Analytic and Asymptotic Approach for SABR

Yasufumi OSAJIMA *
BNP PARIBAS

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*This presentation represents the view of the author and not necessarily those of BNP Paribas Securities. Please address correspondence to: BNP Paribas Securities, Quantitative Research, GranTokyo North Tower 1-9-1, Marunouchi, Chiyoda-ku, Tokyo 100-6740, Japan, E-mail: yasufumi.osajima@japan.bnpparibas.com

Main Issues

- Our main purpose is to give asymptotic expansion formulas of implied volatilities for general diffusion models.

- SABR model is one of stochastic volatility models and popular among practitioners. That is because an accurate asymptotic formula of volatility smile for European call options is known.

- We generalize this formula for general diffusion models. We take an approach based on Malliavin calculus.

- The theory of asymptotic expansions of probability densities based on Malliavin calculus was originated by Bismut [1] and was developed by Watanabe [8] and Kusuoka-Stroock [4].

1. Volatility Skew / Smile

- A European call option is a derivative which at some terminal time T has a value $(X(T) - K)_+$, where $X(T)$ is a underlying asset price at time T and K is a strike rate.

- In financial markets, the Black-Scholes formula has been widely used to price European options. We assume the log-normal model,

$$dX(t) = \sigma X(t) dW(t), \quad X(0) = x_0.$$

Then the (undiscounted) value of a call option with strike K, maturity T is given by

$$C_{BS}(T, K, \sigma) = E[(X(T) - K)_+] = x_0 \Phi(d_1) - K \Phi(d_2),$$

where

$$d_{1,2} = \frac{\log(x_0/K) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

- It is a common practice to quote option prices in terms of 'implied volatility', i.e. given a price $C(T,K)$, the implied volatility is given by

$$C(T,K) = C_{BS}(T,K,\sigma_{BS}(T,K)).$$

- In the original Black-Scholes model, the implied volatility must be constant independent of strike rate.

- But in real financial markets such as foreign exchange options and stock index options, observed implied volatilities depend on strike rate.

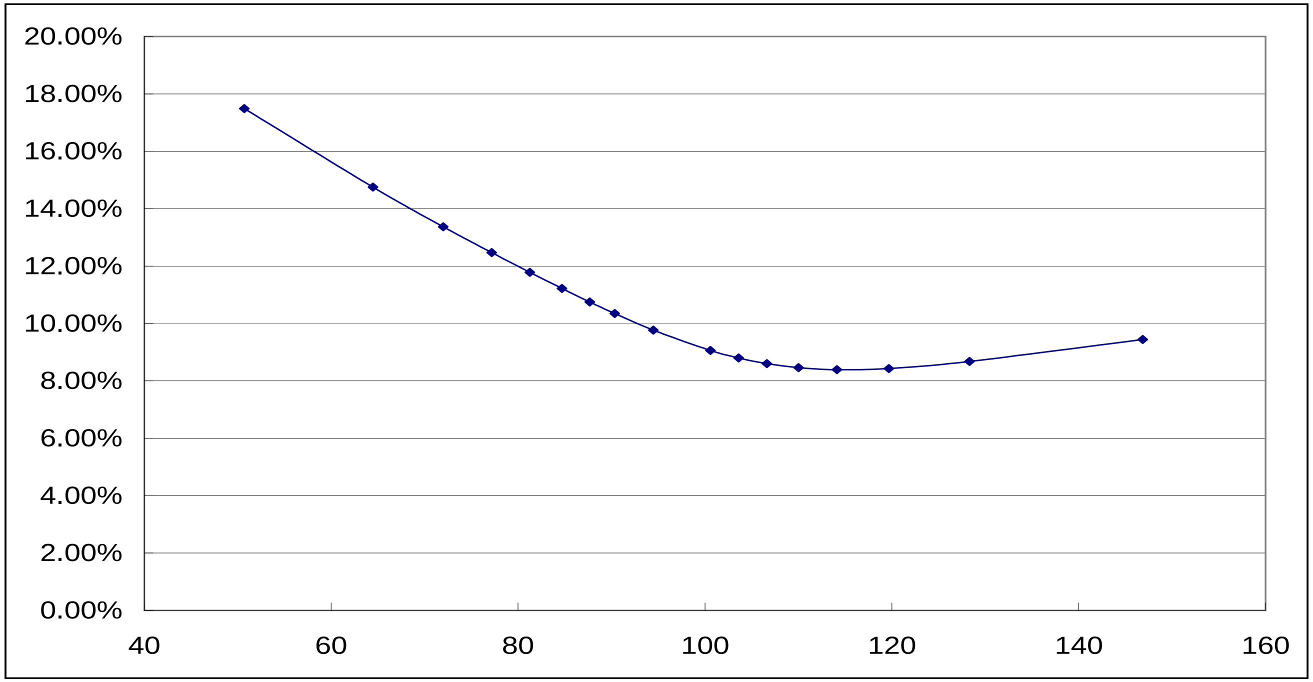

Figure 1: Sample of Implied volatility smile observed in the market

- To price and hedge complex exotic derivatives appropriately, it is necessary to build a model that can calibrate to the volatility smile accurately.

- There are two well-known models to explain these phenomena. The first class of models are called local volatility models for which the volatility is assumed to depend on time and the spot price of the underlying.

- CEV model

$$dX(t) = \alpha X(t)^{\beta} dW(t).$$

The second class of models are stochastic volatility models.

- SABR model

$$dX(t) = \alpha(t)X(t)^{\beta} dW_0(t),$$
$$d\alpha(t) = \nu\alpha(t)(\rho dW_0(t) + \sqrt{1-\rho^2}dW_1(t)).$$

- To calibrate the model to the market, it is necessary to calculate the values of European call options.

2. SABR formula

- $(\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\}_{0 \leqslant t \leqslant T})$ be a complete probability space satisfying the usual hypotheses.

- Under T-forward measure, we assume the following model,

$$
\begin{aligned}
&dX(t) = \varepsilon \alpha(t) \sigma(X(t)) dW_1(t), \\
&d\alpha(t) = \varepsilon \nu \alpha(t) dW_2(t), \\
&d\langle W_1, W_2 \rangle = \rho dt, \;\; X(0) = x_0, \;\; \alpha(0) = \alpha.
\end{aligned}
$$

- Forward value of call option of strike K, maturity T is,

$$
C(T, K) = E^T[(X(T) - K)_+].
$$

- We define

$$G(x) = \int_x^\infty (y - x)\phi(y)dy = \phi(y) - y\Phi(-y),$$

where Φ is the normal distribution and ϕ is Gaussian density.

- When we assume normal model,

$$d\tilde{X}(t) = \sigma_N dW(t), \quad \tilde{X}(0) = x_0,$$

the forward value of call option is

$$C_N(T, K, \sigma_N) = \sigma_N \sqrt{T} G\Big(\frac{K - x_0}{\sigma_N \sqrt{T}}\Big).$$

- We are interested in the implied normal volatility of SABR model, that is, $\sigma_N(K)$ satisfying

$$C(T, K) = C_N(T, K, \sigma_N(K)).$$

- Hagan [2] has calculated the asymptotic expansion of implied normal volatility for SABR model using singular perturbation technique and has obtained the following famous formula;

Theorem. [Hagan] *As $\varepsilon \downarrow 0$, implied normal volatility for SABR model is as follows;*

$$\sigma_N(K) = \frac{\alpha(x_0 - K)}{\int_K^{x_0} \frac{dx}{\sigma(x)}} \left(\frac{\zeta}{\hat{x}(\zeta)} \right)$$

$$\left\{ 1 + \varepsilon^2 \left[\frac{2\gamma_2 - \gamma_1^2}{24} \alpha^2 \sigma^2(X_{av}) + \frac{1}{4} \rho \nu \alpha \gamma_1 \sigma(X_{av}) + \frac{2 - 3\rho^2}{24} \nu^2 \right] T + \cdots \right\}$$

where,

$$x_{av} = \sqrt{x_0 K}, \gamma_1 = \frac{\sigma'(x_{av})}{\sigma(x_{av})}, \gamma_2 = \frac{\sigma''(x_{av})}{\sigma(x_{av})},$$

$$\zeta = \frac{\nu}{\alpha} \frac{x_0 - K}{\sigma(x_{av})}, \ \hat{x}(\zeta) = \log \frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1 - \rho}.$$

3. Energy function of SABR model

- Let H be a Cameron-Martin space. We consider the following associated ODE;

$$\frac{df(t;h)}{dt} = a(t;h)C(f(t;h))(\sqrt{1-\rho^2}\dot{h_1}(t) + \rho\dot{h_2}(t))$$
$$\frac{da(t;h)}{dt} = \nu a(t;h)\dot{h_2}(t),$$
$$f(0,h) = x_0, \ a(0,h) = \alpha.$$

where $h \in H$.

- From Watanabe and Kusuoka-Stroock theory, as $\varepsilon \downarrow 0$, asymptotic expansion of density function is

$$p^{\varepsilon}(t;y) \sim \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{e(y)}{2\varepsilon^2}}(a_0(y) + \varepsilon a_1(y)),$$

where

$$e(y) = \inf\{\frac{1}{2}\int_0^1 |\dot{h}(s)|^2 ds; f(1,h) = y\}.$$

- In this model, we can give the energy term explicitly.

