Random Times, Enlarged Filtrations and Semimartingale Decompositions

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Outline

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References I

[LR1] Li, L. and Rutkowski, M.:
Constructing random times through multiplicative systems.
Working paper, University of Sydney, 2010.

[LR2] Li, L. and Rutkowski, M.:
Progressive enlargements of filtrations and semimartingale decompositions.
Working paper, University of Sydney, 2010.

1. Initial and progressive enlargements of a reference filtration

Enlargements of a filtration

We are given a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with a *reference filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions.

Definition

Let $\mathbb{G}$ be any filtration such that $\mathcal{F}_t \subset \mathcal{G}_t$ for every $t \geq 0$. Then we write $\mathbb{F} \subset \mathbb{G}$ and we say that $\mathbb{G}$ is an *enlargement* of $\mathbb{F}$.

Early studies of enlargements of a filtration undertaken by Barlow-Jeulin-Brémaud-Yor-Jacod-... centered around:

Hypothesis (H):

- Any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.

Hypothesis (H'):

- Any $\mathbb{F}$-semimartingale is a $\mathbb{G}$-semimartingale,
- $\mathbb{G}$-semimartingale decomposition of an $\mathbb{F}$-martingale.

Enlargements through a random time

Definition

By a *random time* τ we mean a random variable on $(\Omega, \mathcal{G}, \mathbb{P})$ taking values in $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$.

We assume that a random variable τ is not an $\mathbb{F}$-stopping time.

In the existing literature, one usually deals with the following two basic types of enlargements:

- initial enlargement,
- progressive enlargement.

Remark

All enlargements are made right-continuous and $\mathbb{P}$-complete.

Initial and progressive enlargements

Definition

The *initial enlargement* of $\mathbb{F}$ through τ is the filtration $\mathbb{G}^*$ defined by $\mathcal{G}_t^* = \sigma(\mathcal{F}_t \cup \sigma(\tau)) := \mathcal{F}_t \vee \sigma(\tau)$ for all $t \in \mathbb{R}_+$.

The initial enlargement is typically used for insider trading.

Definition

The *progressive enlargement* of $\mathbb{F}$ through τ is defined as the smallest enlargement $\mathbb{G}$ of $\mathbb{F}$ such that τ is a $\mathbb{G}$-stopping time.

The progressive enlargement is typically used for credit risk.

Admissible and progressive enlargements

Definition

An *admissible enlargement* $\widetilde{\mathbb{G}}$ of $\mathbb{F}$ associated with τ is any filtration $\widetilde{\mathbb{G}}$ such that $\widetilde{\mathcal{G}}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}$ for all $t \geq 0$.

For instance

$$\widetilde{\mathcal{G}}_t := \left\{ A \in \mathcal{G} \,|\, \exists\, \widetilde{A} \in \mathcal{F}_t \text{ such that } A \cap \{\tau > t\} = \widetilde{A} \cap \{\tau > t\} \right\}$$

Lemma

The progressive enlargement $\mathbb{G}$ satisfies, for all $t \geq 0$,

$$\mathcal{G}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\} = \widetilde{\mathcal{G}}_t \cap \{\tau > t\}$$
$$\mathcal{G}_t \cap \{\tau \leq t\} = \mathcal{G}^*_t \cap \{\tau \leq t\} = (\sigma(\tau) \vee \mathcal{F}_t) \cap \{\tau \leq t\}$$

2. Semimartingale decompositions: classic results

Azéma supermartingale

Definition

The *Azéma supermartingale* of a random time τ is defined by the equality $G_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t)$ for all $t \in \overline{\mathbb{R}}_+$.

Lemma

The process G is a $(\mathbb{P}, \mathbb{F})$-supermartingale that satisfies $0 \leq G_t \leq 1$ for every $t \in \mathbb{R}_+$ and $G_\infty = 0$.

The Azéma supermartingale G is a central object in the studies of enlargements of $\mathbb{F}$ through a random time τ since:

- some (but not all) properties of a random time can be characterized by its Azéma supermartingale,
- semimartingale decomposition of a $(\mathbb{P}, \mathbb{F})$-martingale in enlarged filtration can be written in terms of the Azéma supermartingale.

