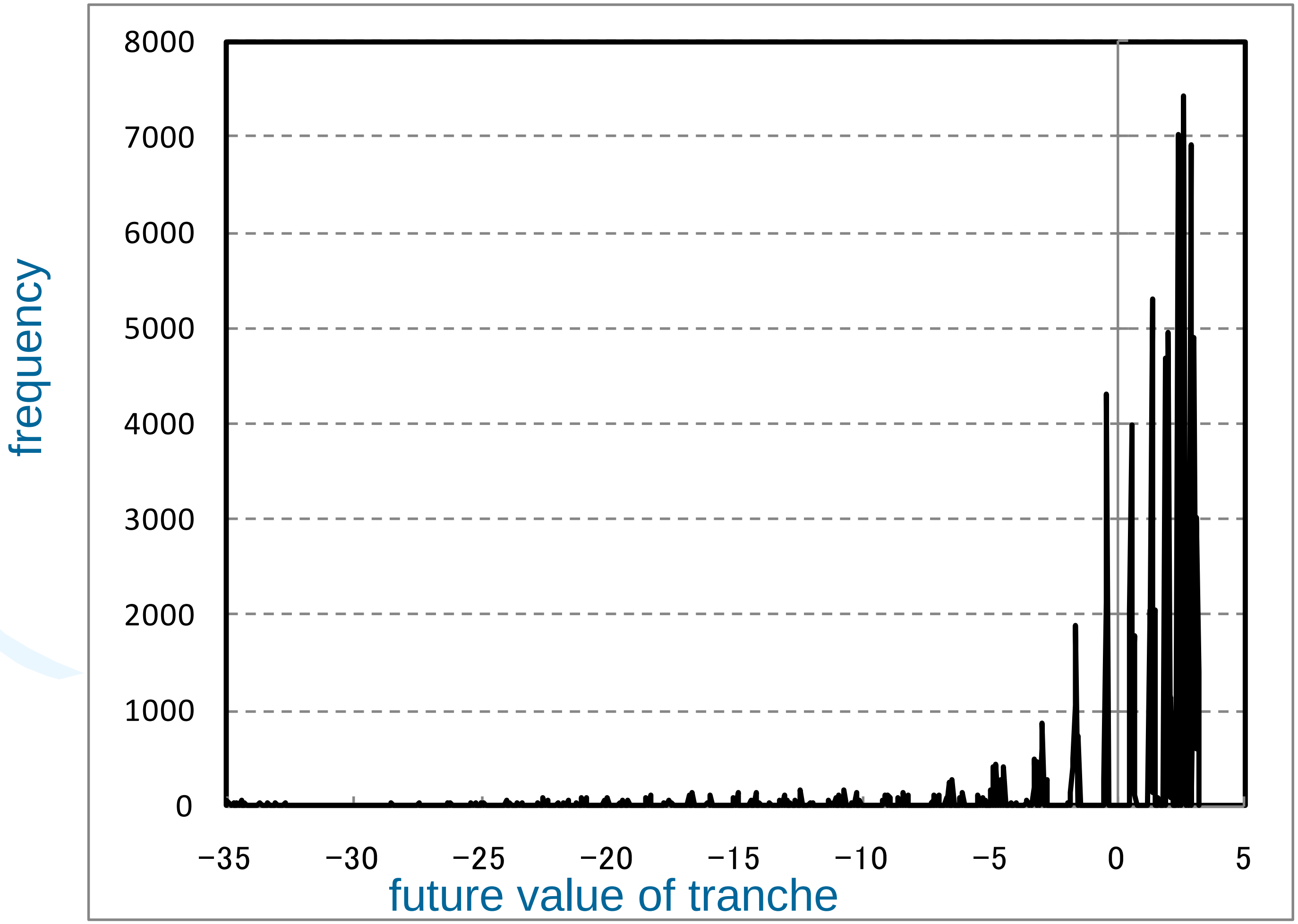
- ✓ Maturity: 5 years.
- ✓ Asset pool: 125 entities.
- ✓ Each asset: face value 10, recovery rate 40%.
- ✓ Credit rating: A … default probability 0.5% / year
  B … default probability 1.0% / year
- ✓ A: 50 entities, B: 75 entities.
- ✓ Interest rates: flat
  default-free 1%, A-rated 2%, B-rated 3%.
- ✓ Tranching: 6 tranches, 0-3, 3-6, 6-9, 9-12, 12-22
- ✓ Coupon rates = fair spreads under this condition.

This is “not” an implied distribution from CDO prices.

