An Extension of CreditGrades model approach with Lévy Processes

Yuji Umezawa

Joint work with
Takaaki Ozeki, Akira Yamazaki, and Daisuke Yoshikawa
(Mizuho-DL Financial Technology Co., Ltd.)

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Contents

Disclaimer

Introduction

Introduction

- It is important to consider the linkeage between credit and equity markets.

Significance of modelling for linkage between credit and equity

- Relative value analysis between credit(CDS etc.) and equity(equity option etc.) markets.
- Covertible bond arbitrage
- Capital structure arbitrage

- The CreditGrades model presented by Finger et al[2002] is one of the most approved approaches to link between credit and equity markets, however there exist some drawbacks.
- We proposes an extended CreditGrades model for pricing equity options and CDSs simultaneously, in order to overcome these drawbacks.

Basics of Credit Models

Basics of Credit Models

Two main types of Credit Models

- **Firm value model**: Fundamental approach for valuing defaultable debt, which based on modeling a stochastic process for the firm's value. "Fundamental" in the sense of linking debt pricing and equity pricing.
  - Classic firm value model (Merton[1976]),
  - CreditGrades model (Finger et al.[2002]),
  - Extended CreditGrades model (Sepp[2006], Ozeki et al.[2010]), etc.
- **Intensity based model**: The default process is usually defined as a one-jump process which can jump from no-default to default, and the probability of a jump in a given time interval is governed by the default intensity.
  - Duffie and Singleton[1999], etc.

Firm value model and B/S

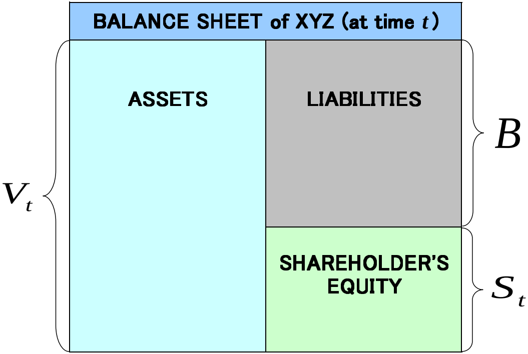

Figure: Relation between a firm value model and B/S

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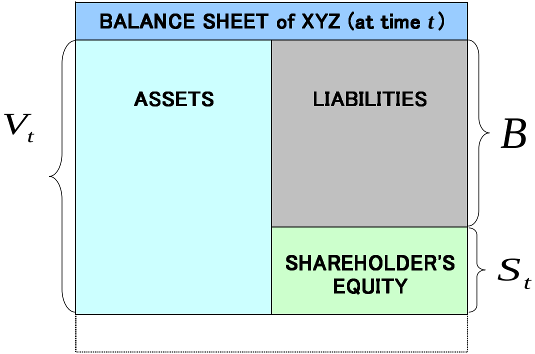

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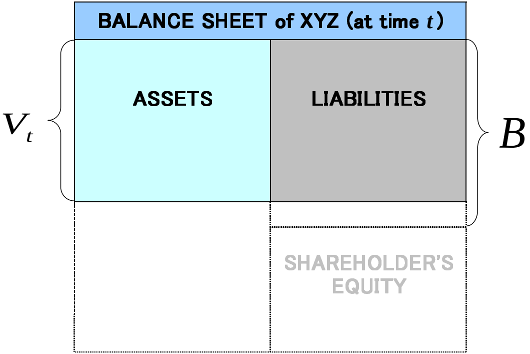

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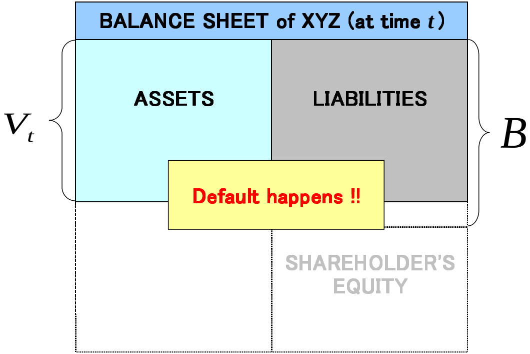

Figure: Relation between a firm value model and B/S

CreditGrades Model

Settings of standard model

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T^*}, \mathbb{Q})$ be a filtered probability space, where T^* is some time horizon, and $\mathbb{Q}$ is a risk neutral probability measure.