Notation and comments

1. Let the Doob-Meyer decomposition of G be $G = M - A$ where $A = H^p$ is the dual $\mathbb{F}$-predictable projection of the process $H_t = \mathbb{1}_{\{\tau \le t\}}$ and M is a $(\mathbb{P}, \mathbb{F})$-martingale.
2. Also $G = \widetilde{M} - \widetilde{A}$ where $\widetilde{A} = H^o$ is the dual $\mathbb{F}$-optional projection of $H_t = \mathbb{1}_{\{\tau \le t\}}$ and $\widetilde{M}$ is an $(\mathbb{P}, \mathbb{F})$-martingale.
3. Equalities $A = \widetilde{A}$ and $M = \widetilde{M}$ hold if all $(\mathbb{P}, \mathbb{F})$-martingales are continuous and/or $\mathbb{P}(\tau = \rho) = 0$ for every $\mathbb{F}$-stopping time ρ (*avoidance property*).
4. Any $\mathbb{F}$-adapted process can be decomposed as follows:

$$X_t = X_{t\wedge\tau-} + \Delta X_\tau \mathbb{1}_{\{\tau \le t\}} + X_{t\vee\tau} - X_\tau$$

where

$$X_{t\wedge\tau-} = X_t \mathbb{1}_{\{\tau > t\}} + X_{\tau-} \mathbb{1}_{\{\tau \le t\}}$$

$\mathbb{G}$-semimartingale decomposition stopped at τ

Recall that $\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$ through τ.

Theorem (Jeulin-Yor (1978))

The process

$$H_t - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, dA_u$$

is a $(\mathbb{P},\mathbb{G})$*-local martingale. If* U *if a* $(\mathbb{P},\mathbb{F})$*-martingale then*

$$U_{t\wedge\tau} - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\langle \widetilde{M}, U\rangle_u$$

is a $(\mathbb{P},\mathbb{G})$*-local martingale.*

This result is valid for any admissible enlargement $\widetilde{\mathbb{G}}$ of $\mathbb{F}$.

Honest times

Definition

A random time is said to be an *honest time* if for every $t > 0$ there exists an $\mathcal{F}_t$-measurable random variable τ_t such that τ is equal to τ_t on the event $\{\tau \le t\}$, that is, $\tau \mathbb{1}_{\{\tau \le t\}} = \tau_t \mathbb{1}_{\{\tau \le t\}}$.

Honest times can also be characterized as follows.

Lemma (Yor (1978))

A random time is an honest time if for every $0 \le s < t$ there exists an event $A_{s,t} \in \mathcal{F}_t$ such that $\mathbb{1}_{\{\tau \le s\}} = \mathbb{1}_{A_{s,t}} \mathbb{1}_{\{\tau \le t\}}$.

We set $F = 1 - G$ where G is the Azéma supermartingale of τ. Hence $F_t = \mathbb{P}\left(\tau \le t \,|\, \mathcal{F}_t\right)$ is a bounded $\mathbb{F}$-submartingale.

$\mathbb{G}$-semimartingale decomposition for honest times

In the case of an honest time the hypothesis (H') is satisfied.

Theorem (Jeulin-Yor (1978))

Let τ be an honest time. If U is a $(\mathbb{P}, \mathbb{F})$-martingale then

$$U_t - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\langle \widetilde{M}, U\rangle_u + \int_{(t\wedge\tau,t]} \frac{1}{F_{u-}}\, d\langle \widetilde{M}, U\rangle_u$$

is a $(\mathbb{P}, \mathbb{G})$-local martingale. If U is a $(\mathbb{P}, \mathbb{F})$-martingale with no jump at τ then

$$U_t - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\langle M, U\rangle_u + \int_{(t\wedge\tau,t]} \frac{1}{F_{u-}}\, d\langle M, U\rangle_u$$

is a $(\mathbb{P}, \mathbb{G})$-local martingale.

Initial times and the hypothesis (H')

Definition

A random time is an *initial time* if there exists a σ-finite measure η on $\mathbb{R}_+$ such that

$$G_{u,t} = \mathbb{P}\left(\tau > u \,|\, \mathcal{F}_t\right) = \int_u^\infty f_{s,t}\, d\eta(s)$$

where $f_{s,t}$ is called the *density process*.

Theorem (Jacod (1985))

If τ is an initial time then the hypothesis (H') holds for the initial enlargement $\mathbb{G}^*$ and thus for any admissible enlargement $\widetilde{\mathbb{G}}$.

In particular, it holds for the progressive enlargement $\mathbb{G}$ of $\mathbb{F}$ through an initial time τ.

$\mathbb{G}$-semimartingale decomposition for initial times

Theorem (Jeanblanc-Le Cam (2009))

Assume τ is an initial time with density $f_{s,t}$ so that

$$G_{u,t} = \mathbb{P}\,(\tau > u \,|\, \mathcal{F}_t) = \int_u^\infty f_{s,t}\, d\eta(s)$$

If U is a $(\mathbb{P}, \mathbb{F})$-martingale with no jump at τ then

$$U_t - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\langle M, U\rangle_u + \int_{(t\wedge s,t]} \frac{1}{f_{s,u-}}\, d\langle M, f_{s,\cdot}\rangle_u \Big|_{s=\tau}$$

is a $(\mathbb{P}, \mathbb{G})$-local martingale.