Distribution of multiplier

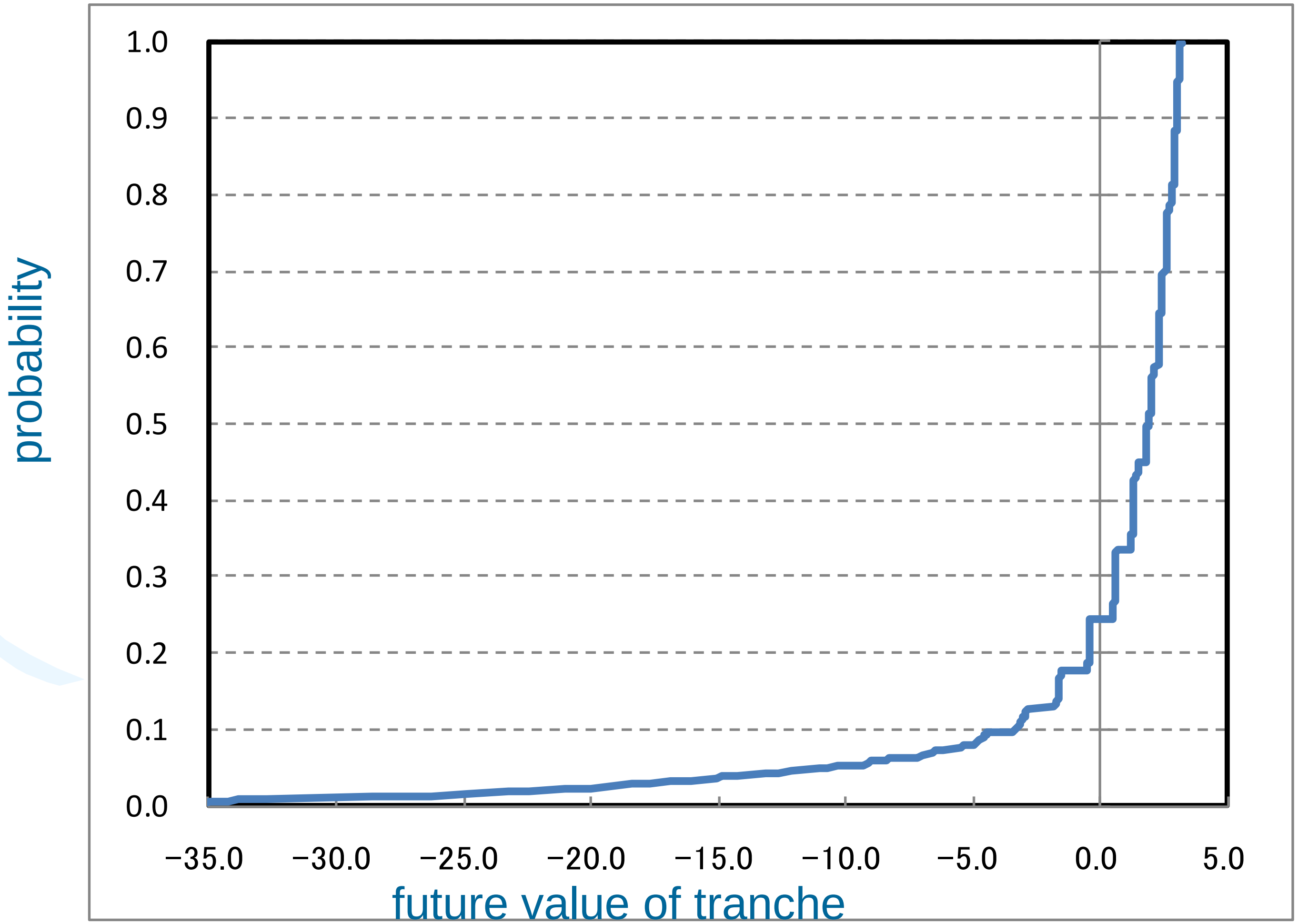


Distribution of future value [3, 6]