- $V_0 := S_0 + B$ is the initial asset value, where S_0 is the initial stock price, and B is the firm's debt. In this model, B is identified with the default barrier for simplicity.
- The firm's asset value follows $V_t = V_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$, where W_t is a standard Brownian motion, so V_t is a continuous process.
- The time τ of the default on time interval $(0, T]$ is defined as

$$\tau = \inf\{t \in (0, T] : V_t \le B\}, \tag{1}$$

- We define the dynamics of the equity price as

$$S_t = \begin{cases} (V_t - B)e^{\int_0^t (r_s - d_s)ds} = (V_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t} - B)e^{\int_0^t (r_s - d_s)ds} & \text{if} \quad t < \tau, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where r_t is a *deterministic* risk-free interest rate, and d_t is a *deterministic* dividend yield of the firm's equity.

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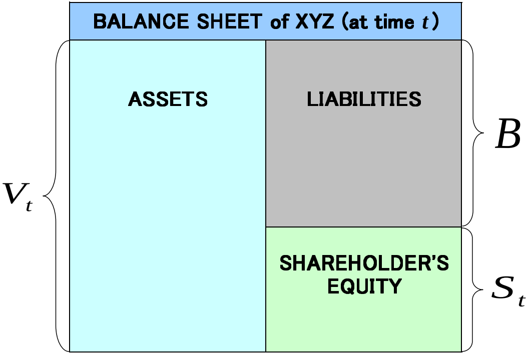
(Reappeared) Firm value model and B/S



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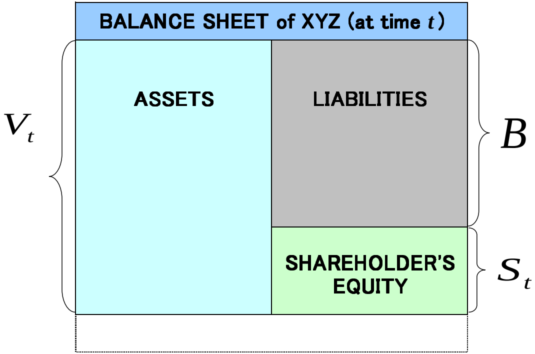

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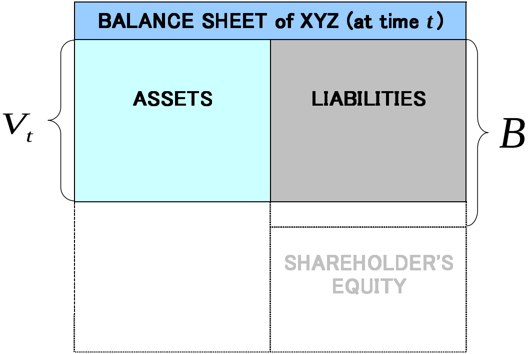

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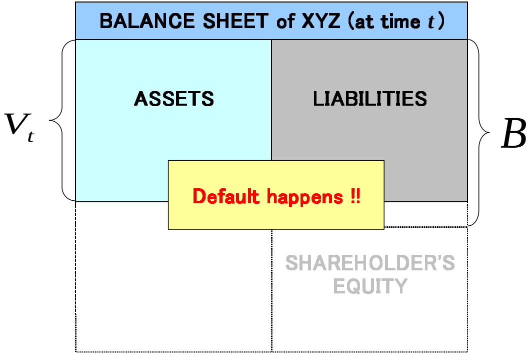

Figure: Relation between a firm value model and B/S

Pricing equity option and CDS

- The equity option price C with strike K and maturity T is given by

$$
\begin{aligned}
C &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_t dt}(S_T - K)^+ \mathbf{1}_{\{\tau > T\}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_t dt}\left(V_0 e^{X_T} - K\right)^+ \mathbf{1}_{\{\tau > T\}}\right], \qquad (3)
\end{aligned}
$$

where $X_t = \sigma W_t - \frac{1}{2}\sigma^2 t$ and the CDS per premium c is given by

$$
c = (1-R)\frac{1 - e^{-\int_0^T r_t dt}\mathbb{Q}(\tau > T) - \int_0^T r_t e^{-\int_0^t r_u du}\mathbb{Q}(\tau > t)dt}{\int_0^T e^{-\int_0^t r_u du}\mathbb{Q}(\tau > t)\, dt}. \qquad (4)
$$

- $\mathbb{Q}(\tau > t)$ is the survival probability until time t (under the risk neutral measure $\mathbb{Q}$).