3. Multiplicative construction of a random time with a given Azéma supermartingale

Random times consistent with G

Assumption

We are given a $(\mathbb{P}, \mathbb{F})$*-supermartingale* G *on* $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ *such that* $0 \leq G_t \leq 1$ *for every* $t \in \mathbb{R}_+$ *and* $G_\infty = 0$*.*

Our goal is to show that given a supermartingale G there exists a random time τ on an extension $(\widehat{\Omega}, \widehat{\mathcal{G}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ of $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ such that:

- the probability measures $\mathbb{P}$ and $\widehat{\mathbb{P}}$ coincide on $\mathbb{F}$,
- G is the Azéma supermartingale of τ so that for all $t \geq 0$

$$G_t = \widehat{\mathbb{P}}(\tau > t \,|\, \mathcal{F}_t)$$

We then say that a random time τ is *consistent* with G.

Related results

Let G be a positive $(\mathbb{P}, \mathbb{F})$-supermartingale such that $0 \leq G \leq 1$.

- If G is a decreasing process then the so-called *canonical construction* gives the existence of τ consistent with G.
- A particular case was examined in Gapeev et al. [GJLR], where it was assumed that the supermartingale G is continuous and $\mathbb{F}$ is the Brownian filtration.
- In working papers by Jeanblanc and Song [JS1, JS2], the authors develop an alternative (related) approach.

Our goals

Using the multiplicative systems introduced in Meyer [M], we will establish:

- the existence of τ with a predetermined supermartingale G satisfying $0 \leq G \leq 1$,
- the uniqueness of the $\mathbb{F}$-conditional distribution of τ in a certain subclass of *admissible* constructions.

Remark

In general, the uniqueness does not hold! It is possible to construct two random times consistent with the same Azéma supermartingale, but having different $\mathbb{F}$-conditional distributions.

Practical importance: the default intensity does not uniquely specify a credit risk model and prices of credit-risky claims.

4. Existence of a random time consistent with G

Notation

For a random time τ given on a space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, we define the $(\mathbb{P}, \mathbb{F})$-submartingale F^τ and the $(\mathbb{P}, \mathbb{F})$-supermartingale G^τ

$$F_t^\tau = \mathbb{P}\,(\,\tau \leq t \,|\, \mathcal{F}_t)\,, \quad G_t^\tau = 1 - F_t^\tau = \mathbb{P}\,(\,\tau > t \,|\, \mathcal{F}_t)$$

Definition

By the $\mathbb{F}$-*conditional distribution* of τ we mean the random field $(F_{u,t}^\tau)_{u,t \in \overline{\mathbb{R}}_+}$ that satisfies, for all $u, t \in \overline{\mathbb{R}}_+$,

$$F_{u,t}^\tau = \mathbb{P}\,(\,\tau \leq u \,|\, \mathcal{F}_t)$$

The $\mathbb{F}$-*conditional survival distribution* of τ is the random field $(G_{u,t}^\tau)_{u,t \in \overline{\mathbb{R}}_+}$ given by, for all $u, t \in \overline{\mathbb{R}}_+$,

$$G_{u,t}^\tau = \mathbb{P}\,(\,\tau > u \,|\, \mathcal{F}_t) = 1 - F_{u,t}^\tau$$

$\mathbb{F}$-conditional distributions

Definition

A random field $(F_{u,t})_{u,t\in\overline{\mathbb{R}}_+}$ on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ is said to be an $\mathbb{F}$*-conditional distribution* of a random time if it satisfies:

- for all $u \in \overline{\mathbb{R}}_+$ and $t \in \overline{\mathbb{R}}_+$, we have $0 \leq F_{u,t} \leq 1$,
- for all $u \in \overline{\mathbb{R}}_+$, the process $(F_{u,t})_{t\in\overline{\mathbb{R}}_+}$ is a $(\mathbb{P}, \mathbb{F})$-martingale,
- for all $t \in \overline{\mathbb{R}}_+$, the process $(F_{u,t})_{u\in\overline{\mathbb{R}}_+}$ is right-continuous, increasing and $F_{\infty,t} = 1$.