Theorem 1. *In SABR model, energy term is*

$$e(K) = \frac{1}{2\nu^2 T}\log(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho})^2 = \frac{\hat{x}(\zeta)^2}{2\nu^2 T},$$

where

$$\zeta = -\frac{\nu}{\alpha}\int_{x_0}^{K}\frac{dz}{C(z)}.$$

Proof. We define

$$B(x) = \int_{x_0}^{x}\frac{dz}{C(z)}.$$

And define

$$q_1(t) = B(f(t;h)),\ q_2(t) = a(t;h).$$

Then q_1, q_2 satisfies the following ODE:

$$\frac{dq_1(t)}{dt} = q_2(t)(\sqrt{1-\rho^2}\dot{h}_1(t) + \rho\dot{h}_2(t)),$$
$$\frac{dq_2(t)}{dt} = \nu q_2(t)\dot{h}_2(t).$$

We define Riemanian metric on $\mathbb{R}^2$ as

$$ds^2 = \sum_{i,j=1}^{2} g_{ij}(q)dq^i dq^j$$

where (g_{ij}) is inverse matrix of (g^{ij}) and

$$\begin{pmatrix} g^{11}(q) & g^{12}(q) \\ g^{21}(q) & g^{22}(q) \end{pmatrix} = \begin{pmatrix} q_2^2 & \rho\nu q_2^2 \\ \rho\nu q_2^2 & \nu^2 q_2^2 \end{pmatrix}$$

Then, We can interpret $e(y)$ as the square of minimum geodesic distance between the point $\{(q_1, q_2) = (0, \alpha)\}$ and the line $\{q_1 = B(K)\}$. We consider the following Hamiltonian

$$H(p,q) = \frac{1}{2}\sum g^{ij}(q)p_i p_j = \frac{1}{2}q_2^2(p_1^2 + 2\nu\rho p_1 p_2 + \nu^2 p_2^2),$$

and the associated Hamilton equation is;

$$\begin{aligned}
\frac{dq_1(t)}{dt} &= q_2(t)^2(p_1(t) + \nu\rho p_2(t)), \\
\frac{dq_2(t)}{dt} &= q_2(t)^2(\nu\rho p_1(t) + \nu^2 p_2(t)), \\
\frac{dp_1(t)}{dt} &= 0, \\
\frac{dp_2(t)}{dt} &= -q_2(t)(p_1(t)^2 + 2\nu\rho p_1(t)p_2(t) + \nu^2 p_2(t)^2)
\end{aligned}$$

with boundary conditions

$$q_1(0) = 0,\ q_1(T) = B(K),\ q_2(0) = \alpha,\ p_2(T) = 0.$$

We can easily check that H, p_1 and $p_1q_1 + p_2q_2$ are the first integrals of this Hamiltonian system i.e.

$$\frac{d}{dt}H(p(t), q(t)) = 0,\ \frac{d}{dt}p_1(t) = 0,\ \frac{d}{dt}(p_1(t)q_1(t) + p_2(t)q_2(t)) = 0.$$

First, we solve p_2 and calculate $p_2(0)$.

$$\frac{dp_2(t)}{dt} = -(p_1(t)^2 + 2\nu\rho p_1(t)p_2(t) + \nu^2 p_2(t)^2)^{1/2}(2H)^{1/2},$$
$$p_2(T) = 0.$$

Using the following indefinite integral

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \log|2ax + b + 2\sqrt{a(ax^2 + bx + c)}|, \quad a > 0,$$

we can solve p_2 as follows;

$$\nu^2 p_2(t) + \nu\rho p_1 + \nu\sqrt{p_1^2 + 2\nu\rho p_1 p_2(t) + \nu^2 p_2(t)^2} = \frac{C}{2}e^{-\sqrt{2H_0}\nu t}.$$

We see that

$$H = \frac{1}{2\nu^2 T^2}\Big\{\log(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho})\Big\}^2.$$

Finally we can calculate the energy as follows:

$$e(K) = \int_0^T H dt = \frac{1}{2\nu^2 T}\Big\{\log(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho})\Big\}^2.$$

$\square$

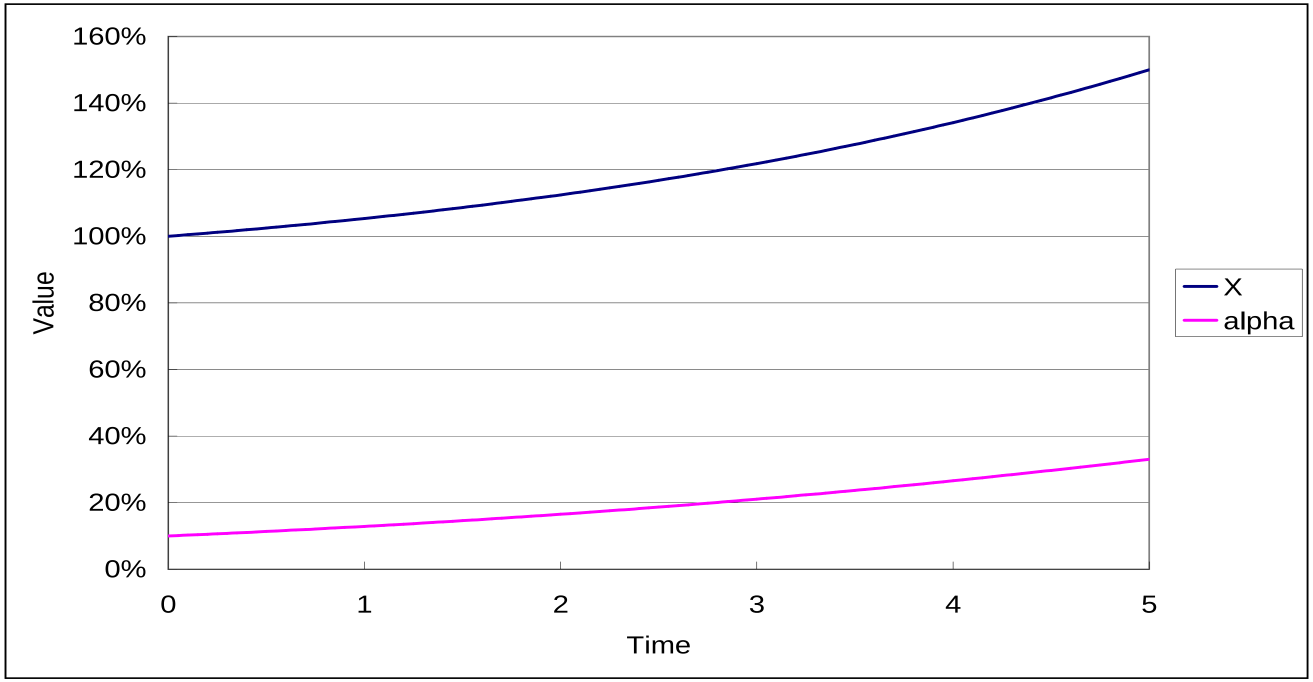

Figure 2: Minimum Energy path of SABR with Positive Rho $x_0 = 1,\ \alpha = 0.1,\ \beta = 0.1,\ \nu = 0.5,\ \rho = 0.8,\ T = 5.$

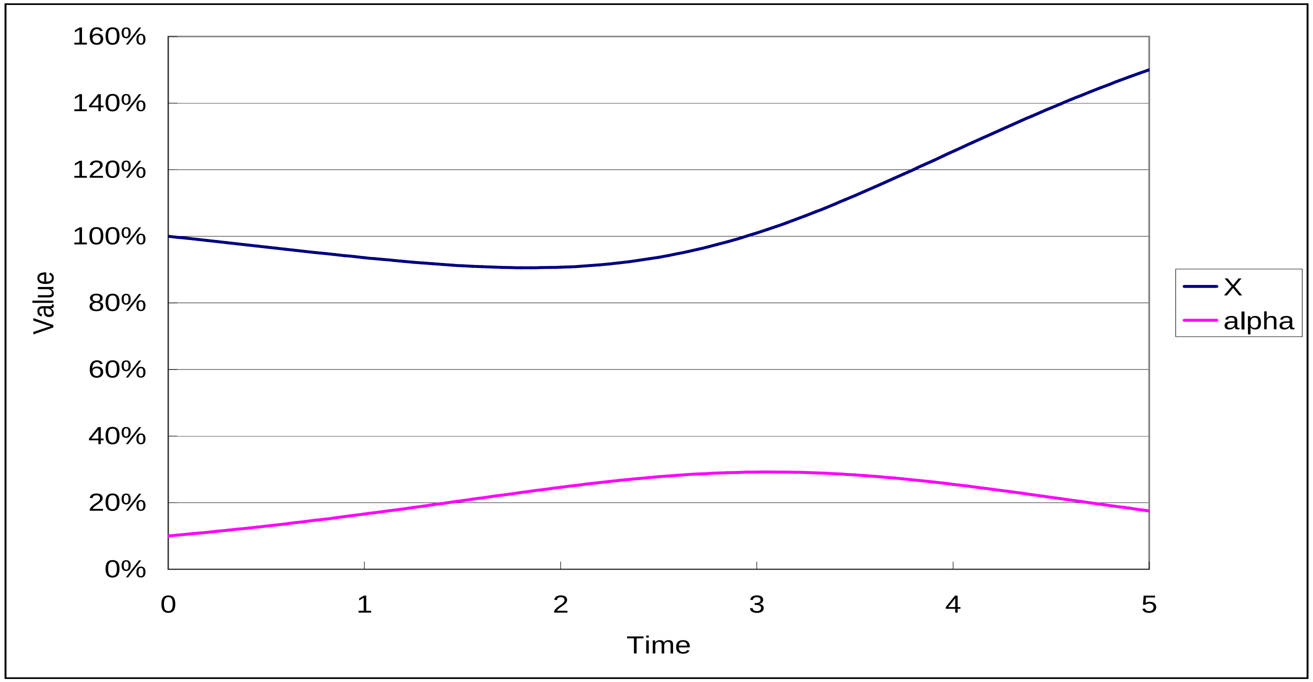

Figure 3: Minimum Energy path of SABR with Positive Rho $x_0 = 1,\ \alpha = 0.1,\ \beta = 0.1,\ \nu = 0.5,\ \rho = -0.8,\ T = 5.$

- Furthermore, we show the following.

Theorem 2. *For any* $K_0 > x_0$,

$$\lim_{\varepsilon \downarrow 0} \sup_{K \in [x_0, K_0]} \left[\varepsilon^2 \log C^{\varepsilon}(T, K) + e(K)\right] = 0$$

Since the energy for the normal model is given by

$$e(K) = \frac{(x_0 - K)^2}{\sigma_N^2},$$

the implied normal volatility for SABR model satisfies

$$\lim_{\varepsilon \downarrow 0} \left| \frac{\sigma_N(K)}{\varepsilon} - \frac{\alpha(K - x_0)}{\int_{x_0}^{K} \frac{dz}{C(z)}} \left(\frac{\zeta}{\hat{x}(\zeta)} \right) \right| = 0.$$

This is the initial term of SABR formula.