(Cont'd) Pricing equity option and CDS

- In the standard CreditGrades model, the analytical formulae of equity option prices and survival probabilities can be obtained. (Finger et al. [2002], Sepp [2006])
- The equity option prices are given by

$$\begin{aligned} C^{\sigma} =& C^{\sigma}_{BS}(T, S_0 + B, K + B, \bar{r}, \bar{d}) \\ & - \frac{S_0 + B}{B} C^{\sigma}_{BS}\left(T, \frac{B^2}{S_0 + B}, K + B, \bar{r}, \bar{d}\right), \end{aligned} \tag{5}$$

$$\text{where} \quad \bar{r} = \frac{1}{T}\int_0^T r_t dt, \quad \bar{d} = \frac{1}{T}\int_0^T d_t dt,$$

  and $C^{\sigma}_{BS}(T, S, K, r, d)$ is the Black-Scholes price of a call option.
- The survival probability of the standard model with an asset volatility σ can be calculated by the following formula:

$$\mathbb{Q}(\tau > T; \sigma) = \mathcal{N}\left(\frac{\log\left(\frac{S_0+B}{B}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - \frac{S_0 + B}{B}\mathcal{N}\left(\frac{\log\left(\frac{B}{S_0+B}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right), \tag{6}$$

  where $\mathcal{N}(\cdot)$ is the cumulative distribution function of standard normal distribution.

Features and drawbacks

- The equity volatility σ_t^S is given by the following local volatility function:

$$\sigma_t^S = \sigma \frac{S_t + B}{S_t} = \sigma \left(1 + \frac{B}{S_t} \right). \tag{7}$$

  Therefore, the standard model can describe the implied equity volatility skew naturally. However, it is not able to reflect unpredictable credit events into the implied volatility, since σ_t^S depends only on the leverage ratio of the firm's debt.

- In this model, the default events are predictable, since the model assumes a continuous dynamics. As a result, if the current value of the firm is remote from the barrier, both the default probability and the credit spread in short-term are close to zero.
- Introducing a stochastic default barrier considered to be a solution to this problem, however it is difficult to choose the appropriate distribution of the stochastic behavior of the barrier, since it is usually unobservable.

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Extended Model

Features

Features of Extended Model

- The firm's asset value process include jumps, so is *not* predictable.
- The model describes more realistic firm's value dynamics.
- The model can generates a realistic value for short term CDS spread.
- The model can fits to a realistic implied volatility surface.

Examples: Modelling including jumps in the firm's value model

- Sepp [2006]: Extended CreditGrades model
- Cariboni and Schoutens [2007]
- Madan and Schoutens [2008], etc.

Settings of extended model

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T^*}, \mathbb{Q})$ be a filtered probability space, where T^* is some time horizon, and $\mathbb{Q}$ is a risk neutral probability measure.

Settings of Extended Model

- $V_0 := S_0 + B$ is the initial asset value, where S_0 is the initial stock price, and B is the firm's debt. In this model, B is identified with the default barrier for simplicity.
- The firm's asset value follows $V_t = V_0 e^{X_t}$, where X_t is a Lévy process.
- The time τ of the default on time interval $(0, T]$ is defined as

$$\tau = \inf\{t \in (0, T] : V_t \le B\}, \tag{8}$$

- We define the dynamics of the equity price as

$$S_t = \begin{cases} (V_t - B)e^{\int_0^t (r_s - d_s)ds} = (V_0 e^{X_t} - B)e^{\int_0^t (r_s - d_s)ds} & \text{if} \quad t < \tau, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

where r_t is a *deterministic* risk-free interest rate, and d_t is a *deterministic* dividend yield of the firm's equity.

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Examples of Choosing Lévy process X_t

Examples

$X_t =: \mu t + \sigma W_t + \bar{Z}_t,$

- standard model: $\bar{Z}_t = 0,$
- Exponential Jump (EJCG): $\bar{Z}_t = -\sum_{j=1}^{N_t} Y_j,$
  where $N_t,\ t \geq 0$ denote Poisson process with intensity λ, and $(Y_j)_{j\in\mathbb{N}}$ are i.i.d. random variable according to exponential distribution with parameter a.
- Gamma Jump (GJCG): $\bar{Z}_t = -G_t,$
  where the Lévy density of G_t is given by the following gamma distribution:

$$\Pi^G(x) = \frac{\lambda e^{-ax}}{x}\mathbf{1}_{x>0}, \tag{10}$$

- Inverse Gaussian Jump (IGJCG): $\bar{Z}_t = -I_t,$
  where the Lévy density of I_t is given by the following inverse gaussian distribution:

$$\Pi^I(x) = \frac{\lambda \exp\left(-\frac{1}{2}a^2x\right)}{\sqrt{2\pi}x^{3/2}}\mathbf{1}_{x>0}, \tag{11}$$