Lemma (L.-R. (2010))

For any given $\mathbb{F}$-conditional distribution $F_{u,t}$, there exists a random time τ on an extension of $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ such that $\widehat{\mathbb{P}}(\tau \leq u \,|\, \mathcal{F}_t) = F_{u,t}$ for all $u, t \in \overline{\mathbb{R}}_+$. Moreover, the equality $\widehat{\mathbb{P}}\,|_{\mathcal{F}_t} = \mathbb{P}\,|_{\mathcal{F}_t}$ holds for all $t \in \mathbb{R}_+$.

Extended canonical construction

Proof.

The extended canonical construction runs as follows:

1. We extend the space by setting

$$\widehat{\Omega} = \Omega \times [0,1], \; \widehat{\mathcal{G}} = \mathcal{G} \otimes \mathcal{B}[0,1], \; \widehat{\mathbb{P}} = \mathbb{P} \times \lambda$$

   where λ is the Lebesgue measure on $[0,1]$. Then the equality $\widehat{\mathbb{P}}\,|_{\mathcal{F}_t} = \mathbb{P}\,|_{\mathcal{F}_t}$ holds for all $t \in \mathbb{R}_+$.

2. We set $U(x) = x$ where $x \in [0,1]$. Then U is a uniformly distributed random variable on $(\widehat{\Omega}, \widehat{\mathbb{P}})$ independent of $\mathcal{F}_\infty$.

3. We define the random time τ by the formula

$$\tau = \inf \{ t \in \mathbb{R}_+ : F_{t,\infty} \geq U \}$$

   Then $G_t^\tau = 1 - F_{t,t}$.

□

Step 3 in the proof

Proof.

The random time τ constructed in Step 3 satisfies

$$\{\tau \le u\} = \{F_{u,\infty} \ge U\}$$

Since U is independent of $\mathcal{F}_\infty$, we obtain

$$\begin{aligned}\widehat{\mathbb{P}}\,(\,\tau \le u \,|\, \mathcal{F}_\infty) &= \widehat{\mathbb{P}}\,(\,F_{t,\infty} \ge U \,|\, \mathcal{F}_\infty) \\ &= \widehat{\mathbb{P}}(U \le x)\big|_{x=F_{u,\infty}} = F_{u,\infty}\end{aligned}$$

Consequently,

$$F^\tau_t = \widehat{\mathbb{P}}\,(\,\tau \le t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(F_{t,\infty} \,|\, \mathcal{F}_t) = F_{t,t}$$

We conclude that the equality $G^\tau_t = 1 - F_{t,t}$ holds. □

Multiplicative system

Remark

It remains to construct $F_{u,t}$ consistent with a given in advance process G in the sense that $G_t = 1 - F_{t,t}$ for all $t \in \mathbb{R}_+$. To this end, we will use the concept of a multiplicative system.

Definition (Meyer (1979))

A random field $(C_{u,t})_{u,t\in\overline{\mathbb{R}}_+}$ is called a *multiplicative system* if it satisfies:

1. if $u \leq s \leq t$ then $C_{u,s}C_{s,t} = C_{u,t}$ and $C_{u,t} = 1$ if $u \geq t$,
2. for every u, the process $(C_{u,t})_{t\in\overline{\mathbb{R}}_+}$ is $\mathbb{F}$-predictable and decreasing in t,
3. for every t, the process $(C_{u,t})_{u\in\overline{\mathbb{R}}_+}$ is a right-continuous and increasing in u (not necessarily $\mathbb{F}$-adapted).

Multiplicative system associated with Y

- Let $Y = (Y_t)_{t \in \overline{\mathbb{R}}_+}$ be a positive $(\mathbb{P}, \mathbb{F})$-submartingale.
- In our case, we set $Y = F$.

Definition (Meyer (1979))

A multiplicative system $(C_{u,t})_{u,t \in \overline{\mathbb{R}}_+}$ is said to be *associated with a* $(\mathbb{P}, \mathbb{F})$-submartingale Y if for all $t \in \overline{\mathbb{R}}_+$

$$\mathbb{E}_{\mathbb{P}}\left(C_{t,\infty} Y_\infty \,|\, \mathcal{F}_t \right) = Y_t$$

Note that $C_{t,\infty}$ is bounded and Y_∞ is integrable (by definition).

Theorem (Meyer (1979))

Any positive submartingale $Y = (Y_t)_{t \in \overline{\mathbb{R}}_+}$ admits an associated multiplicative system.

Existence of a multiplicative system

Proof.