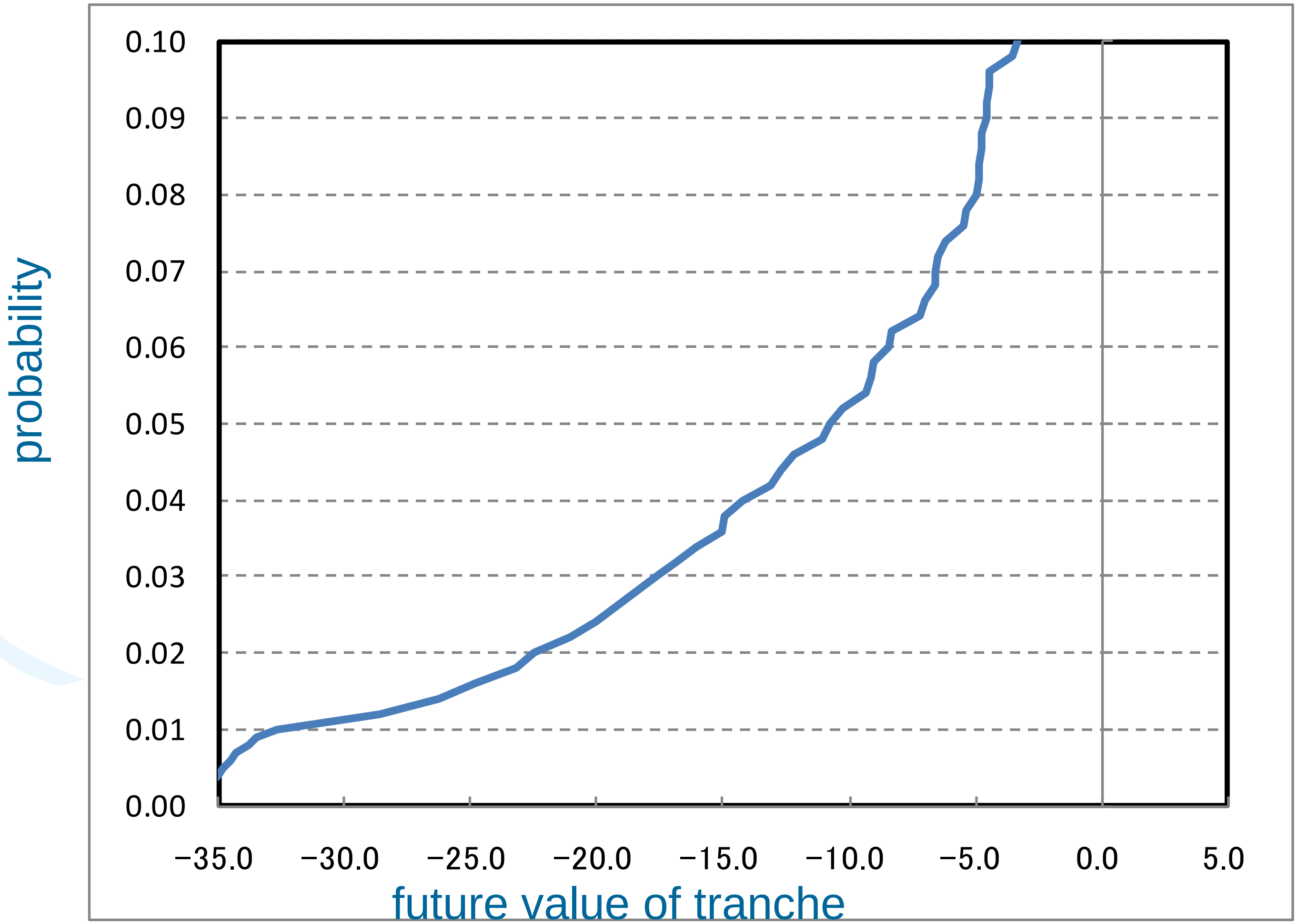


Distribution function [0, 3] (1)

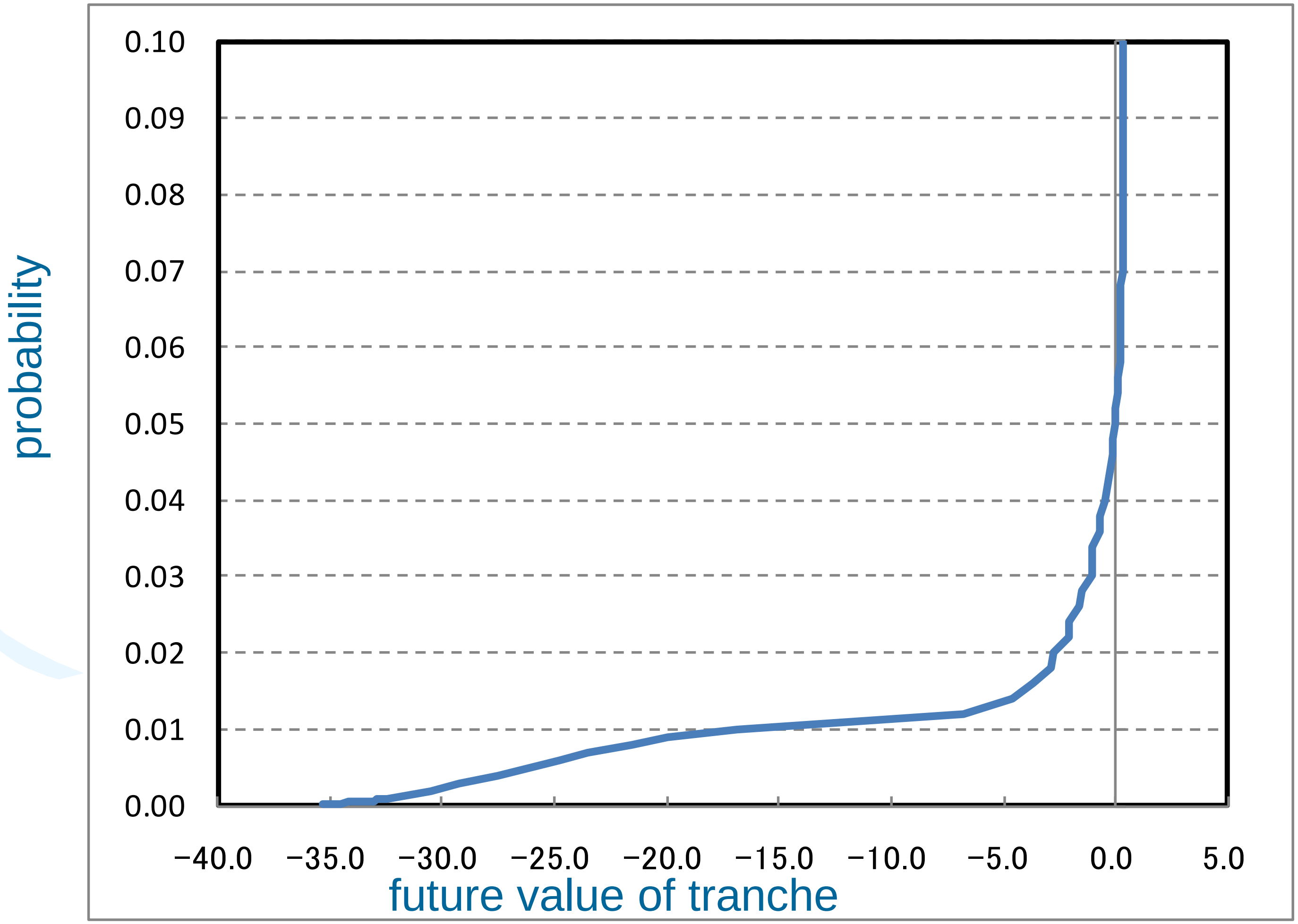
Distribution function [3, 6] (1)