(cont'd) Examples of Choosing Lévy process X_t

Characteristic Exponent $\psi_X(\theta)$

$$\exp\{-t\psi_X(\theta)\} := \mathbb{E}\left[e^{i\theta X_t}\right],$$

$$\begin{aligned} X_t &= \mu t + \sigma W_t + \bar{Z}_t =: \mu t + Z_t, \qquad (12) \\ \mu &= \psi_Z(-i), \quad \text{(Condition for Martingale property)} \end{aligned}$$

Characteristic exponent for each process

- standard model: $\psi_Z(\theta) = \frac{1}{2}\sigma^2\theta^2$,
- Exponential Jump (EJCG): $\psi_Z(\theta) = \frac{1}{2}\sigma^2\theta^2 + \lambda\left(1 - \frac{a}{a+i\theta}\right)$,
- Gamma Jump (GJCG): $\psi_Z(\theta) = \frac{1}{2}\sigma^2\theta^2 + \lambda\log\left(1 + \frac{i\theta}{a}\right)$,
- Inverse Gaussian Jump (IGJCG): $\psi_Z(\theta) = \frac{1}{2}\sigma^2\theta^2 + \lambda\left(\sqrt{a^2 + 2i\theta} - a\right)$,

Pricing equity option and CDS

- The equity option price C with strike K and maturity T is given by

$$\begin{aligned} C &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_t dt}(S_T - K)^+ \mathbf{1}_{\{\tau > T\}}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_t dt}\left(V_0 e^{X_T} - K\right)^+ \mathbf{1}_{\{\tau > T\}}\right], \end{aligned} \tag{13}$$

and the CDS per premium c is given by

$$c = (1-R)\frac{1 - e^{-\int_0^T r_t dt}\mathbb{Q}(\tau > T) - \int_0^T r_t e^{-\int_0^t r_u du}\mathbb{Q}(\tau > t)dt}{\int_0^T e^{-\int_0^t r_u du}\mathbb{Q}(\tau > t)\, dt}. \tag{14}$$

- Note that
$\{\tau > T\} = \left\{\min_{0\leq s\leq T} V_s > B\right\} = \left\{\min_{0\leq s\leq T} X_s > \log\left(\frac{S_0}{S_0 + B}\right)\right\}.$
- If the joint distribution of $(X_T, N_T) := (X_T, \min_{0\leq s\leq T} X_s)$ are known, this expectations can be calculated.

Wiener-Hopf Factorization

Wiener-Hopf Factorization (Sato, 1999)

Factorization identities of the Laplace fransform (in t) of the distribution X_t of a Lévy process by using the Laplace transforms (in t) of the distributions of the supremum process, infimum process, etc.

Wiener-Hopf Factorization for a Minumum Process

The Laplace transform in t of the joint characteristic function of $(N_t, X_t - N_t)$ is given by

$$q \int_0^{+\infty} e^{-qt} \mathbb{E}\left[e^{ixN_t + iy(X_t - N_t)}\right] dt = \Phi^+_{q,X}(y)\Phi^-_{q,X}(x),$$

for any $q > 0$ and $x,\ y \in \mathbf{R}$, where $N_t = \min_{0 \leq s \leq t} X_s$, and

$$\Phi^+_{q,X}(\theta) = \exp\left\{\int_0^{+\infty} t^{-1} e^{-qt} dt \int_0^{+\infty} \left(e^{i\theta x} - 1\right) dF_{X_t}(x)\right\},$$

$$\Phi^-_{q,X}(\theta) = \exp\left\{\int_0^{+\infty} t^{-1} e^{-qt} dt \int_{-\infty}^{0} \left(e^{i\theta x} - 1\right) dF_{X_t}(x)\right\}.$$

(Cont'd) Wiener-Hopf Factorization

- We cannot get the closed form of distribution function F_X in general.

Spectrally Negative Lévy Processes

- The Lévy Processes that have only negative jumps.
- Suitable for modelling of credit events.
- Wiener-Hopf factors $\Phi^+_{q,X}, \Phi^-_{q,X}$ are given by simple forms.

Wiener-Hopf Factor of Spectrally Negative Lévy Processes

$$\begin{aligned}\Phi^+_{q,X}(\theta) &= \frac{\eta_q}{\eta_q - i\theta}, \\ \Phi^-_{q,X}(\theta) &= \frac{q\left(\eta_q - i\theta\right)}{\eta_q\left(q + \psi_X(\theta)\right)},\end{aligned} \tag{15}$$

where η_q is the unique positive real root of $q + \psi_X(-i\eta_q) = 0$.