- Assume first that Y is bounded below by $\epsilon > 0$. Then it suffices to take

$$C_{u,t} = \exp\left(- \int_{(u,t]} \frac{dB^c_s}{{}^pY_s} \right) \prod_{u<s\le t} \left(1 - \frac{\Delta B_s}{{}^pY_s} \right)$$

  where B is the increasing process which generates the positive supermartingale $X_t = Y_\infty - Y_t$, that is,

$$X_t = \mathbb{E}_{\mathbb{P}}\left(B_\infty \,|\, \mathcal{F}_t \right) + B_t$$

  and ${}^pY$ is the $\mathbb{F}$-predictable projection of Y.
- The general case is established by passing to the limit as $\epsilon \to 0$.

□

Construction of an $\mathbb{F}$-conditional distribution

Using Meyer's theorem, we can establish the following result.

Lemma (L.-R. (2010))

Let $(F_t)_{t\in\overline{\mathbb{R}}_+}$ be a submartingale such that $0 \leq F \leq 1$. We define the random field $(F_{u,t})_{u,t\in\overline{\mathbb{R}}_+}$ by setting

$$F_{u,t} = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(F_u \,|\, \mathcal{F}_t\right), & t \in [0, u), \\ C_{u,t}F_t, & t \in [u, \infty], \end{cases}$$

where $C_{u,t}$ is any multiplicative system associated with F. Then $F_{u,t}$ is an $\mathbb{F}$-conditional distribution of a random time and $F_{t,t} = F_t$.

Multiplicative construction of a random time

Theorem (L.-R. (2010))

Let G be a $(\mathbb{P}, \mathbb{F})$-supermartingale such that $0 \leq G \leq 1$.

1. *The random field $F_{u,t} = C_{u,t}(1 - G_t)$ is an $\mathbb{F}$-conditional distribution for all $u \leq t$.*
2. *The extended canonical construction yields a random time τ on the extended space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ such that for all $u \leq t$*

$$\widehat{\mathbb{P}}\,(\tau > u \,|\, \mathcal{F}_t) = 1 - C_{u,t}(1 - G_t)$$

3. *For all $u \leq t$ we have*

$$\widehat{\mathbb{P}}\,(\tau \leq u \,|\, \mathcal{F}_t) = C_{u,t}\,\widehat{\mathbb{P}}\,(\tau \leq t \,|\, \mathcal{F}_t)$$

5. Uniqueness of a random time consistent with G

Equivalence of admissible constructions

Definition

An *admissible construction* of a random time τ is a pair $(\tau, \mathbb{Q})$ defined on any extension $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, \widetilde{\mathbb{F}})$ of the filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ such that:

- $\mathbb{Q}\,(\tau > t \,|\, \mathcal{F}_t) = G_t$ for all $t \in \mathbb{R}_+$,
- the restriction of $\mathbb{Q}$ to $\mathbb{F}$ equals $\mathbb{P}$.

Definition

Two construction $(\tau, \mathbb{Q})$ and $(\widehat{\tau}, \widehat{\mathbb{Q}})$ are *equivalent* if the $\mathbb{F}$-conditional distributions of τ and $\widehat{\tau}$ are indistinguishable.

Uniqueness of $F_{u,t}$ for $t \le u$

- For any admissible construction $(\tau, \mathbb{Q})$, the $\mathbb{F}$-conditional distribution of $F_{u,t}$ for $u \ge t$ is fixed since $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ for all $t \ge 0$ and thus for all $u \ge t$

$$F_{u,t} = \mathbb{Q}\,(\,\tau \le u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\,(\,F_{u,u} \,|\, \mathcal{F}_t) = 1 - \mathbb{E}_{\mathbb{P}}\,(\,G_u \,|\, \mathcal{F}_t)$$

- Hence in any construction one has essentially the freedom to choose the conditional distribution $F_{u,t}$ for $t > u$, as long as $F_{t,t} = 1 - G_t$.
- The canonical construction of τ yields for $t \ge u$

$$F_{u,t} = \mathbb{Q}\,(\,\tau \le u \,|\, \mathcal{F}_t) = \mathbb{Q}\,(\,\tau \le u \,|\, \mathcal{F}_t) = F_{u,u}$$

but it can only be applied when G is decreasing.

Hypotheses (H), (HP) and (DP)

Definition

A pair $(\tau, \mathbb{Q})$ is said to satisfy:

- the *hypothesis* (H) if for all $0 \leq u \leq s \leq t$

$$F_{u,s} = F_{u,t}$$

- the *hypothesis* (HP) (or the *proportionality* property) if for all $0 \leq u < s < t$

$$F_{u,s} F_{s,t} = F_{s,s} F_{u,t}$$

- the *hypothesis* (DP) (or the *decreasing proportionality* property) if for all $0 \leq u < s < t$

$$F_{u,s} F_{t,t} \geq F_{s,s} F_{u,t}$$

Uniqueness of $F_{u,t}$ for $t > u$

Proposition (L.-R. (2010))

The following implications are valid:

$$(H) \implies (HP) \implies (DP)$$

The following result is used to establish the uniqueness of $\mathbb{F}$-conditional distribution.