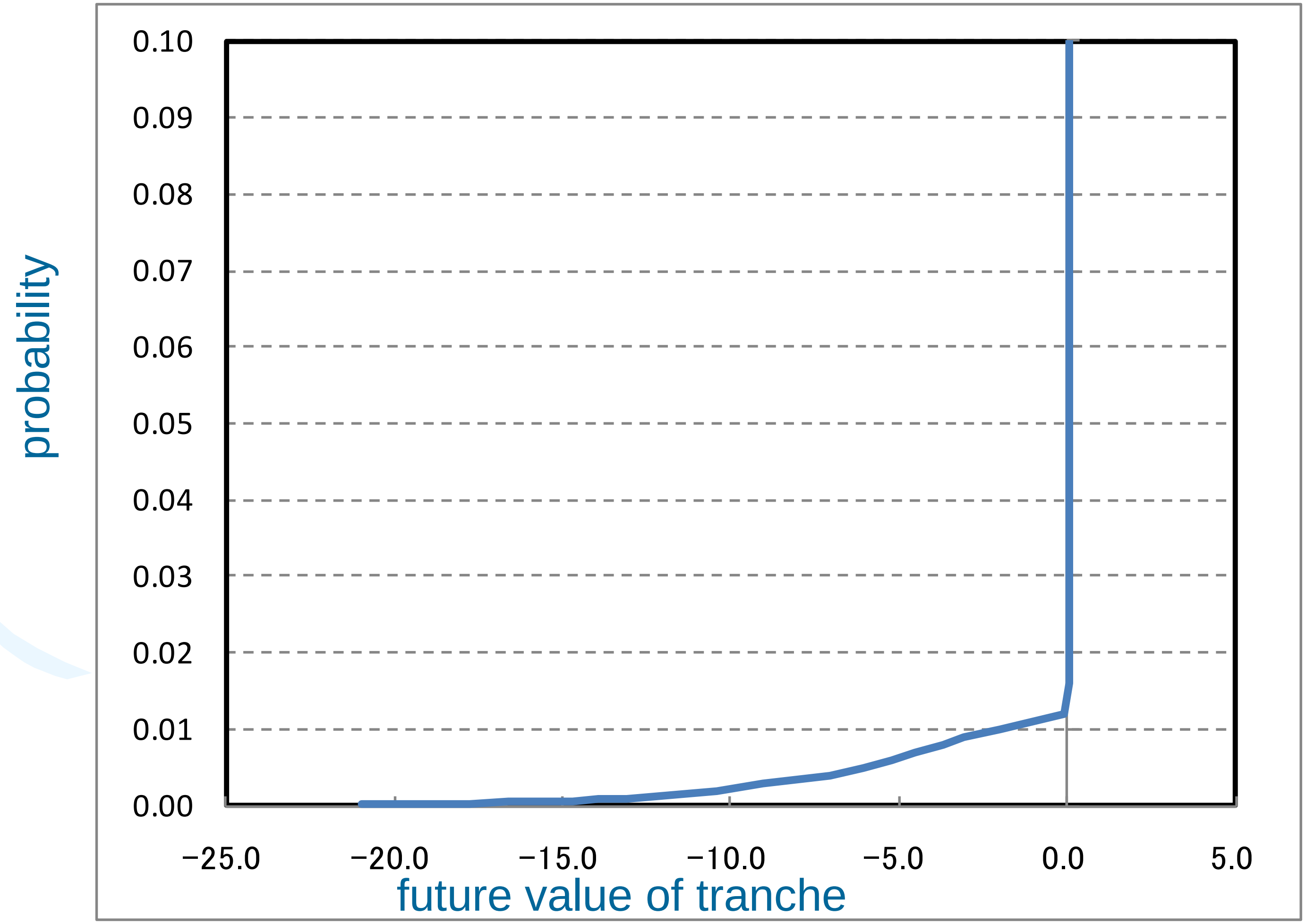
Distribution function [3, 6] (2)



Distribution function [6, 9]



Distribution function [9, 12]



Distribution function [12, 22]