Wiener-Hopf Factorization: standard model and EJCG

- In case of the Standard model, the equation $q + \psi_X(-i\eta_q) = 0$ can be rewritten as the following two degree polynomial equation:

$$\sigma^2\eta_q^2 - \sigma^2\eta_q - 2q = 0. \tag{16}$$

Then η_q can be obtained analytically by solving the above equation, that is,

$$\eta_q = \frac{1}{2} + \frac{1}{\sigma}\sqrt{\frac{\sigma^2}{4} + 2q}. \tag{17}$$

- In case of the Exponential Jump Model (EJCG), the equation $q + \psi_X(-i\eta_q) = 0$ can be rewritten as the following third degree polynomial equation:

$$\sigma^2\eta_q^3 + (a\sigma^2 + 2\mu)\eta_q^2 + 2(a\mu - \lambda - q)\eta_q - 2aq = 0. \tag{18}$$

Then η_q can be obtained analytically by the Cardano formula.

- In other cases (GJCG,IGJCG), numerical computation such as the Newton method is needed in order to obtain the value of η_q.

Pricing formula for equity option

The equity option price C with strike K and maturity T is given by the following representation:

$$C = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_t dt}(S_T - K)^+ \mathbf{1}_{\{\tau > T\}}\right] = e^{-\int_0^T d_t dt}(S_0 + B)f(T,k,b) + C^\sigma, \tag{19}$$

where

$$f(T,k,b) := \frac{1}{2\pi i}\int_{\varsigma - i\infty}^{\varsigma + i\infty} e^{qT}\frac{1}{(2\pi)^2}\iint_{\mathbf{R}^2} e^{-(iu+\alpha)k-(iv+\beta)b}\kappa(q,u,v)dudvdq$$

$$\kappa(q,u,v) := \frac{\Phi^+_{q,X}(u - i[\alpha+1])\Phi^-_{q,X}(u+v-i[\alpha+\beta+1])}{q(iu+\alpha)(iv+\beta)(iu+\alpha+1)} - \frac{\Phi^+_{q,Y}(u - i[\alpha+1])\Phi^-_{q,Y}(u+v-i[\alpha+\beta+1])}{q(iu+\alpha)(iv+\beta)(iu+\alpha+1)},$$

$$k := \log\left(\frac{Be^{\int_0^T(r_t-d_t)dt} + K}{(S_0+B)e^{\int_0^T(r_t-d_t)dt}}\right), \quad b := \log\left(\frac{B}{S_0+B}\right).$$

$\Phi^\pm_{q,X}(\cdot)$ and $\Phi^\pm_{q,Y}(\cdot)$ denote the Wiener-Hopf factors of the Lévy process X_t and a Gaussian process $Y_t := \sigma W_t - \frac{1}{2}\sigma^2 t$ respectively.

Pricing formula for CDS

The survival probability $\mathbb{Q}(\tau > t)$ is given by the following representation:

$$\mathbb{Q}(\tau > t) = g(t, b) + \mathbb{Q}(\tau > t; \sigma), \tag{20}$$

where

$$\begin{aligned} g(t, b) &:= \frac{1}{2\pi i} \int_{\varsigma - i\infty}^{\varsigma + i\infty} e^{qt} \frac{1}{2\pi} \int_{\mathbf{R}} e^{-(iu+\alpha)b} \xi(q, u) du dq, \\ \xi(q, u) &:= \frac{\Phi^{-}_{q,X}(u - i\alpha) - \Phi^{-}_{q,Y}(u - i\alpha)}{q(iu + \alpha)}, \\ b &:= \log\left(\frac{B}{S_0 + B}\right). \end{aligned} \tag{21}$$

The CDS per premium c can be calculated by

$$c = (1 - R) \frac{1 - e^{-\int_0^T r_t dt} \mathbb{Q}(\tau > T) - \int_0^T r_t e^{-\int_0^t r_u du} \mathbb{Q}(\tau > t) dt}{\int_0^T e^{-\int_0^t r_u du} \mathbb{Q}(\tau > t) \, dt}. \tag{22}$$

Derivation of pricing formula for equity option

- The equity call option price C can be expressed as follows:

$$\begin{aligned} C &= \mathbb{E}\left[e^{-\int_0^T r_t dt}(S_T - K)^+ \mathbf{1}_{\{\tau > T\}}\right] \\ &= \mathbb{E}\left[e^{-\int_0^T r_t dt}\left(\tilde{V}_0 e^{X_T} - \tilde{K}\right)^+ \mathbf{1}_{\{\min_{0 \le s \le T} X_s > b\}}\right] \\ &= e^{-\int_0^T r_t dt}\tilde{V}_0\, \mathbb{E}\left[\left(e^{X_T} - e^k\right)^+ \mathbf{1}_{\{N_T^X > b\}}\right], \end{aligned} \tag{23}$$

where $k := \log(\tilde{K}/\tilde{V}_0)$ and $N_T^X := \min_{0 \le s \le T} X_s$.