4. Kusuoka-Stroock theory

- Let $(\Theta, \|\cdot\|_\Theta)$ be a separable Banach space and $(H, \|\cdot\|_H)$ be a separable Hilbert space such that H is a dense subspace of Θ and the inclusion map is continuous.

- Let μ_s, $s \in [0,\infty)$, be the (necessarily unique) probability measure on $(\Theta, \mathcal{B}_\Theta)$ with the property that

$$\int_\Theta \exp[\sqrt{-1}\langle u, \theta\rangle]\mu_s(d\theta) = \exp(-\frac{s}{2}\|u\|_H^2),\ u \in \Theta^*.$$

Then (Θ, H, μ_1) is an abstract Wiener space in the sense of L. Gross.

- Define $\mathcal{F}C^\infty_\nearrow([0,\infty)\times\Theta; E)$ to be the space of $f : [0,\infty) \to E$ for which there exists an $n \in \mathbb{N}$, an $\tilde{f} \in C^\infty_\nearrow(\mathbb{R}^{1+n})$ and a continuous linear map $A : \Theta \to \mathbb{R}^n$ such that

$$f(s,\theta) = \tilde{f}(s, A\theta),\ (s,\theta) \in [0,\infty)\times\Theta.$$

- Define an operator $D : \mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta; E) \to \mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta; \mathcal{H}(E))$ by

$$Df(s,\theta)(h) = \lim_{\tau\to 0}\frac{f(s,\theta+\tau h)-f(s,\theta)}{\tau},\ (s,\theta)\in[0,\infty)\times\Theta \ \mathit{and}\ h\in H.$$

- For any complete orthonormal basis $\{h_i\}\subset H$, the Laplacian is given by

$$\Delta f(s,\theta) = trace_H D^2 f(s,\theta) \equiv \sum_i D^2 f(s,\theta)(h_i,h_i)\in E$$

is well defined.

- Define the heat operator $\mathcal{A} : \mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta; E) \to \mathcal{F}C^\infty_{\nearrow}([0,\infty)\times\Theta; E)$ by

$$\mathcal{A}f(s,\theta) = \frac{\partial f}{\partial s}(s,\theta) + \frac{1}{2}\Delta f(s,\theta),\ (s,\theta)\in[0,\infty)\times\Theta.$$

- We define $e : \mathbb{R}^N \to [-\infty, \infty]$ by

$$e(x) \equiv \inf\{\frac{\|h\|_H^2}{2} - f(0,h); F(0,h) = x\}, \qquad x \in \mathbb{R}^N.$$

- We also assume the following.
(A2) For each $y \in Y$,

$$M(y) \equiv \{h \in H; F(0,h) = y\} \neq \emptyset$$

and that

$$e(y) = \frac{\|h(y)\|^2}{2} - f(0, h(y))$$

for precisely one $h(y) \in M(y)$.

- Here we omit several assumptions. We define

$$\begin{aligned} A(s,\theta) &= DF(s,\theta)DF(s,\theta)^* \\ &= ((DF_i(s,\theta), DF_j(s,\theta))_H)_{1 \leqq i,j \leqq N} \end{aligned}$$

and assume the following.

(A5) For any $p \in [1, \infty)$

$$\overline{\lim_{s\downarrow 0}}\, s \log\left(\int_\Theta |\det A(s,\theta)|^{-p} \mu_s(d\theta)\right) \leqq 0.$$

Then Kusuoka-Stroock [4] proved the following.

Theorem. [Kusuoka-Stroock] *For each $s \in (0,1]$, a signed measure $P_s(\cdot)$ on $\mathbb{R}^N$ given by*

$$P_s(\Gamma) = \int_{F(s,\theta)\in\Gamma} g(s,\theta) \exp\left(\frac{f(s,\theta)}{s}\right) \mu_s(d\theta),\ \Gamma \in \mathcal{B}(\mathbb{R}^N),$$

admits a smooth density $p_s(\cdot)$. Moreover, there exist sequence $\{a_n\}_{n=0}^\infty \subseteq C(Y;\mathbb{R})$ and $\{K_n\}_{n=0}^\infty \subseteq (0,\infty)$ with the property that, for every $n \in \mathbb{N}$,

$$\left|(2\pi s)^{N/2} e^{e(y)/s} p_s(y;0) - \sum_{m=0}^{n} s^{m/2} a_m(y)\right| \leqq K_n s^{(n+1)/2},\ (s,y) \in (0,1] \times Y.$$

Our result is the following.

Theorem 3. *e is smooth in the neighborhood of Y and*

$$a_0(y) = (\det \nabla^2 e(y))^{1/2} \mathrm{det}_2(I_H - B(y))^{-1/2} \exp\Big(\sum_{i=1}^{N} \frac{\partial e}{\partial y_i}(y) \mathcal{A} F^i(0, h(y)) + \mathcal{A} f(0, h(y)) \Big)$$

for $y \in Y$, where

$$B(y) \equiv \sum_{i=1}^{N} \frac{\partial e}{\partial y_i}(y) D^2 F^i(0, h(y)) + D^2 f(0, h(y)), \qquad y \in Y.$$

Here det_2 is called Carleman-Fredholm determinant defined by

$$\mathrm{det}_2(I_H - B) \equiv \det(I_H - B) \cdot e^{trace_H B}.$$

5. Application to Stochastic Differential Equations

- We apply Kusuoka-Stroock-O's theorem to diffusion models.

- Let $\{W^1(t), \cdots, W^d(t)\ ;\ t \in [0,T]\}$ be a d-dimensional Brownian motion. Let $X_\varepsilon(t), t \in [0,T], \varepsilon \in (0,1]$, be the solution to the stochastic differential equation,

$$dX_\varepsilon^i(t) = \sum_{k=1}^{d} \varepsilon V_k^i(t, X_\varepsilon(t))dW^k(t) + V_0^i(t, X_\varepsilon(t))dt, \quad 1 \le i \le N,$$
$$X_\varepsilon(0) = x_0 = (x_0^1, \ldots, x_0^N), \quad x_0 \in \mathbb{R}^N,$$

where $V_0, \cdots, V_d \in C_b^\infty([0,T] \times \mathbb{R}^N; \mathbb{R}^N)$. We assume

$$V_0^1 \equiv 0,$$

and the ellipticity of $V_1, \cdots, V_d$, at x_0, i.e. there exists a constant

$\delta > 0$ such that

$$\sum_{k=1}^{d} V_k(0, x_0) \otimes V_k(0, x_0) \geq \delta I,$$

where I denotes the identity matrix.

- We investigate the distribution of $X_\varepsilon^1(T)$. From the ellipticity condition, the law of $X_\varepsilon^1(T)$ is absolutely continuous and has a smooth density $p_\varepsilon(y)$.

- Let H be the Cameron-Martin space. We consider the associated ordinary differential equation

$$\frac{d}{dt} y^i(t; h) = \sum_{k=1}^{d} V_k^i(t, y(t; h)) \dot{h}^k(t) + V_0^i(t, y(t; h)), \quad t \in [0, T], \quad h \in H,$$

$$y(0; h) = x_0, \quad x_0 \in \mathbb{R}^n.$$

- The energy function $e : \mathbb{R} \to \mathbb{R}$ is given by

$$e(y) = \inf\Big\{\frac{1}{2}\sum_{i=1}^{d}\int_0^T |\dot{h}^i(s)|^2 ds; h \in H,\ y^1(T;h) = y\Big\}.$$

- Here we will give the asymptotic expansion of the energy for general diffusion models.

- Corresponding to $\varepsilon = 0$, define a flow $\phi : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$\begin{aligned}&\frac{d}{dt}\phi(t,x) = V_0(t,\phi(t,x)), \quad t \in [0,T],\\ &\phi(0,x) = x.\end{aligned}$$

- Define the push-forward of the vector field V by the map ϕ_t.

$$\tilde{V}_k^i(t,y) = \sum_{j=1}^{N} \frac{\partial \phi^i}{\partial x^j}(-t,\phi(t,y))V_k^j(t,\phi(t,y)), \quad 1 \le i \le N, \quad 1 \le k \le d,$$

- Define $(g^{ij})_{1 \le i,j \le N} : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ corresponding to Riemannian metric by

$$g^{ij}(t,x) = \sum_{k=1}^{d} \tilde{V}_k^i(t,x)\tilde{V}_k^j(t,x),\ 1 \le i,j \le N.$$

- Define the generating operator $L_t,\ t \in [0,T]$ by

$$(L_t f)(x) = \frac{1}{2}\sum_{i,j=1}^{N} g^{ij}(t,x)\frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{N} b^i(t,x)\frac{\partial f}{\partial x^i}(x),$$

where $b \in C_b^\infty([0,T] \times \mathbb{R}^N; \mathbb{R}^N)$ is given by

$$b^i(t,y) = \frac{1}{2}\sum_{k,l=1}^{N}\sum_{m=1}^{d} \frac{\partial^2 \phi^i}{\partial x^k \partial x^l}(-t,\phi(t,y))V_m^k(t,\phi(t,y))V_m^l(t,\phi(t,y)).$$

- Define linear operators V and Γ by

$$
\begin{aligned}
(Vf)(t,x) &\equiv \sum_{i=1}^{N} g^{1i}(t,x) \int_t^T \frac{\partial f}{\partial x^i}(s,x)ds, \\
\Gamma(f,g)(x) &\equiv \sum_{i,j=1}^{N} \int_0^T g^{ij}(t,x)\Big(\int_t^T \frac{\partial f}{\partial x^i}(s,x)ds\Big)\Big(\int_t^T \frac{\partial g}{\partial x^j}(s,x)ds\Big)dt.
\end{aligned}
$$

- Define

$$
\begin{aligned}
b_1 &= \int_0^T g^{11}(t,x_0)dt, \\
b_2 &= \frac{3}{2}\int_0^T (Vg^{11})(t,x_0)dt, \\
b_3 &= 2\int_0^T (V^2 g^{11})(t,x_0)dt + \frac{1}{2}\Gamma(g^{11},g^{11})(x_0).
\end{aligned}
$$

We note that these parameters are determined by ‘geometric structure' of the model.