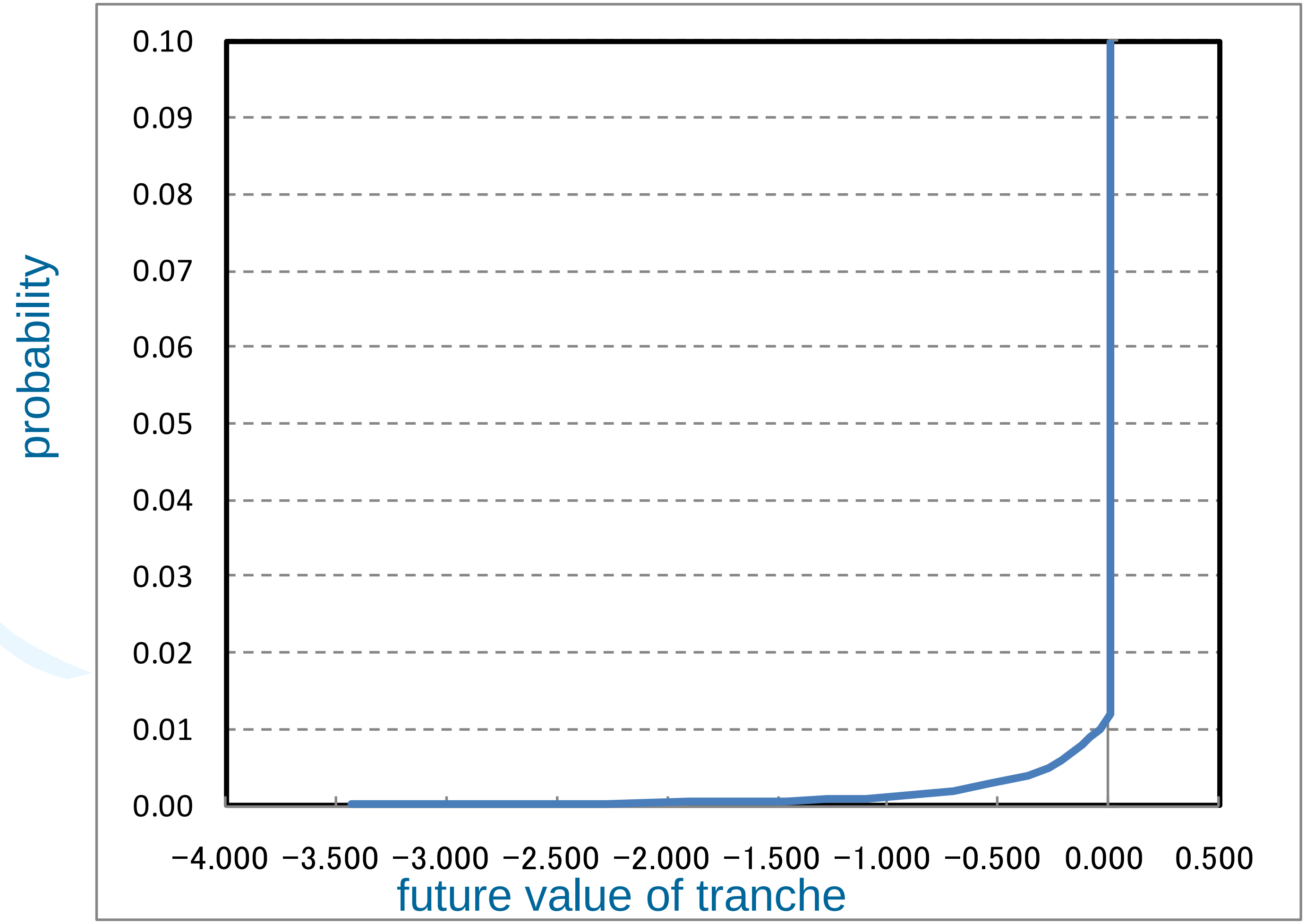


Table of VaR, ES (1)

attach-detach		0-3	3-6	6-9	9-12	12-22
initial face value		37.5	37.5	37.5	37.5	125.0
average		0.0	0.0	0.0	0.01	0.001
standard deviation		12.4	5.8	2.8	0.86	0.085
VaR	95.0%	22.5	10.7	0.1	-0.08	-0.005
	99.0%	30.1	32.7	16.9	2.02	0.039
	99.5%	31.7	34.9	26.2	6.08	0.278
	99.9%	33.6	35.8	32.4	13.12	1.105
Expected	95.0%	27.1	21.7	6.9	1.46	0.098
Shortfall	99.0%	31.8	34.7	26.3	7.39	0.509
	99.5%	32.8	35.5	30.1	10.53	0.868
	99.9%	33.6	36.2	33.9	16.73	2.081

100α%-VaR = average – 100(1-α)-percentile
100α%-ES = average – conditional expectation under 100(1-α)-percentile

Summary : CDO

- Each tranche has a left long-tail.
- The left tail grows drastically near (or under) 1% confidence level in the mezzanine tranches.
  - ✓ This method can give a common shock with a certain confidence level to many tranches.
  - ✓ It implies that this model might succeed the description of the Armageddon factor.
  - ✓ This is "market-implied" Armageddon factor ?

  - ✓ However, in a sense, these results is only natural.
  - ✓ And, this tendency depends strongly on the distribution of the multiplier.

Summary : CDO (cont.)

- As in Table, VaRs under high confidence levels of equity [0-3] is lower than those of mezzanine [3-6].
  - ✓ This is because we consider the effect of the coupon up to the risk horizon. The coupon of the equity is high.
  - ✓ And, this might be partly because this model can give a common shock with a certain confidence level to many tranches.

☆ I'm sorry that my numerical results are wrong in my article. These results shown here would be correct, I hope.

4. Numerical Example: a Bond Portfolio

risk horizon = 1 year

Monte Carlo (100,000 scenarios)