Theorem (Meyer (1979))

If $C_{u,t}$ and $\overline{C}_{u,t}$ are two multiplicative systems associated with a given positive submartingale $Y = (Y_t)_{t \in \overline{\mathbb{R}}_+}$ then the random fields $C_{u,t}Y_t$ and $\overline{C}_{u,t}Y_t$ are indistinguishable.

Uniqueness of $F_{u,t}$ for $t > u$

Proposition (L.-R. (2010))

Two admissible constructions with a given supermartingale G are equivalent if:

- *the hypothesis (HP) holds for $(\mathbb{Q}, \tau)$ and $(\widehat{\mathbb{Q}}, \widehat{\tau})$,*
- *for any fixed $u \geq 0$, the $\mathbb{F}$-adapted processes*

$$C_{u,t} = \frac{\mathbb{Q}\,(\tau \leq u \,|\, \mathcal{F}_t)}{\mathbb{Q}\,(\tau \leq t \,|\, \mathcal{F}_t)} \qquad \widehat{C}_{u,t} = \frac{\widehat{\mathbb{Q}}\,(\widehat{\tau} \leq u \,|\, \mathcal{F}_t)}{\widehat{\mathbb{Q}}\,(\widehat{\tau} \leq t \,|\, \mathcal{F}_t)}$$

are $\mathbb{F}$-predictable in $t \in \overline{\mathbb{R}}_+$.

Comments

- Any random time τ constructed through the multiplicative approach satisfies the hypothesis (*HP*).
- Any honest time τ satisfies the hypothesis (*HP*). Honest time are $\mathcal{F}_\infty$-measurable so they are not obtained through the multiplicative approach.
- Given a supermartingale G, we can also produce a random time τ consistent with G for which the hypothesis (*HP*) fails to hold.

Non-uniqueness of $F_{u,t}$ for $t > u$

Example

Let M be a continuous, positive, square-integrable martingale.

- Consider the $\mathbb{F}$-conditional distribution

$$\widehat{F}_{u,t} = 1 - \exp\left(-uM_t - \frac{1}{2}u^2\langle M, M\rangle_t\right)$$

  The extended canonical construction yields a random time $\widehat{\tau}$ for which the hypothesis (HP) is not valid since for $u < t$

$$\mathbb{P}\,(\widehat{\tau} \leq u \,|\, \mathcal{F}_t) = \widehat{F}_{u,t} \neq C_{u,t}\widehat{F}_{t,t} = C_{u,t}\,\mathbb{P}\,(\widehat{\tau} \leq t \,|\, \mathcal{F}_t)$$

- The multiplicative approach gives $F_{u,t}$ consistent with $G = \widehat{G}$ for which the hypothesis (HP) holds.
- Hence the extended canonical construction yields two random times with the same Azéma supermartingale, but with different $\mathbb{F}$-conditional distributions $F_{u,t}$ and $\widehat{F}_{u,t}$ for $t > u$.

6. Separability of a conditional distribution

Separability of a conditional distribution

Definition

We say that an $\mathbb{F}$-conditional distribution $F_{u,t}$ is *completely separable* if there exists a positive $(\mathbb{P}, \mathbb{F})$-martingale X and a positive, $\mathbb{F}$-adapted, increasing process Y such that $F_{u,t} = Y_u X_t$ for every $u, t \in \mathbb{R}_+$ such that $u \leq t$.

Separability of $F_{u,t}$ is a weaker form of complete separability.

Proposition (L.-R. (2010))

- *If the $\mathbb{F}$-conditional distribution of τ is separable and $F_0 = 0$ then the hypothesis (HP) holds.*
- *If the $\mathbb{F}$-conditional distribution $F_{u,t} > 0$ satisfies the hypothesis (HP) then the random field $F_{u,t}$ is separable.*

Multiplicative approach

Proposition (L.-R. (2010))