Theorem 4. *There are constants $r_0 > 0$ and $C_0 > 0$ such that energy e satisfies*

$$\left| e(y) - \left[\frac{1}{2b_1}(y - x_0^1)^2 - \frac{b_2}{3b_1^3}(y - x_0^1)^3 + \left(-\frac{b_3}{4b_1^4} + \frac{b_2^2}{2b_1^5} \right)(y - x_0^1)^4 \right] \right| \le C_0 |y - x_0^1|^5,$$

Proof. We define Hamiltonian $\mathcal{H} : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{H}(t,x,p) = \frac{1}{2} \sum_{i,j=1}^{N} g^{ij}(t,x) p_i p_j + \sum_{i=1}^{N} b^i(t,x) p_i.$$

We assume h_0 attains the minimum and satisfies

$$h_0 = \sum_{k=1}^{N} \lambda_k D y^k(1; h_0) + D f(h_0).$$

Let

$$x(t) = y(t; h_0),$$
$$p_i(t) = \sum_{j=1}^{N} \bar{J}_i^j(t; h_0)\Big(\sum_{k=1}^{N} J_j^k(1; h_0)\lambda_k\Big), \quad 1 \leq i \leq N.$$

Then (x, p) satisfies the Hamilton equation,

$$\frac{d}{dt}x^i(t) = \frac{\partial}{\partial p_i}\mathcal{H}(t, x(t), p(t)),$$
$$\frac{d}{dt}p_i(t) = -\frac{\partial}{\partial x^i}\mathcal{H}(t, x(t), p(t)), \quad t \in [0, 1], \quad 1 \leq i \leq N,$$
$$x(0) = x_0, \quad x_0 \in \mathbb{R}^n.$$

Then the energy is given by

$$e(x) = \sum_{i,j=1}^{N} \int_0^1 g^{ij}(t, x(t))p_i(t)p_j(t)dt.$$

The solution can be written as

$$x^i(t;w) = x_0^i + \sum_{j=1}^{N} \int_0^t g^{ij}(s, x(s;w)) p_j(s;w) ds,$$

$$p_i(t;w) = p_i(1;w) + \frac{1}{2} \sum_{j,r=1}^{N} \int_t^1 \frac{\partial g^{jr}}{\partial x^i}(s, x(s;w)) p_j(s;w) p_r(s;w) ds.$$

where

$$\lambda_i = \begin{cases} w & (i=1), \quad w \in \mathbb{R} \\ 0 & (2 \leq i \leq N). \end{cases}$$

In our case,

$$h_0(x) = argmin\{\frac{1}{2}\|h\|^2 \ ; y^1(1;h) = x\}$$

Therefore we have $h_0(x) = \lambda(x) Dy^1(1;h_0)$ and furthermore we have $\lambda(x) = \frac{\partial e}{\partial x}$. We calculate the asymptotic expansion inductively. □

6. Energy function of SABR model

– On the other hand, we will calculate the asymptotic expansion using Theorem 4. The parameters are given by

$$
\begin{aligned}
b_1 &= \alpha^2\sigma(x_0)^2 T, \\
b_2 &= \frac{3}{2}\sigma(x_0)^3\alpha^3(\alpha\sigma'(x_0)+\nu\rho)T^2, \\
b_3 &= \Big(\frac{8}{3}\alpha^6\sigma(x_0)^4\sigma'(x_0)^2+\frac{2}{3}\alpha^6\sigma(x_0)^5\sigma''(x_0)+6\nu\rho\sigma(x_0)^4\sigma'(x_0)\alpha^5 \\
&\quad +2\nu^2\rho^2\sigma(x_0)^4\alpha^4+\frac{2}{3}\alpha^4\sigma(x_0)^4\nu^2\Big)T^3, \\
L &= \frac{\alpha^2\sigma(x_0)^2T^2}{2}\Big(\alpha^2(\sigma'(x_0)^2+\sigma(x_0)\sigma''(x_0))+4\nu\rho\alpha\sigma'(x_0)+\nu^2\Big).
\end{aligned}
$$

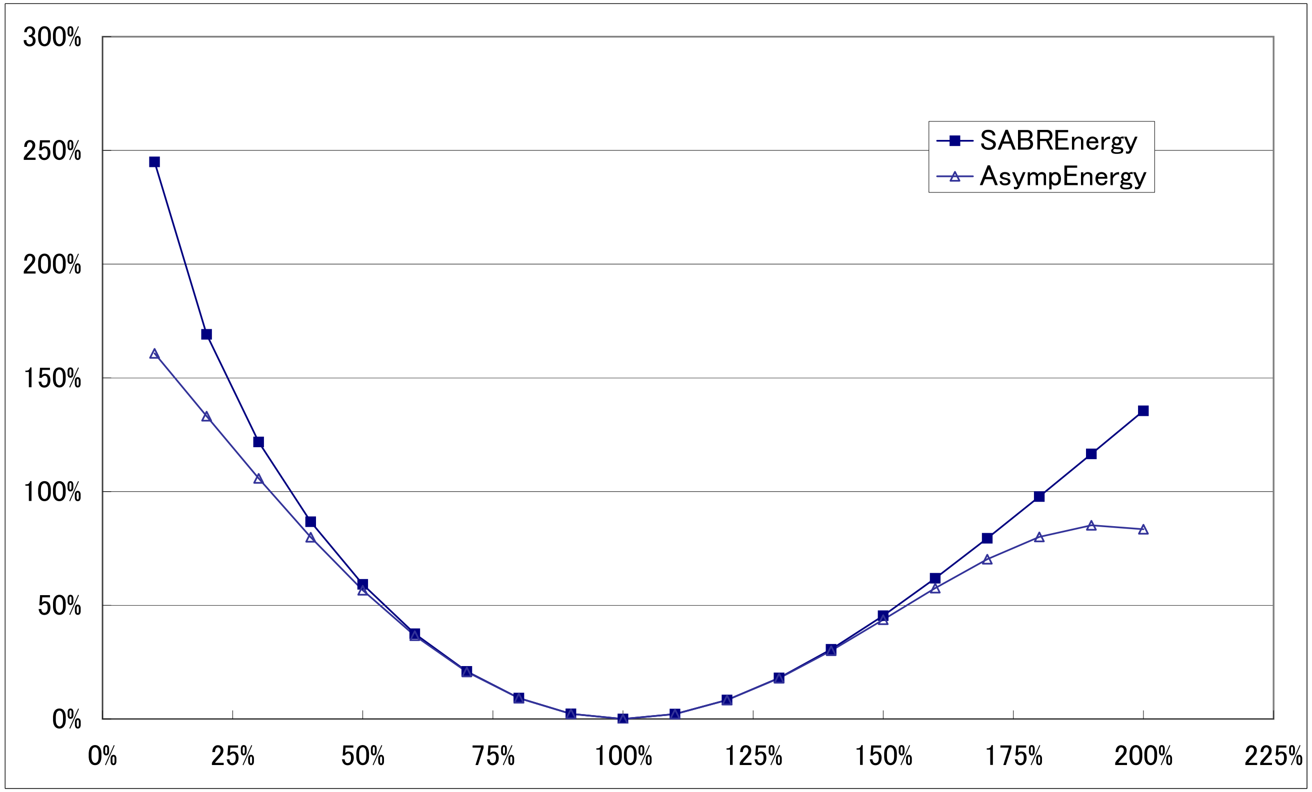

Figure 4: energy function of SABR model, asymptotic expansion vs analytic formula

7. Asymptotic expansion of probability density

- Using Theorem 3, we determine the asymptotic expansion to the higher order. Our main result is the following.

Theorem 5. *There is a constant $r_0, C_1, C_2 > 0$ such that the probability density $p_\varepsilon(y)$ satisfies following.*

$$\left|(2\pi\varepsilon^2)^{\frac{1}{2}}\exp\Big(\frac{e(y)}{\varepsilon^2}\Big)p_\varepsilon(y) - a_0(y) - \varepsilon^2 a_2(y)\right| \le \varepsilon^4 C_1, \quad y \in [x_0^1 - r_0, x_0^1 + r_0].$$

Here, a_0 and a_2 are continuous functions such that

$$\left|a_0(y) - \Big(\frac{\partial^2 e(y)}{\partial y^2}\Big)^{\frac{1}{2}}\exp\Big(\frac{L(y-x_0^1)^2}{2b_1^2}\Big)\right| \le C_2|y - x_0^1|^3,$$

$$a_2(x_0^1) = \frac{1}{\sqrt{b_1}}\Big(-\frac{L}{2b_1} - \frac{5}{6}\frac{b_2^2}{b_1^3} + \frac{3}{4}\frac{b_3}{b_1^2}\Big),$$

where

$$L=\int_{0<u<t<T}L_u(g^{11}(t,\cdot))(x_0)dudt.$$

Proof.

– We define the Wiener functionals
$F^i:(0,1)\times\Theta\times[x_0^1-r_0,x_0^1+r_0]\to\mathbb{R},\ 1\le i\le N$ by

$$F^i(s,\theta,y)=X_s^i(1,\theta)-y.$$

– The asymptotic expansion of heat operator is given by

$$\mathcal{A}F^1(0,h_0(y),y)\approx\frac{(y-x_0^1)}{2b_1}\Big\{\sum_{i=1}^N\int_0^1\int_0^t b^i(u,x_0)\nabla_i g^{11}(t,x_0)dudt$$
$$+\sum_{k=1}^d\sum_{i,j=1}^N\int_0^1 V_k^1(t,x_0)\nabla_{i,j}^2V_k^1(t,x_0)\Big(\int_0^t g^{ij}(u,x_0)du\Big)dt\Big\}.$$

– $\det_2(I_H - B(y))$ is given by Hilbert-Schmidt norm of D^2F^1.
– We have

$$\begin{aligned}&\|D^2F^1(0,0,x_0)\|_{HS}^2\\&= 2\sum_{m=1}^{d}\sum_{l_1,l_2=1}^{N}\int_0^1\int_0^t g^{l_1l_2}(u,x_0)\nabla_{l_1}V_m^1(t,x_0)\nabla_{l_2}V_m^1(t,x_0)dudt.\end{aligned}$$

– We can calculate $a_2(x_0^1)$ from $\int_{-\infty}^{\infty} p_\varepsilon(y)dy = 1$.