- Similarly, the call price C^σ of the standard CreditGrades model with a Gaussian process $Y_t := \sigma W_t - \frac{1}{2}\sigma^2 t$ is given by

$$C^\sigma = e^{-\int_0^T r_t dt}\tilde{V}_0\, \mathbb{E}\left[\left(e^{Y_T} - e^k\right)^+ \mathbf{1}_{\{N_T^Y > b\}}\right], \tag{24}$$

where $N_T^Y := \min_{0 \le s \le T} Y_s$.

(Cont'd) Derivation of pricing formula for equity option

- We calculate the difference between the call price of the Extended model and that of the standard model.

$$
\begin{aligned}
f(T,k,b) &:= \frac{C - C^{\sigma}}{e^{-\int_0^T r_t dt}\tilde{V}_0} \\
&= \mathbb{E}\left[\left(e^{X_T} - e^k\right)^+ \mathbf{1}_{\{N_T^X > b\}}\right] - \mathbb{E}\left[\left(e^{Y_T} - e^k\right)^+ \mathbf{1}_{\{N_T^Y > b\}}\right] \\
&= \left(e^x - e^k\right)^+ \mathbf{1}_{\{y > b\}} \left\{\rho_{X_T,N_T^X}(x,y) - \rho_{Y_T,N_T^Y}(x,y)\right\},
\end{aligned} \tag{25}
$$

where $\rho_{X,Z}(\cdot,\cdot)$ denote the joint density function of (X,Z).

- Consider the Fourier transform of the function $(k,b) \mapsto e^{\alpha k + \beta b} f(T,k,b)$:

$$
\begin{aligned}
F(T,u,v) &:= \iint_{\mathbf{R}^2} e^{iuk+ivb} e^{\alpha k + \beta b} f(T,k,b) dkdb \\
&= \frac{\Psi_{X_T,N_T^X}(u - i\alpha - i, v - i\beta) - \Psi_{Y_T,N_T^Y}(u - i\alpha - i, v - i\beta)}{(iu+\alpha)(iv+\beta)(iu+\alpha+1)},
\end{aligned} \tag{26}
$$

where $\Psi_{X,Z}(\cdot,\cdot)$ denote the joint characteristic function of (X,Z).

(Cont'd) Derivation of pricing formula for equity option

- $\alpha, \beta > 0$ is used in order to avoid the singularity at $u = 0$ and $v = 0$ on the integrands in the Fourier inversion.(Carr and Madan[1999])
- Next, consider the Laplace transform of the function $F(T, u, v)$

$$\begin{aligned}\kappa(q,u,v) &:= \int_0^{+\infty} e^{-qT} F(T,u,v)\,dT \\ &= \frac{1}{q(iu+\alpha)(iv+\beta)(iu+\alpha+1)} \\ &\times \Big\{ \Phi^+_{q,X}(u - i[\alpha+1])\Phi^-_{q,X}(u+v-i[\alpha+\beta+1]) \\ &\qquad - \Phi^+_{q,Y}(u - i[\alpha+1])\Phi^-_{q,Y}(u+v-i[\alpha+\beta+1]) \Big\},\end{aligned} \tag{27}$$

- Then, the difference between the call price of the Extended model and that of the standard model is obtained by the inverse Laplace transform and the inverse Fourier transform numerically.
- Since the call price of the standard model can be analytically calculated,we can get that of Extended model.

Derivation of pricing formula for CDS

- The procedure is similar to that of calculating the call option price.
- Consider the the difference between the survival probability of the Extended model and that of the standard model:

$$\begin{aligned} g(b,t) &:= \mathbb{Q}(\tau > t) - \mathbb{Q}(\tau > t; \sigma) = \mathbb{E}\left[\mathbf{1}_{\{N_t^X > b\}} - \mathbf{1}_{\{N_t^Y > b\}}\right], \\ &= \int_{\mathbf{R}} \left(\mathbf{1}_{\{y>b\}}\rho_{N_t^X}(y) - \mathbf{1}_{\{y>b\}}\rho_{N_t^Y}(y)\right) dy, \end{aligned} \tag{28}$$

and the Fourier transform of function $b \mapsto e^{\alpha b} g(b,t)$:

$$\begin{aligned} G(u,t) &= \int_{\mathbf{R}} e^{iub} e^{\alpha b} g(b,t) db \\ &= \frac{\Psi_{N_t^X}(u - i\alpha) - \Psi_{N_t^Y}(u - i\alpha)}{iu + \alpha}. \end{aligned} \tag{29}$$