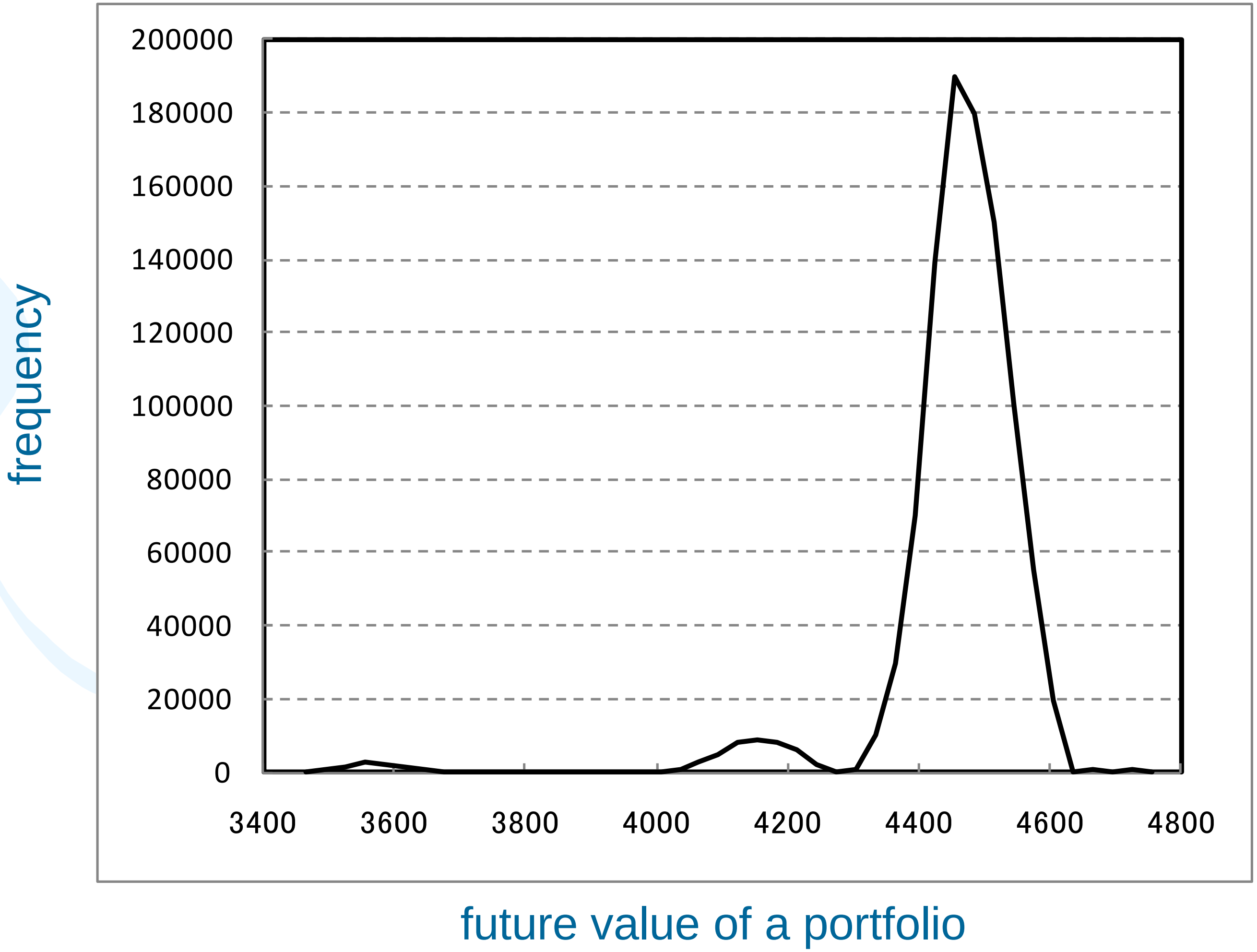
Bonds setting

- ✓ Portfolio: 500 corporate discount bonds.
- ✓ A-rated: 200 bonds, B-rated: 300 bonds.
- ✓ Maturity: 5 years.
- ✓ Face value: 10, Recovery Rate: 40%.

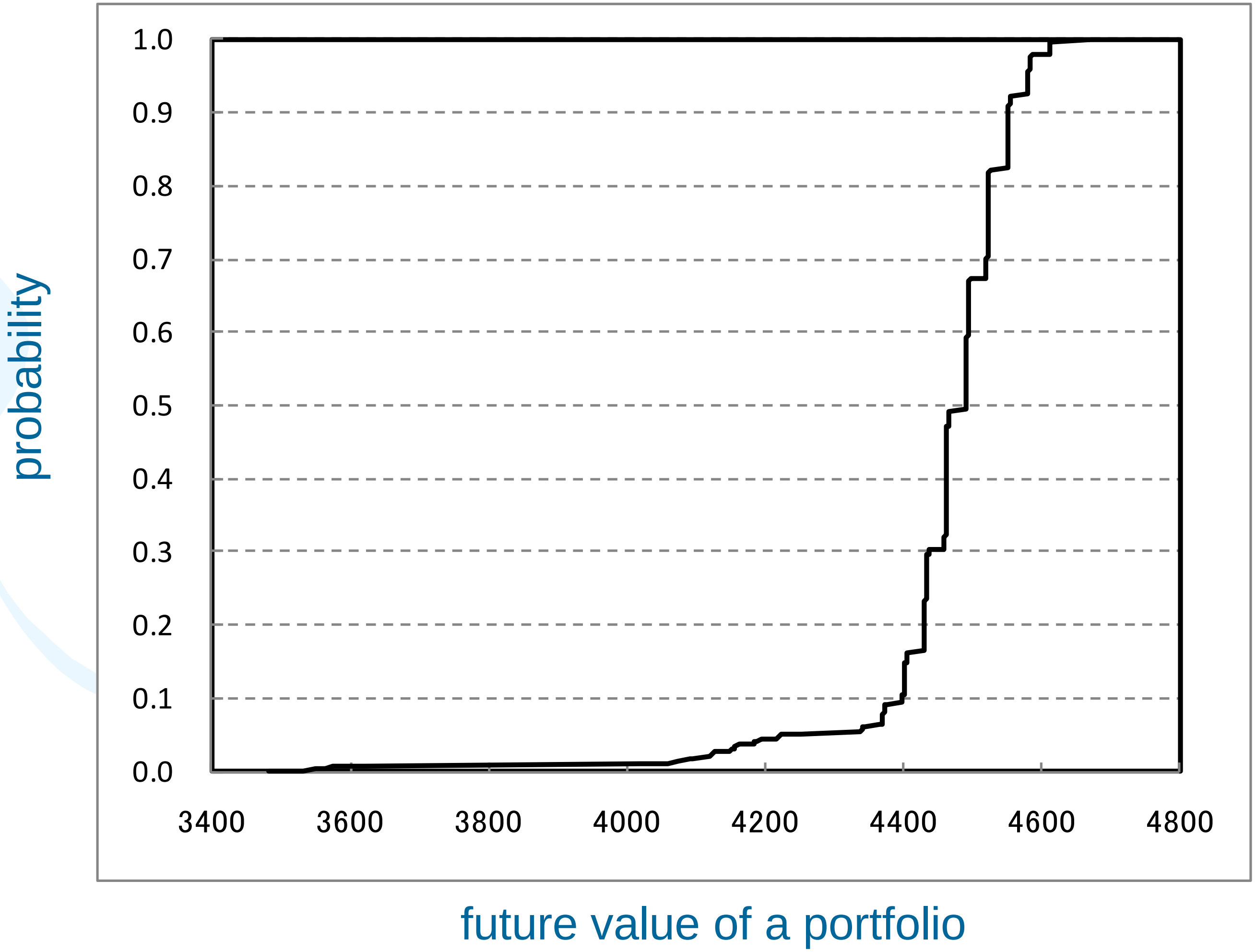
- ✓ Default probabilities: same in Section 3.
- ✓ Interest rates: same in Section 3.
- ✓ Distribution of multiplier: same in Section 3.