If $G_t < 1$ for $t \geq 0$ then $F_{u,t}$ obtained through the multiplicative approach is completely separable: for all $0 \leq u \leq t$

$$F_{u,t} = \frac{F_t\,\mathcal{E}_t\left(\int_{(0,\cdot]}({}^pF_s)^{-1}\,dA_s\right)}{\mathcal{E}_u\left(\int_{(0,\cdot]}({}^pF_s)^{-1}\,dA_s\right)} = Y_uX_t$$

where the strictly positive, increasing process Y is given by

$$Y_u = \left[\mathcal{E}_u\left(\int_{(0,\cdot]}({}^pF_s)^{-1}\,dA_s\right)\right]^{-1}$$

and the strictly positive $(\mathbb{P},\mathbb{F})$-martingale X equals

$$X_t = F_t\,\mathcal{E}_t\left(\int_{(0,\cdot]}({}^pF_s)^{-1}\,dA_s\right)$$

7. Semimartingale decompositions: new results

Standing assumptions

In this part, we make the following standing assumptions:

1. We consider a random time τ on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ such that G is the Azéma supermartingale of τ.
2. The Doob-Meyer decomposition of G is denoted as $G = M - A$.
3. We denote by $F_{u,t}$ the $\mathbb{F}$-conditional distribution of τ under $\mathbb{P}$, that is, $F_{u,t} = \mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t)$ for all $u, t \geq 0$.
4. For simplicity, we assume that a $(\mathbb{P}, \mathbb{F})$-martingale U is continuous at τ.

$\mathbb{G}$-Semimartingale decomposition: general case

Recall that for every $s \geq 0$ the process $(F_{s,u})_{u \geq s}$ is a bounded, positive $(\mathbb{P}, \mathbb{F})$-martingale.

Theorem (L.-R. (2010))

Assume that the hypothesis (HP) holds and the $\mathbb{F}$-conditional distribution satisfies $0 < F_{u,t} \leq 1$ for every $0 < u \leq t$. If U is a $(\mathbb{P}, \mathbb{F})$-local martingale then

$$U_t - \int_{(0,t\wedge\tau]} (G_u)^{-1} \, d[U, G]_u - \int_{(t\wedge s,t]} (F_{s,u})^{-1} \, d[U, F_{s,\cdot}]_u \Big|_{s=\tau}$$

is a $(\mathbb{P}, \mathbb{G})$-local martingale.

$\mathbb{G}$-Semimartingale decomposition: separable case

Recall that the separability of $F_{u,t}$ is (almost) equivalent to (*HP*).

Corollary (L.-R. (2010))

Assume that the $\mathbb{F}$-conditional distribution of τ is completely separable, that is, the $\mathbb{F}$-conditional distribution of τ is given by $F_{u,t} = Y_u X_t$ for every $0 \leq u \leq t$. If U is a $(\mathbb{P}, \mathbb{F})$-martingale then

$$U_t - \int_{(0,t\wedge\tau]} (G_u)^{-1} \, d[U, G]_u - \int_{(t\wedge\tau,t]} (X_u)^{-1} \, d[U, X]_u$$

is a $(\mathbb{P}, \mathbb{G})$-local martingale.

$\mathbb{G}$-Semimartingale decomposition: multiplicative case

The next corollary is comparable to the case of an honest time.

Corollary (L.-R. (2010))

Assume that $G_t < 1$ and for every $t > 0$ and τ was constructed using the multiplicative approach. If U is a $(\mathbb{P}, \mathbb{F})$-martingale then

$$U_t - \int_{(0,t\wedge\tau]} (G_u)^{-1}\, d[U,M]_u + \int_{(t\wedge\tau,t]} (F_u)^{-1}\, d[U,M]_u$$

is a $(\mathbb{P}, \mathbb{G})$-local martingale.

Remark

Similar decomposition (but with the predictable bracket) was obtained by Jeanblanc and Song in [JS1] under continuity assumptions.

$\mathbb{G}$-Semimartingale decomposition: density hypothesis

Definition

The field $F_{u,t}$ satisfies the *density hypothesis* if there exists an $\mathbb{F}$-adapted, increasing process D and, for every $s \geq 0$, a $(\mathbb{P}, \mathbb{F})$-martingale $(m_{s,t})_{t \geq s}$ such that

$$F_{u,t} = \int_{[0,u]} m_{s,t} \, dD_s$$

Corollary (L.-R. (2010))

Under the density hypothesis, if U is a $(\mathbb{P}, \mathbb{F})$-martingale then

$$U_t - \int_{(0,t\wedge\tau]} (G_u)^{-1} \, d[U, G]_u - \int_{(t\wedge s,t]} (m_{s,u})^{-1} \, d[U, m_{s,\cdot}\,]_u \Big|_{s=\tau}$$

is a $(\mathbb{P}, \mathbb{G})$-local martingale.