(Cont'd) Derivation of pricing formula for CDS

- Next, consider the Laplace transform of the function $t \mapsto G(u,t)$:

$$\begin{aligned}\xi(q,u) &= \int_0^{+\infty} e^{-qt} G(u,t)dt. \\ &= \frac{1}{q(iu+\alpha)} \left\{ \Phi^-_{q,X}(u-i\alpha) - \Phi^-_{q,Y}(u-i\alpha) \right\}.\end{aligned} \tag{30}$$

- Then, the difference between the call price of the survival probability and that of the standard model is obtained by the inverse Laplace transform and the inverse Fourier transform numerically.
- Since the survival probability of the standard model can be analytically calculated, we can get that of Extended model.

Gaver-Stehfest algorithm

For function $\tilde{f}(\cdot)$ of certain class defined on $[0, \infty)$, the inverse Laplace transform f of $\tilde{f}$ is given by

$$f(t) := \lim_{n \to \infty} f_n(t), \tag{31}$$

where

$$\begin{aligned} f_n(t) &= \frac{\ln 2}{t} \frac{(2n)!}{n!(n-1)!} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \tilde{f}\left((n+k) \frac{\ln 2}{t} \right), \\ \tilde{f}(t) &= \int_0^{\infty} e^{-ts} f(s) ds. \end{aligned} \tag{32}$$

Using an n-point Richardson extrapolation, $f(t)$ is approximated by $f_n^*(t)$ for sufficiently large n, where

$$f_n^*(t) = \sum_{k=1}^{n} (-1)^{n-k} \frac{k^n}{k!(n-k)!} f_k(t). \tag{33}$$

Numerical Examples

(Reappeared) settings of extended model

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T^*}, \mathbb{Q})$ be a filtered probability space, where T^* is some time horizon, and $\mathbb{Q}$ is a risk neutral probability measure.

Settings of Extended Model

- $V_0 := S_0 + B$ is the initial asset value, where S_0 is the initial stock price, and B is the firm's debt. In this model, B is identified with the default barrier for simplicity.
- The firm's asset value follows $V_t = V_0 e^{X_t}$, where X_t is a Lévy process.
- The time τ of the default on time interval $(0, T]$ is defined as

$$\tau = \inf\{t \in (0, T] : V_t \le B\}, \tag{34}$$

- We define the dynamics of the equity price as

$$S_t = \begin{cases} (V_t - B)e^{\int_0^t (r_s - d_s)ds} = (V_0 e^{X_t} - B)e^{\int_0^t (r_s - d_s)ds} & \text{if} \quad t < \tau, \\ 0 & \text{otherwise,} \end{cases} \tag{35}$$

where r_t is a *deterministic* risk-free interest rate, and d_t is a *deterministic* dividend yield of the firm's equity.

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where r_t is a *deterministic* risk-free interest rate, and d_t is a *deterministic* dividend yield of the firm's equity.

Models for numerical examples

Models

- Exponential Jump (EJCG): $\psi_Z(\theta) = \frac{1}{2}\sigma^2\theta^2 + \lambda\left(1 - \frac{a}{a+i\theta}\right)$,
- Gamma Jump (GJCG): $\psi_Z(\theta) = \frac{1}{2}\sigma^2\theta^2 + \lambda \log\left(1 + \frac{i\theta}{a}\right)$,
- Inverse Gaussian Jump (IGJCG): $\psi_Z(\theta) = \frac{1}{2}\sigma^2\theta^2 + \lambda\left(\sqrt{a^2 + 2i\theta} - a\right)$,

$$
\begin{aligned}
X_t &= \mu t + \sigma W_t + \bar{Z}_t =: \mu t + Z_t, \\
\mu &= \psi_Z(-i), \quad \text{(Condition for Martingale property)}
\end{aligned}
$$

- In the case where $\lambda = 0$, these models is identical to standard CreditGrades model, i.e. $\bar{Z}_t = 0$.