$\square$

8. Malliavin derivative along the minimum energy path

Malliavin Derivative of X is given by

$$
\begin{aligned}
D(X_t^i(s, h_0 + w_s)) = &\int_0^t \nabla_j V_k^i(r, X_r(s, h_0 + w_s)) D(X_r^j(s, h_0 + w^s)) dw_s^k(r) \\
&+ \int_0^{\cdot \wedge t} V_k^i(r, X_r(s, h_0 + w_r^s)) dr \\
&+ \int_0^t \nabla_j V_k^i(r, X_r(s, h_0 + w_r^s)) D(X_r^j(s, h_0 + w^s)) \dot{h}_0^k(r) dr \\
&+ \int_0^t \nabla_j V_0^i(r, X_r(s, h_0 + w_r^s)) D(X_r^j(s, h_0 + w^s)) dr,
\end{aligned}
$$

therefore by taking the limit of $s \downarrow 0$, we have the ODE

$$
\begin{aligned}
D(X_t^i(0, h_0))[k_1] = &\int_0^t V_k^i(r, y(r; h_0)) \dot{k}_1(r) dr \\
&+ \int_0^t \nabla_j V_k^i(r, y(r; h_0)) D(X_r^j(0, h_0))[k_1] \dot{h}_0^k(r) dr.
\end{aligned}
$$

If we define J as the solution of the ODE

$$dJ_j^i(t) = \sum_{j=1}^{d} \sum_{k=1}^{N} \nabla_\alpha V_k^i(t, y(t; h_0)) \dot{h}^k(t) J_j^\alpha(t)$$

$$J_j^i(0) = \delta_j^i$$

Then we can write the Malliavin derivative with J as following;

$$J^{-1}(t) D X_t(0, h_0)[k_1] = \sum_{\alpha=1}^{d} \int_0^t J^{-1}(s) V_\alpha(r, y(r; h_0)) \dot{k}_1^\alpha(r) dr.$$

9. Second Malliavin Derivative D^2 and heat operator $\mathcal{A}$

The Second Malliavin derivative of X can be written as

$$
\begin{aligned}
D^2(X^i_t(0,h_0))[k_1][k_2] &= \int_0^t \nabla_j V^i_k(r,y(r;h_0))DX^j_r(0,h_0)[k_1]\dot{k}_2(r)dr \\
&+ \int_0^t \nabla_j V^i_k(r,y(r;h_0))DX^j_r(0,h_0)[k_2]\dot{k}_1(r)dr \\
&+ \int_0^t \nabla_l\nabla_j V^i_k(r,y(r;h_0))D(X^j_r(0,h_0))[k_1]D(X^l_r(0,h_0))[k_2]\dot{h}^k_0(r)dr \\
&\int_0^t \nabla_j V^i_k(r,y(r;h_0))D^2(X^j_r(0,h_0))[k_1][k_2]\dot{h}^k_0(r)dr
\end{aligned}
$$

Then the solution of ODE is given by

$$
\begin{aligned}
J^{-1}(t)D^2(X_t^i(0,h_0))[k_1][k_2] &= \int_0^t J^{-1}(r)\nabla_j V_k^i(r,y(r,h_0))DX_r^j(0,h_0)[k_1]\dot{k}_2(r)dr \\
&+ \int_0^t J^{-1}(r)\nabla_j V_k^i(r,X_r(0,h_0))DX_r^j(0,h_0)[k_2]\dot{k}_1(r)dr \\
&+ \int_0^t J^{-1}(r)\nabla_l\nabla_j V_k^i(r,X_r(0,h_0))D(X_r^j(0,h_0))[k_1]D(X_r^l(0,h_0))[k_2]\dot{h}_0^k(r)dr
\end{aligned}
$$

Next we consider heat operator $\mathcal{A}$.

$$
\mathcal{A}f(s,\theta) = [\frac{\partial f}{\partial s} + \frac{1}{2}trace_H D^2 f](s,\theta)
$$

take the limit of $s \downarrow 0$, we have

$$\mathcal{A}(X_t^i(0,h_0)) = \int_0^t \nabla_j V_k^i(r, y(r,h_0))\mathcal{A}X_r^j(0,h_0)\dot{h}_0^k(r)dr$$
$$+ \frac{1}{2}\int_0^t \nabla_l \nabla_j V_k^i(r, y(r;h_0)) \left(D(X_r^j(0,h_0)), D(X_r^l(0,h_0) \right) \dot{h}_0^k(r)dr$$

where $\left(DX_r^i, DX_r^l\right)$ is Malliavin covariance which is given by

$$\sigma(t) = (\sigma^{ij}(t)) = (DX^i(t), DX^j(t))_{H^*} = J(t)\tilde{\sigma}(t)^t J(t),$$

$$\tilde{\sigma}(t) = \sum_{\alpha=1}^d \int_0^t J^{-1}(s)V_\alpha(X(s)) \otimes J^{-1}(s)V_\alpha(X(s))ds.$$

Therefore we have

$$J^{-1}(t)\mathcal{A}(X_t^i(0,h_0)) = \int_0^t J^{-1}(r)\nabla_l\nabla_j V_k^i(r, y(r;h_0))\sigma^{jl}(r,h_0)\dot{h}_0^k(r)dr.$$

There are several cases that we can solve these ODE analytically.

10. Homogeneous Local Volatility model

Here we consider the following model

$$dX_t = \sigma(X_t)dW_t,$$
$$X_0 = x_0.$$

We define the function $\beta : \mathbb{R} \to \mathbb{R}$ by

$$\beta(y) = \int_{x_0}^{y} \frac{dx}{\sigma(x)},$$

then we have the following theorem.

Theorem 6. *The energy, a_0 and $a_2(x_0)$ are given as follows;*

$$e(y) = \frac{1}{2T}\beta(y)^2, \quad a_0(y) = \frac{1}{\sqrt{T}}\frac{1}{\sigma(y)}\sqrt{\frac{\sigma(x_0)}{\sigma(y)}},$$
$$a_2(x_0) = \frac{\sigma(x_0)\sqrt{T}}{4}\left(\frac{\sigma''(x_0)}{\sigma(x_0)} - \left(\frac{\sigma'(x_0)}{\sigma(x_0)}\right)^2\right).$$

The asymptotic expansion of probability density is given by

$$p_\varepsilon(T,y) = \frac{1}{\sqrt{2\pi\varepsilon^2 T}\sigma(y)}\sqrt{\frac{\sigma(x_0)}{\sigma(y)}}\exp(-\frac{\beta(y)^2}{2T})\cdot$$
$$\left(1+\varepsilon^2 T\frac{\sigma(y)\sigma(x_0)}{4}\sqrt{\frac{\sigma(y)}{\sigma(x_0)}}\left(\frac{\sigma''(x_0)}{\sigma(x_0)}-\frac{1}{2}\left(\frac{\sigma'(x_0)}{\sigma(x_0)}\right)^2\right)\right).$$

Proof. It is easy to show that h_0 which attains the minimum

$$e(y) = \inf\Big\{\frac{1}{2}\int_0^T |\dot{h}(r)|^2 dr; h \in H,\ y(T;h) = y\Big\}, \tag{1}$$

is given by

$$h_0(t) = \beta(y)\frac{t}{T}.$$

Therefore we have the energy $e(y)$. The solution of ODE

$$\frac{d}{dt}y(t;h_0) = \sigma(y(t;h_0))\dot{h}_0(t),$$
$$y(0,x) = x_0.$$

is given by

$$y(t;h_0) = \beta^{-1}(h_0(t)).$$

We can give $a_2(x_0)$ from the coefficients.

$$b_1 = \sigma(x_0)^2 T$$
$$b_2 = \frac{3}{2}\sigma(x_0)^3\sigma'(x_0)T^2,$$
$$b_3 = \Big(\frac{8}{3}\sigma(x_0)^4\sigma'(x_0)^2 + \frac{2}{3}\sigma(x_0)^5\sigma''(x_0)\Big)T^3,$$
$$L = \Big(\frac{1}{2}\sigma(x_0)^2\sigma'(x_0)^2 + \frac{1}{2}\sigma(x_0)^3\sigma''(x_0)\Big)T^2.$$

We will give $a_0(y)$ in the next 2 chapters.