8. Girsanov's theorem

Radon-Nikodym density

Lemma (L.-R. (2010))

Let the $\mathbb{F}$-conditional distribution of τ under $\mathbb{P}$ be separable. We define the process $Z^{\mathbb{G}}$ by the formula

$$Z_t^{\mathbb{G}} = \widetilde{Z}_t \mathbb{1}_{\{\tau > t\}} + \widehat{Z}_{\tau,t} \mathbb{1}_{\{\tau \le t\}}$$

where $\widehat{Z}_{u,t} = \frac{F_{u,u}}{F_{u,t}}$ *and*

$$\widetilde{Z}_t = G_t^{-1} \left(1 - \int_{(0,t]} \widehat{Z}_{u,t} \, dF_{u,t} \right) = G_t^{-1} \left(1 - \mathbb{E}_{\mathbb{P}} \left(\widehat{Z}_{\tau,t} \mathbb{1}_{\{\tau \le t\}} \,\middle|\, \mathcal{F}_t \right) \right)$$

Then the process $Z^{\mathbb{G}}$ is a $(\mathbb{P}, \mathbb{G})$-local martingale.

Application of Girsanov's theorem

The hypothesis (HP) implies that the hypothesis (H) holds under an equivalent probability measure.

Proposition (L.-R. (2010))

Assume that:

- *the conditional distribution of the random time τ under $\mathbb{P}$ is separable so that the hypothesis (HP) is satisfied,*
- *the process $Z^{\mathbb{G}}$ is a positive $(\mathbb{P}, \mathbb{G})$-martingale with the property that $\mathbb{E}_{\mathbb{P}}(Z^{\mathbb{G}}_t \mid \mathcal{F}_t) = 1$ for $t \in \mathbb{R}_+$.*

Then there exists an equivalent probability measure $\widehat{\mathbb{P}}$ such that the hypothesis (H) holds under $\widehat{\mathbb{P}}$.

Concluding remarks

Remarks:

1. In the recent paper by Coculescu et al. [CJN], the existence of an equivalent probability measure for which the hypothesis (*H*) holds was shown to be a sufficient condition for a model with enlarged filtration to be arbitrage-free, provided that the corresponding model based on the filtration $\mathbb{F}$ enjoys this property.
2. The last proposition shows that if the hypothesis (*HP*) holds then, under mild technical assumptions, the result from [CJN] can be applied to progressive enlargement.
3. Our results can also be used for modeling asymmetric information (weak insider trading) in a progressive enlargement setting.

Selected references

References I

[Y] Yor, M.:
Grossissement d'une filtration et semi-martingales: théorèmes généraux.
In: *Séminaire de Probabilités XII, Lecture Notes in Mathematics 649*. Springer, Berlin, 1978, pp. 61–69.

[JY] Jeulin, T. and Yor, M.:
Grossissement d'une filtration et semi-martingales: formules explicites.
In: *Séminaire de Probabilités XII, Lecture Notes in Mathematics 649*. Springer, Berlin, 1978, pp. 78–97.

[J] Jacod, J.:
Grossissement initial, hypothèse (H') et théorème de Girsanov.
In: *Grossissements de filtrations: exemples et applications, Lecture Notes in Mathematics 1118*, T. Jeulin and M. Yor (eds). Springer, Berlin, 1985, pp. 15–35.

References II

- [M] Meyer, P.A.:
  Représentations multiplicatives de sousmartingales d'après Azéma.
  In: *Séminaire de Probabilités XIII, Lecture Notes in Mathematics 721.* Springer, Berlin, 1979, pp. 240–249.
- [JL] Jeanblanc, M. and Le Cam, Y.:
  Progressive enlargement of filtration with initial times.
  Stochastic Processes and their Applications 119 (2009), 2523–2543.
- [GJLR] Gapeev, P., Jeanblanc, M., Li, L., and Rutkowski, M.:
  Constructing random times with given survival processes and applications to valuation of credit derivatives.
  In: *Contemporary Quantitative Finance*, C. Chiarella and A. Novikov, eds., Springer, Berlin, 2010.

References III

- [CJN] Coculescu, D., Jeanblanc, M., and Nikeghbali, A.:
  Default times, non arbitrage conditions and change of probability measures.
  Working paper, 2009.
- [EJJ] El Karoui, N., Jeanblanc, M., and Jiao, Y.:
  What happens after a default: the conditional density approach.
  Working paper, 2009.
- [JS1] Jeanblanc, M. and Song, S.:
  An explicit model of default time with given survival probability.
  Working paper, University of Evry, 2010.
- [JS2] Jeanblanc, M. and Song, S.:
  Default times with given survival probability and their $\mathbb{F}$-martingale decomposition formula.
  Working paper, University of Evry, 2010.