Parameters

Parameters

Using parameters are as follows:

$$\begin{aligned} \sigma &= 0.2, \quad \lambda = 0.00,\ 0.25,\ 0.50,\ 1.00, \\ a &= 10(\text{EJCG}),\ 8(\text{GJCG}),\ 4(\text{IGJCG}), \\ S_0 &= 100,\ B = 100,\ r_t = d_t = 0,\ \text{for all } t \geq 0. \end{aligned}$$

- The greater λ is, the more jump occurs.
- The leverage ratio in each model equals to $B/S_0 = 1.0$.

Numerical Examples of option implied volatility by EJCG

Figure: Implied Volatilities on the 3-Month Options by EJCG

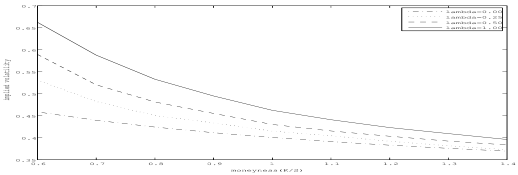
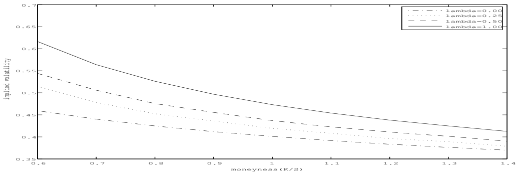


Figure: Implied Volatilities on the 6-Month Options by EJCG



Numerical Examples of option implied volatility by GJCG

Figure: Implied Volatilities on the 3-Month Options by GJCG

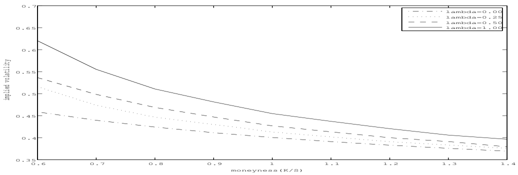
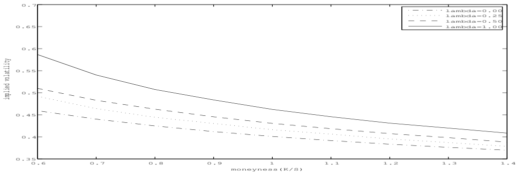


Figure: Implied Volatilities on the 6-Month Options by GJCG



Numerical Examples of option implied volatility by IGJCG

Figure: Implied Volatilities on the 3-Month Options by IGJCG

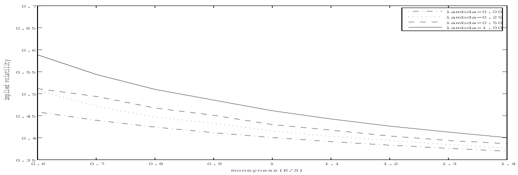
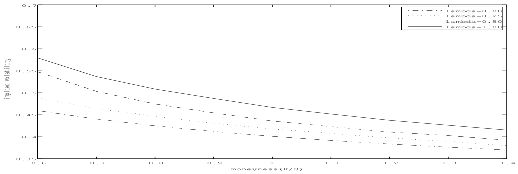


Figure: Implied Volatilities on the 6-Month Options by IGJCG



Interpretation for numerical result: option implied volatility

- (Remind) The standard model can describe the implied equity volatility skew naturally. However, it is not able to reflect unpredictable credit events into the implied volatility, since the local volatility function σ_t^S depends only on the leverage ratio of the firm's debt.
- Different volatility skew curves of the firms can be obtained using different types of our extended models with suitable parameters, even if the leverage ratios are the same values.

Numerical Examples: CDS Premium

Table: CDS per Premiums (bp)

Model	λ	1-Year	2-Year	3-Year	4-Year	5-Year
Standard	0.00	6	59	126	185	209
EJCG	0.25	24	95	169	221	252
	0.50	45	136	210	261	293
	1.00	96	212	289	331	347
GJCG	0.25	25	92	153	190	209
	0.50	42	119	181	217	234
	1.00	79	175	236	268	278
IGJCG	0.25	22	91	152	191	210
	0.50	37	118	182	219	236
	1.00	71	172	239	273	283

Interpretation for numerical result: CDS premium

- (Remind) In the Standard model, the default events are predictable, since the model assumes a continuous dynamics. As a result, if the current value of the firm is remote from the barrier, both the default probability and the credit spread in short-term are close to zero.
- Our Extended models are able to generate higher short-term spreads without a stochastic default barrier.

Conclusion

Conclusion

- Reviewing the basics.
- Reviewing the CreditGrades model: features and drawbacks.
- Introducing Extended model.
- Confirming the features of Extended Model by Numerical Examples.
- Future interest: calibrating real market data and doing empirical analysis.

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Thank you for your kind attention!!