11. Malliavin calculus of local volatility model

Next we calculate Malliavin derivative along this minimum energy path;

$$D(X_t(0,h_0))[k_1] = \int_0^t \sigma(y(r;h_0))\dot{k}_1(r)dr + \int_0^t \sigma'(y(r;h_0))D(X_r(0,h_0))[k_1]\dot{h}_0(r)dr.$$

First we define $J : [0,T] \times \mathbb{R} \to \mathbb{R}$ as the solution of ODE

$$\begin{cases} \frac{d}{dt}J(t,x) = \sigma'(y(r;h_0))J(t,x)\dot{h}_0(t) \\ J(0,x) = 1. \end{cases}$$

This is easily solved as

$$J(t,x) = \frac{\sigma(y(t;h_0))}{\sigma(x_0)}$$

Then the Malliavin derivative is given by

$$\begin{aligned} J^{-1}(t)DX(t)[k_1] &= \int_0^t J^{-1}(r)\sigma(y(t;h_0))\dot{k}_1(r)dr \\ &= \sigma(x)\int_0^t \dot{k}_1(r)dr = \sigma(x)k_1(t) \end{aligned}$$

Therefore we have

$$DX(t)[k_1] = \sigma(y(t;h_0))k_1(t).$$

The Malliavin covariance is given by

$$(DX_t, DX_t)_{H^*} = J^2(t)\int_0^t J^{-2}(r)\sigma^2(y(r;h_0))dr = t\sigma^2(y(t;h_0)).$$

Proposition. *The heat operator for local volatility model is written as*

$$\mathcal{A}X_T(0,h_0) = \frac{T}{2}\left(\sigma(y)\sigma'(y) - \frac{\sigma(y)}{\beta(y)}\log\left(\frac{\sigma(y)}{\sigma(x_0)}\right)\right).$$

Proof. Since the heat operator satisfies the ODE

$$\frac{d}{dt}\mathcal{A}X_t(0,h_0) = \sigma'(y(t;h_0))\mathcal{A}X_t(0,h_0)\dot{h}_0(t) + \frac{1}{2}\sigma''(y(t;h_0))\left(DX_t, DX_t\right)\dot{h}_0(t).$$

This solution is written as

$$\mathcal{A}X_t(0,h_0) = J(t,x)\int_0^t J(r,x)^{-1}\frac{r}{2}\sigma''(y(r;h_0))\sigma^2(y(r;h_0))\dot{h}_0(r)dr$$
$$= \frac{T}{2}\frac{\sigma(y(t;h_0))}{\beta(y)}\int_{y(0;h_0)}^{y(t;h_0)}\left(\int_x^\theta \frac{dy}{\sigma(y)}\right)\sigma''(\theta)d\theta.$$

In particular,

$$\mathcal{A}X_T(0,h_0) = \frac{T}{2}\frac{\sigma(K)}{\int_{x_0}^K \frac{dy}{\sigma(y)}}\int_{x_0}^K\left(\int_{x_0}^\theta \frac{dy}{\sigma(y)}\right)\sigma''(\theta)d\theta.$$

□

Next we will give the second Malliavin Derivative and det_2.

Proposition 1. *Carleman determinant det_2 for local volatility model is given as follows;*

$$\det{}_2(I_H - B(y)) = (1-\beta(y)\sigma'(y))\exp(\beta(y)\sigma'(y)).$$

Proof. The ODE for D^2 is easily solved as following.

$$
\begin{aligned}
J^{-1}(t)D^2(X_t(0,h_0))[k_1][k_2] &= \int_0^t J^{-1}(r)\sigma'(r,y(r;h_0))DX_r(0,h_0)[k_1]\dot{k}_2(r)dr \\
&+ \int_0^t J^{-1}(r)\sigma'(r,y(r;h_0))DX_r(0,h_0)[k_2]\dot{k}_1(r)dr \\
&+ \int_0^t J^{-1}(r)\sigma''(r,y(r;h_0))D(X_r(0,h_0))[k_1]D(X_r(0,h_0))[k_2]\dot{h}_0^k(r)dr \\
&= \sigma(x_0)\int_0^t \frac{d}{dr}\left(\sigma'(y(r;h_0)k_1(r)k_2(r)\right)dr
\end{aligned}
$$

Therefore we have

$$
D^2(X_T(0,h_0))[k_1][k_2] = \sigma(y)\sigma'(y)k_1(T)k_2(T).
$$

Next we will calculate $\det_2(I_H - B(y))$ where

$$
B(y) = \frac{\partial e(y)}{\partial y} D^2 X_T(0,h_0).
$$

Since

$$\langle B(y)k_1, k_2 \rangle_{H*} = \int_0^T \frac{d}{dt}(B(y)k_1)(t)\frac{d}{dt}k_2(t)dt$$
$$= \frac{\partial e(y)}{\partial y}\sigma(y)\sigma'(y)k_1(T)k_2(T),$$

we have

$$B(y)k_1 = \frac{\partial e(y)}{\partial y}\sigma(y)\sigma'(y)t\int_0^T \dot{k}_1(r)dr$$
$$= \frac{\partial e(y)}{\partial y}\sigma(y)\sigma'(y)T\,\langle k_1, l\rangle\, l$$

where $l(t) = \frac{t}{\sqrt{T}}$. Therefore we can write B as

$$B(y) = \frac{\partial e(y)}{\partial y}\sigma(y)\sigma'(y)T \cdot l \otimes l.$$

Therefore we have

$$\begin{aligned}\det_2(I_H - B(y)) &= \prod (1-\lambda_k)e^{\lambda_k} \\ &= (1 - \frac{\partial e(y)}{\partial y}\sigma(y)\sigma'(y)T)\exp(\frac{\partial e(y)}{\partial y}\sigma(y)\sigma'(y)T).\end{aligned}$$

□

Proof.[Proof of theorem] Now we have all components of a_0;

$$a_0(y) = \sqrt{\frac{\partial^2 e(y)}{\partial y^2}}\det_2(I_H - B(y))^{-1/2}\exp(\frac{\partial e(y)}{\partial y}\mathcal{A}X^1(0, h(y)),$$

and

$$e(y) = \frac{\beta(y)^2}{2T}, \frac{\partial e(y)}{\partial y} = \frac{\beta(y)}{T\sigma(y)}, \frac{\partial^2 e(y)}{\partial y^2} = \frac{1-\beta(y)\sigma'(y)}{T\sigma(y)^2},$$

we have our statement. □

12. Piecewise constant local volatility model

We consider the following model

$$dX_t = \sigma(X_t)dW_t, \quad \sigma(x) = \begin{cases} \sigma_1 & (x \geq H) \\ \sigma_0 & (x < H). \end{cases}$$

Theorem 7. *The probability density at time T for $x_0 > H$ is given as follows*

$$\begin{aligned} p(T, x_0, y) &= \frac{1}{\sigma_1}\phi(T, \frac{x_0 - y}{\sigma_1}) \\ &+ \frac{\sigma_1}{\sigma_0 + \sigma_1}\int_0^T q(T - s, \frac{x_0 - H}{\sigma_0}) \left(\frac{1}{\sigma_1}\phi(s, \frac{y - H}{\sigma_1}) - \frac{1}{\sigma_0}\phi(s, \frac{y - H}{\sigma_0})\right) ds \\ &+ \frac{\sigma_0\sigma_1}{\sigma_0 + \sigma_1}\int_0^T \phi(T - s, \frac{x - H}{\sigma_0}) \left(\frac{1}{\sigma_1^2}q(s, \frac{y - H}{\sigma_1}) - \frac{1}{\sigma_0^2}q(s, \frac{y - H}{\sigma_0})\right) ds \end{aligned}$$

where

$$\phi(t, x) = \frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t}), \quad q(t, x) = \frac{1}{\sqrt{2\pi t}}\frac{x}{t}\exp(-\frac{x^2}{2t})$$

Proof. Apply Laplace transform to the fundamental solution and apply the Kac's theorem. $\square$

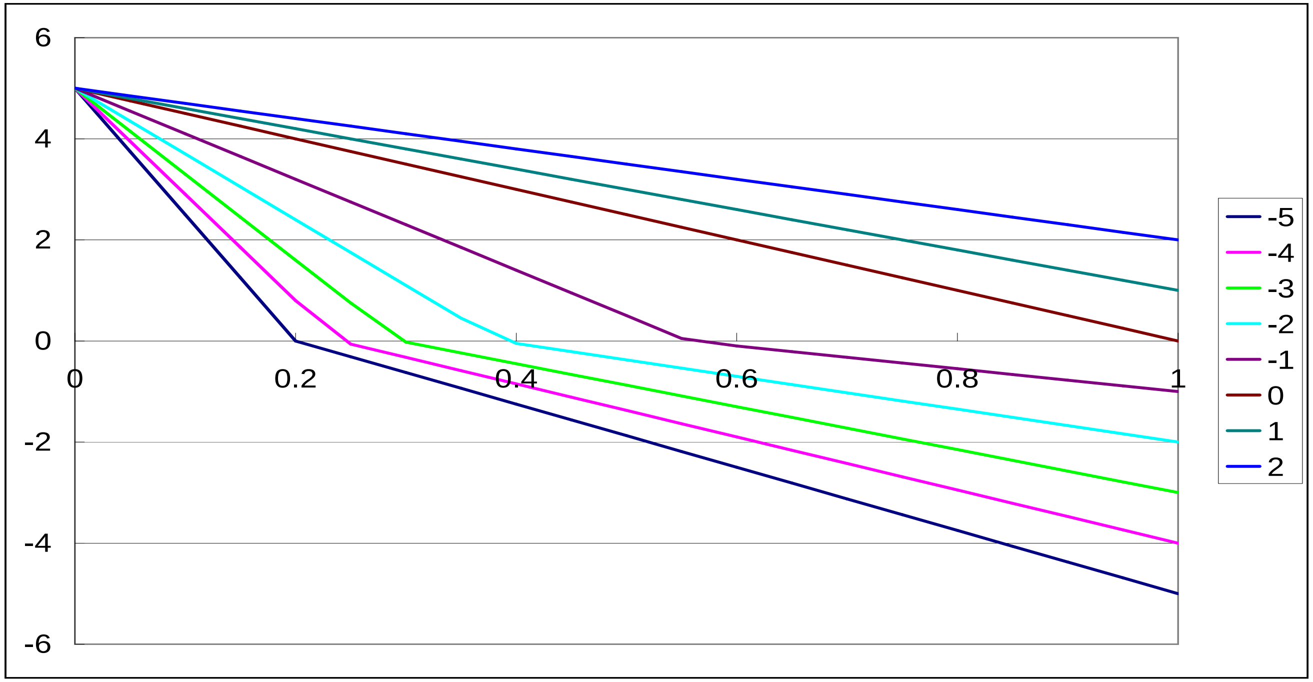

Figure 5: Minimum Energy Path for PCLV model with $H = 0, x_0 = 5, \sigma_0 = 1.0, \sigma_1 = 4.0, T = 1.0.$