This application might be controversial, I think.

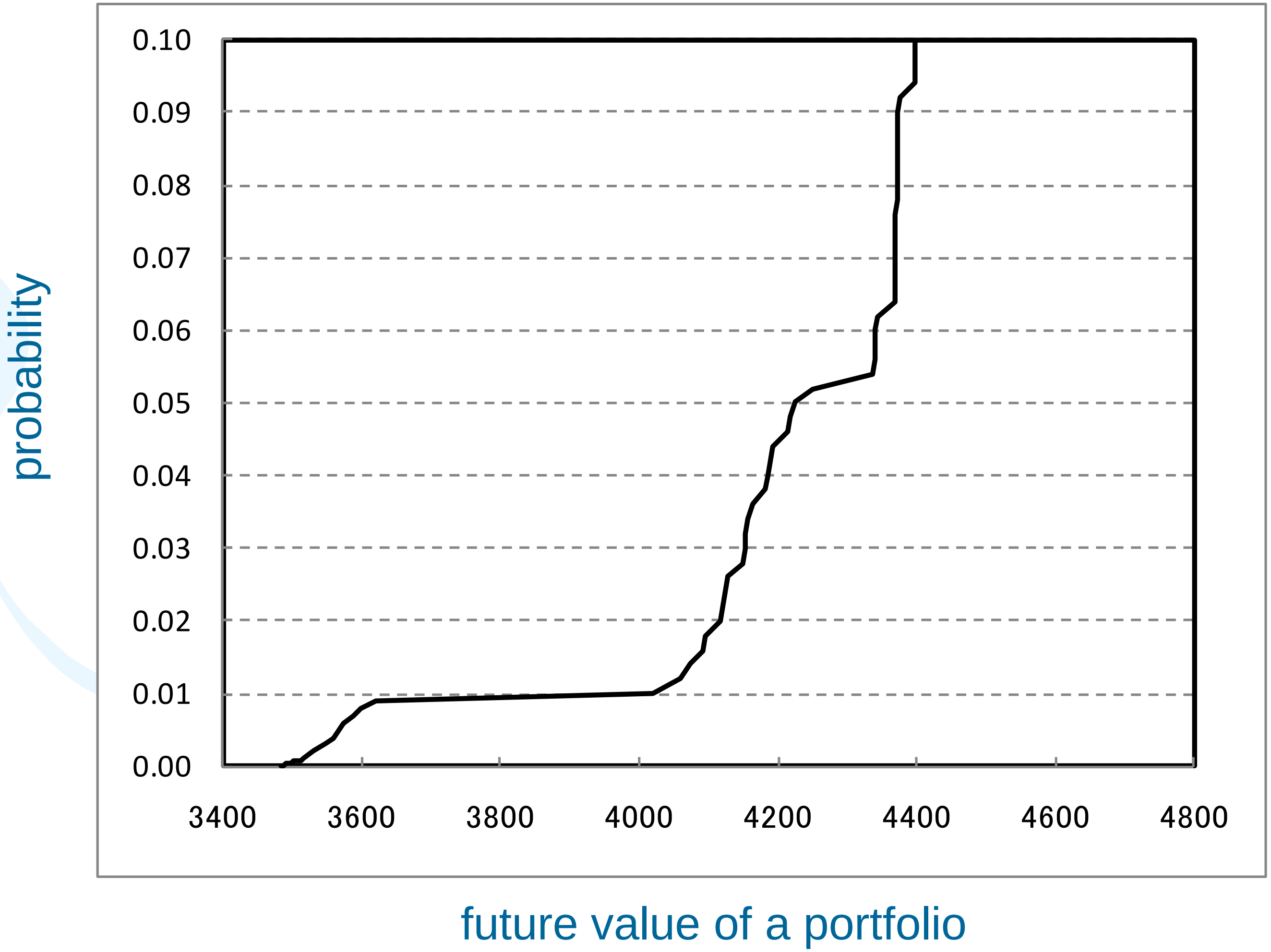
Distribution of future value



Distribution function (1)



Distribution function (2)