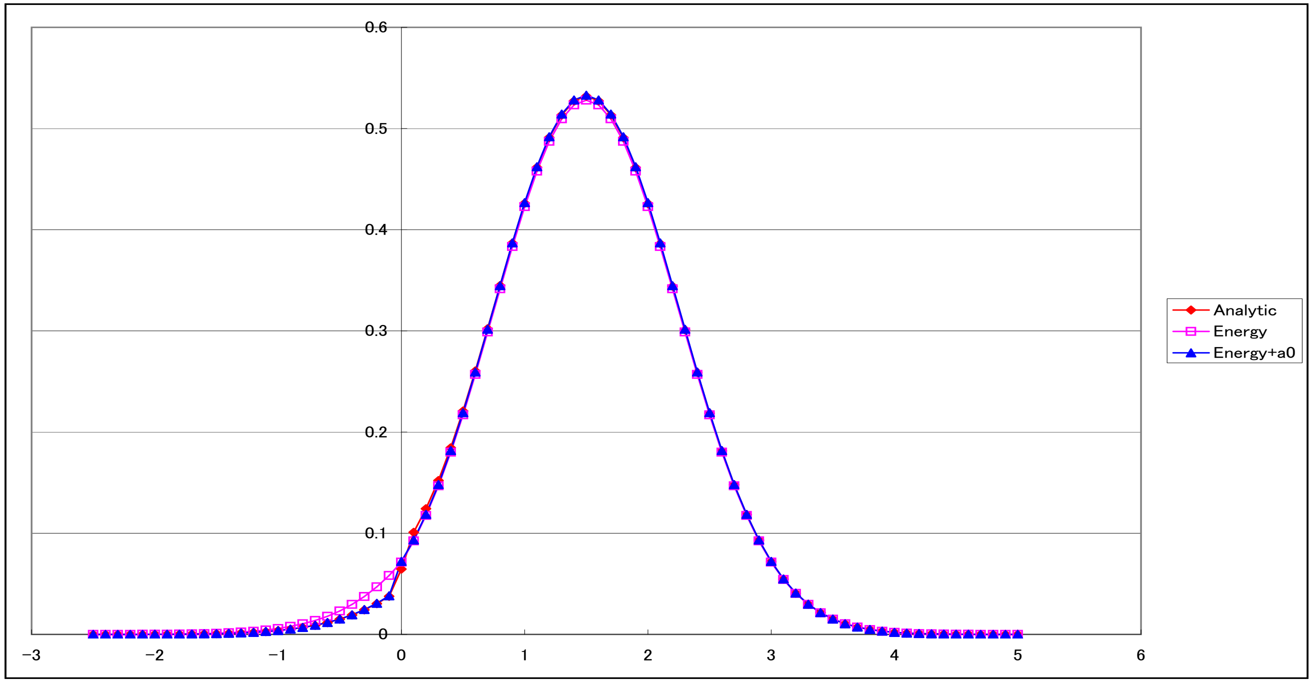

Figure 6: Probability Density of PCLV model, asymptotic expansion vs analytic with $H = 0, x_0 = 1.5, \sigma_0 = 1.0, \sigma_1 = 0.75, T = 1.0$.

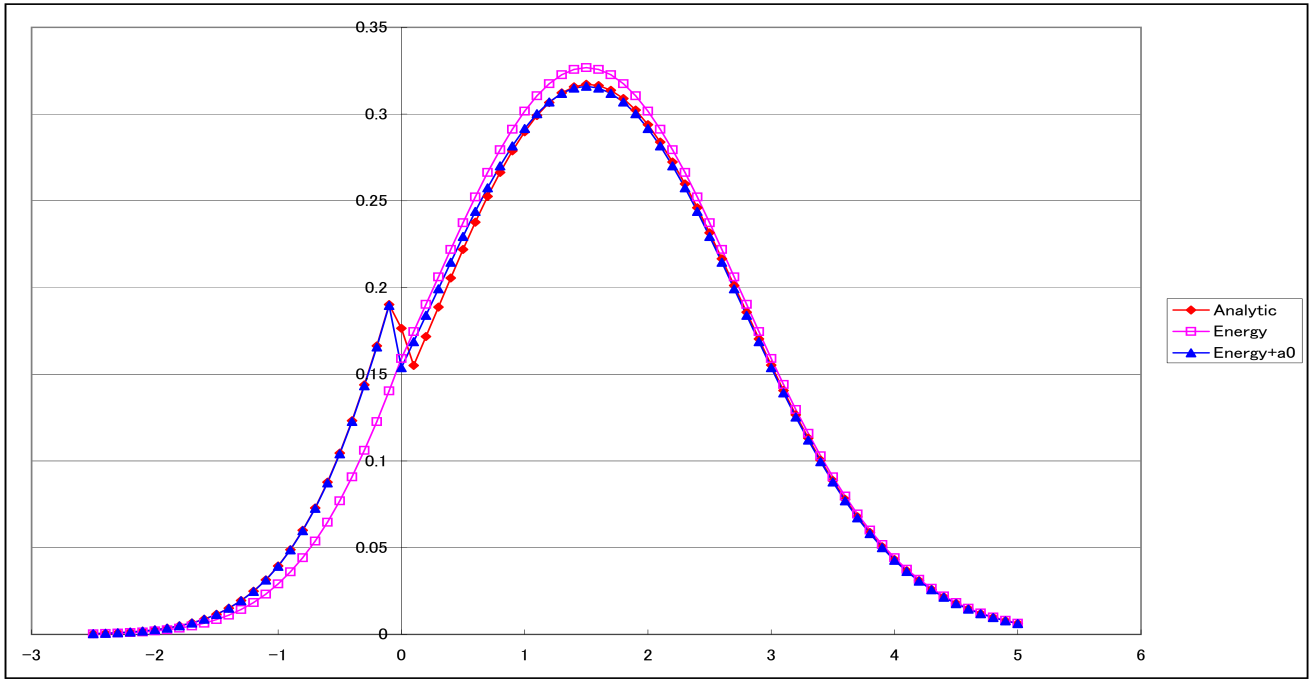

Figure 7: Probability Density of PCLV model, asymptotic expansion vs analytic with $H = 0, x_0 = 1.5, \sigma_0 = 1.0, \sigma_1 = 1.25, T = 1.0.$

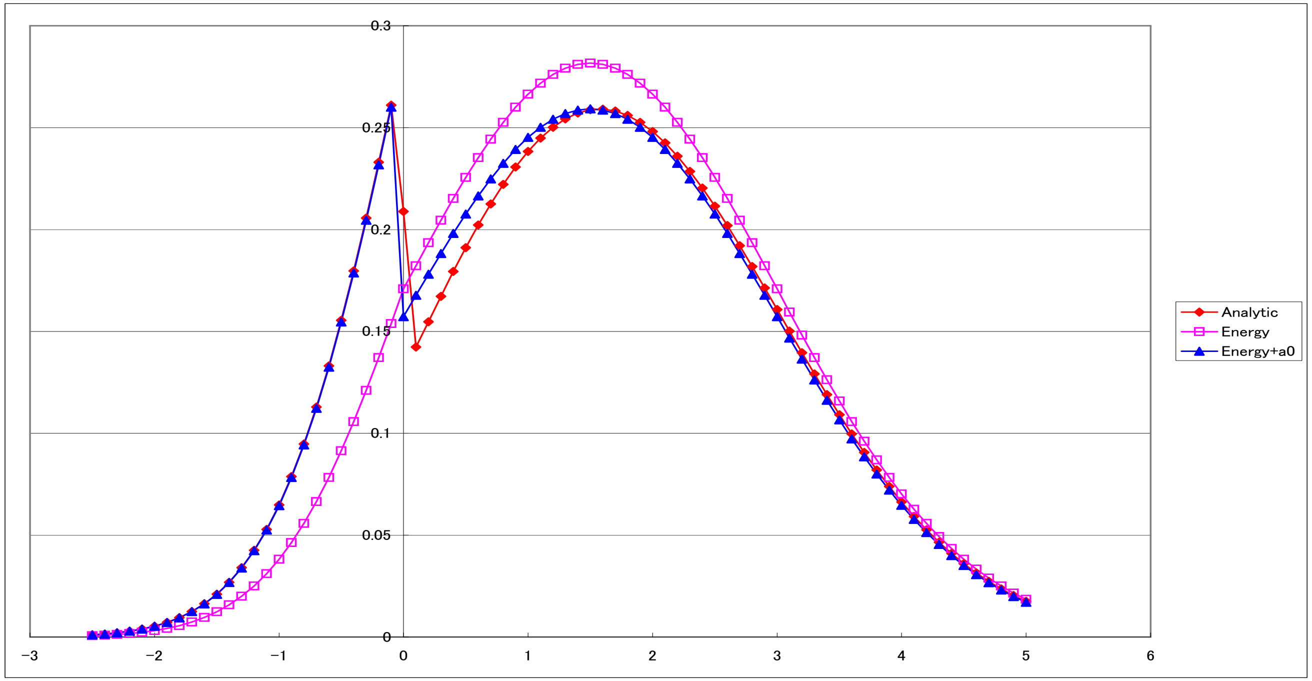

Figure 8: Probability Density of PCLV model, asymptotic expansion vs analytic with $H = 0, x_0 = 1.5, \sigma_0 = 1.0, \sigma_1 = 1.5, T = 1.0$.

13. Generalized SABR Formula

- We regard X^1_ε as the underlying of options. Then the forward value of a call option of strike rate K and maturity T is given by

$$C_\varepsilon(T,K) = E[(X^1_\varepsilon(T) - K)^+], \quad \varepsilon \in (0,1].$$

- We calculate the asymptotic expansion of call option value and implied volatilities.

- We define smooth functions $\varphi_n \in C^\infty_b([0,\infty))$, $n \geq 0$, by

$$\varphi_n(x) = \int_0^\infty z^n \exp(-xz - \frac{z^2}{2})dz, \quad x \geq 0.$$

- We define

$$e(x) = \frac{1}{2}\Big(\int_{x^1_0}^x \frac{dy}{q(y)}\Big)^2, \quad x \in [x^1_0 - r_0, x^1_0 + r_0].$$

Theorem 8. *There is a constant C_1 such that the value of the call option with strike rate K, maturity T satisfies*

$$\Big|\sqrt{2\pi}\exp(\frac{e(K)}{\varepsilon^2})C_\varepsilon(T,K) - \varepsilon a_0(K)q(K)^2\varphi_1\big(\frac{\sqrt{2e(K)}}{\varepsilon}\big)(1+R_2(\varepsilon,K))\Big| \le C_1\varepsilon^4,$$

where

$$R_2(\varepsilon,K) = \varepsilon q(K)\Big(\frac{a_0'(K)}{a_0(K)} + \frac{3}{2}\frac{q'(K)}{q(K)}\Big)\frac{\varphi_2(\sqrt{2e(K)}/\varepsilon)}{\varphi_1(\sqrt{2e(K)}/\varepsilon)} + \varepsilon^2 q(K)^2\Big[\frac{1}{2}\frac{a_0''(K)}{a_0(K)}$$
$$+ 2\frac{a_0'(K)}{a_0(K)}\frac{q'(K)}{q(K)} + \frac{7}{6}\Big(\frac{q'(K)}{q(K)}\Big)^2 + \frac{2}{3}\frac{q''(K)}{q(K)}\Big]\frac{\varphi_3(\sqrt{2e(K)}/\varepsilon)}{\varphi_1(\sqrt{2e(K)}/\varepsilon)} + \varepsilon^2\frac{a_2(K)}{a_0(K)}.$$

Proof. We define $g:\mathbb{R}\to\mathbb{R}$ by $e(g(x)) = \frac{x^2}{2}$, then we have

$$C_\varepsilon(T,K) = \int_K^\infty (y-K)p_\varepsilon(y)dy = \int_K^\infty (y-K)\big(\frac{1}{2\pi\varepsilon^2}\big)^{\frac{1}{2}}\exp(-\frac{e(y)}{\varepsilon^2})a_\varepsilon(y)dy,$$

$$C_\varepsilon(T,K) = \int_{g^{-1}(K)}^{\infty} (g(x)-K)\big(\frac{1}{2\pi\varepsilon^2}\big)^{\frac{1}{2}} \exp(-\frac{x^2}{2\varepsilon^2}) a_\varepsilon(g(x)) g'(x) dx.$$

Let $A_\varepsilon(x) = a_\varepsilon(g(x))g'(x)$ and putting $x = \varepsilon z + g^{-1}(K)$, we have

$$\begin{aligned}
&\exp\big(\frac{e(K)}{\varepsilon^2}\big) C_\varepsilon(T,K) \\
&= \int_0^\infty \big(g(\varepsilon z + g^{-1}(K)) - K\big)\frac{1}{\sqrt{2\pi}} \exp\big(-\frac{z^2}{2} - \frac{z g^{-1}(K)}{\varepsilon}\big) A_\varepsilon(\varepsilon z + g^{-1}(K)) dz \\
&\approx \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\big(-\frac{z^2}{2} - \frac{z g^{-1}(K)}{\varepsilon}\big) \sum_{\substack{n,m\geq 0 \\ 2n+m+1\leq N}} c_{n,m}(g^{-1}(K))\varepsilon^{2n+m+1} z^{m+1} dz \\
&= \sum_{\substack{n,m\geq 0 \\ 2n+m+1\leq N}} c_{n,m}(g^{-1}(K))\varepsilon^{2n+m+1} \frac{1}{\sqrt{2\pi}} \varphi_{m+1}\big(\frac{g^{-1}(K)}{\varepsilon}\big).
\end{aligned}$$

In the case $N=2$, we can calculate the asymptotic expansion of $c_{0,0}, c_{0,1}, c_{0,2}, c_{1,0}$. $\square$

The asymptotic expansion of the implied normal volatilities are given by the following.

Theorem 9. *The asymptotic expansion of implied normal volatilities are given by*

$$\left|\Big(\frac{\varepsilon|K-x_0^1|}{\sqrt{2e(K)T}}\Big)^{-1}\sigma_N(T,K)-\exp(J)\right|\le C(\varepsilon+|K-x_0^1|)^3,\quad K\in[x_0^1,K_1],$$

where

$$J=\frac{|K-x_0^1|^2}{b_1^2}\Big(\frac{L}{2}+\frac{1}{6}\frac{b_2^2}{b_1^2}-\frac{1}{4}\frac{b_3}{b_1}\Big)\varphi_1(\frac{\sqrt{2e(K)}}{\varepsilon})+\frac{\varepsilon^2}{b_1}\Big(-\frac{L}{2}-\frac{5}{6}\frac{b_2^2}{b_1^2}+\frac{3}{4}\frac{b_3}{b_1}\Big)\varphi_1(\frac{\sqrt{2e(K)}}{\varepsilon})$$
$$+\frac{\varepsilon}{\sqrt{b_1}}\frac{|K-x_0^1|}{b_1}\Big(L+\frac{2}{3}\frac{b_2^2}{b_1^2}-\frac{3}{4}\frac{b_3}{b_1}\Big)\varphi_2(\frac{\sqrt{2e(K)}}{\varepsilon})+\frac{\varepsilon^2}{b_1}\Big(\frac{L}{2}+\frac{b_2^2}{2b_1^2}-\frac{b_3}{2b_1}\Big)\varphi_3(\frac{\sqrt{2e(K)}}{\varepsilon}).$$

Calculating the Taylor expansion of exponent part, we obtained our formula. □

For SABR model, Taylor expansion of $\exp(J)$ around x_0 obtain

$$\sigma_N(T,K) = \frac{K - x_0}{\sqrt{2e_{sabr}(K)T}}\Big(1 + \Big[\frac{2\sigma(x_0)\sigma''(x_0) - \sigma'(x_0)^2}{24}\alpha^2 + \frac{1}{4}\rho\nu\alpha\sigma'(x_0) + \frac{2-3\rho^2}{24}\nu^2\Big]T\Big).$$

This is almost the same as original SABR formula.
We compare Implied volatility smile for asymptotic expansion (using analytic energy), asymptotic expansion (using asymptotic energy) and Monte Carlo simulation.

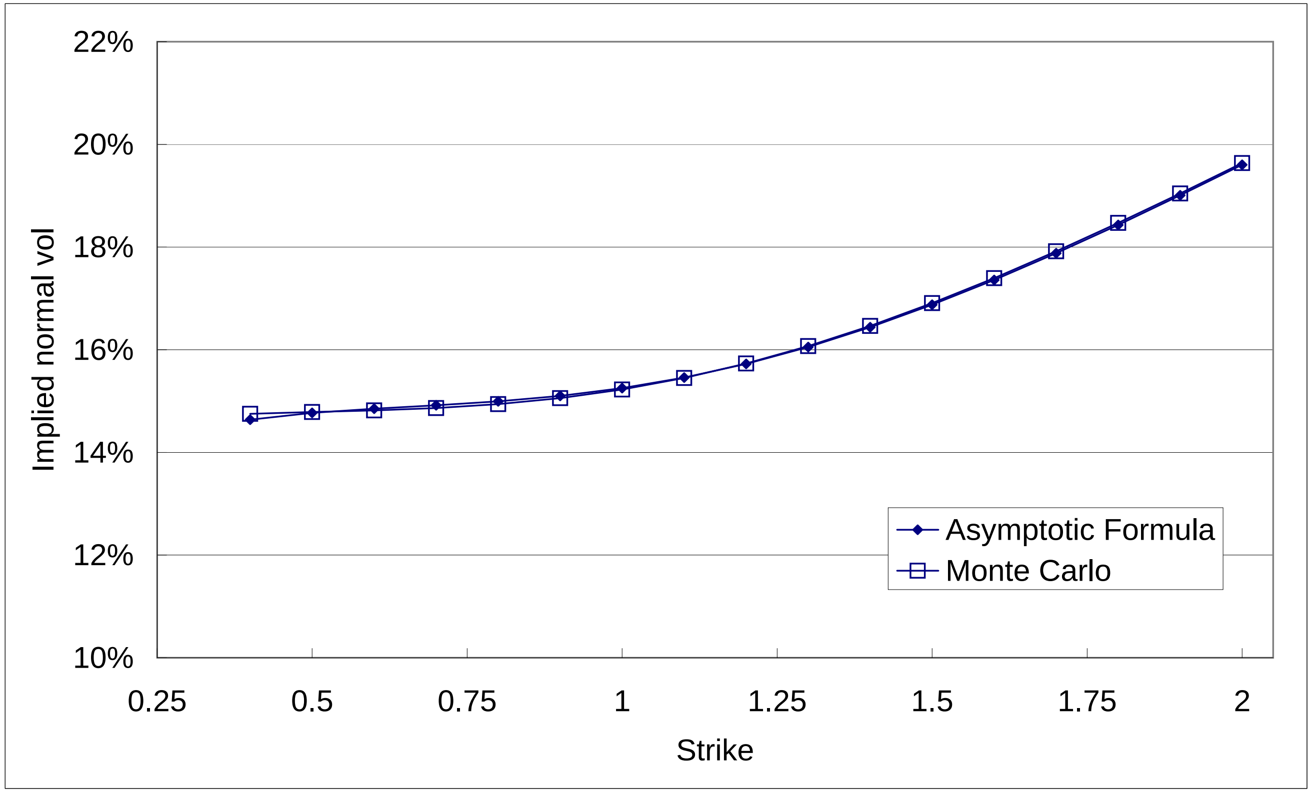

Figure 9: Implied volatility smile of SABR model, asymptotic expansion vs Monte Carlo simulation with $x_0 = 1,\ \alpha = 0.15,\ \beta = 0.5,\ \nu = 0.2,\ \rho = -0.2,\ T = 10.$

14. Conclusion

- We gave the asymptotic expansion of probability density of one component.

- The initial term is given by the 'energy of path'. For some cases, we can give the analytical formula of energy, e.g. SABR model. But it is not the case in general. So we gave the asymptotic expansion of the energy.

- The first order of the expansion is given by Kusuoka-Stroock's asymptotic expansion theory based on Malliavin calculus. For local volatility model, we give the analytical formula.

- We apply the result to the asymptotic expansion of call options.

- Finally we gave the asymptotic expansion of implied volatilities for general diffusion models. When we applied the formula for SABR model, it coincides Hagan's original formula.

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