Expected Shortfall

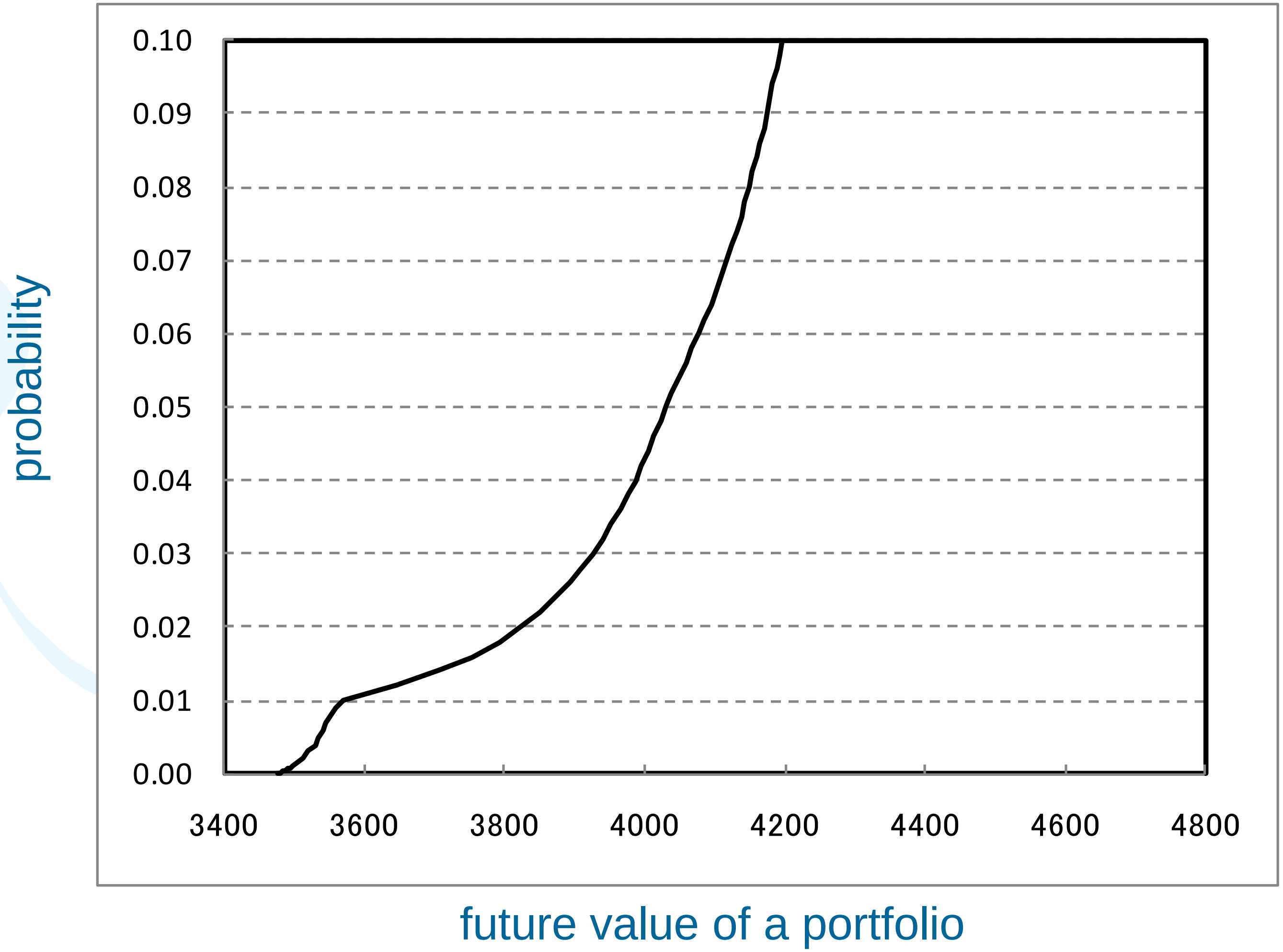


Table of VaR, ES (2)

average		4,460.8
standard deviation		125.1
VaR	95.0%	237.7
	99.0%	441.8
	99.5%	895.3
	99.9%	945.6
Expected Shortfall	95.0%	431.0
	99.0%	892.4
	99.5%	924.9
	99.9%	963.3

100α%-VaR = average – 100(1-α)-percentile
100α%-ES = average – conditional expectation under 100(1-α)-percentile

Summary: bond portfolio

- Distribution of future value of the portfolio has a left long-tail.
- VaR moves drastically near 5% and 1%.
  - ✓ These results strongly depend on the distribution of the multiplier κ.
- ES moves more smoothly than VaR.
  - ✓ ES is more preferable in practice (?)

5. Concluding Remarks

Some comments

- According to the numerical results of Hull and White (2006), there exists a small probability that the hazard rates will grow much larger than usual based on the market prices of CDOs, before the credit crunch after 2007.
- Our model can reflect the probability on the risk evaluation of a portfolio. That is, the latent fear of the market participants can be reflected on the risk evaluation.
- In other words, the stress scenarios implied by the market data can be includes with their probabilities in our model.

- Although the implied copula is a static model, our extension can include stochastic behavior of mean hazard rates.

References

- Hull, J. and A., White, “Valuing credit derivatives using an implied copula approach,” Journal of Derivatives, 14(2), 2006.
- Hull, J. and A., White, “An improved implied copula model and its application to the valuation of bespoke CDO tranches,” Journal of Investment Management, Forthcoming.
- Hull, J. and A. White, “Valuation of a CDO and n-th to default CDS without Monte Carlo simulation,“ Journal of Derivatives, 12(2), 2004.
- Kijima, M., and Y. Muromachi, “Evaluation of a credit risk of a portfolio with stochastic interest rate and default processes,” Journal of Risk, 3(1), 2000.

Thank you for your attention.

Appendix.

Calibration of $\eta_s(t)$

- As in the implied copula model, the distribution $(\eta_s(t), P\{S=s\})$ is calibrated based on the market prices of CDO tranches.

- Please see the details in Hull and White (2006).