On the design of catastrophe bonds

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Cat bonds

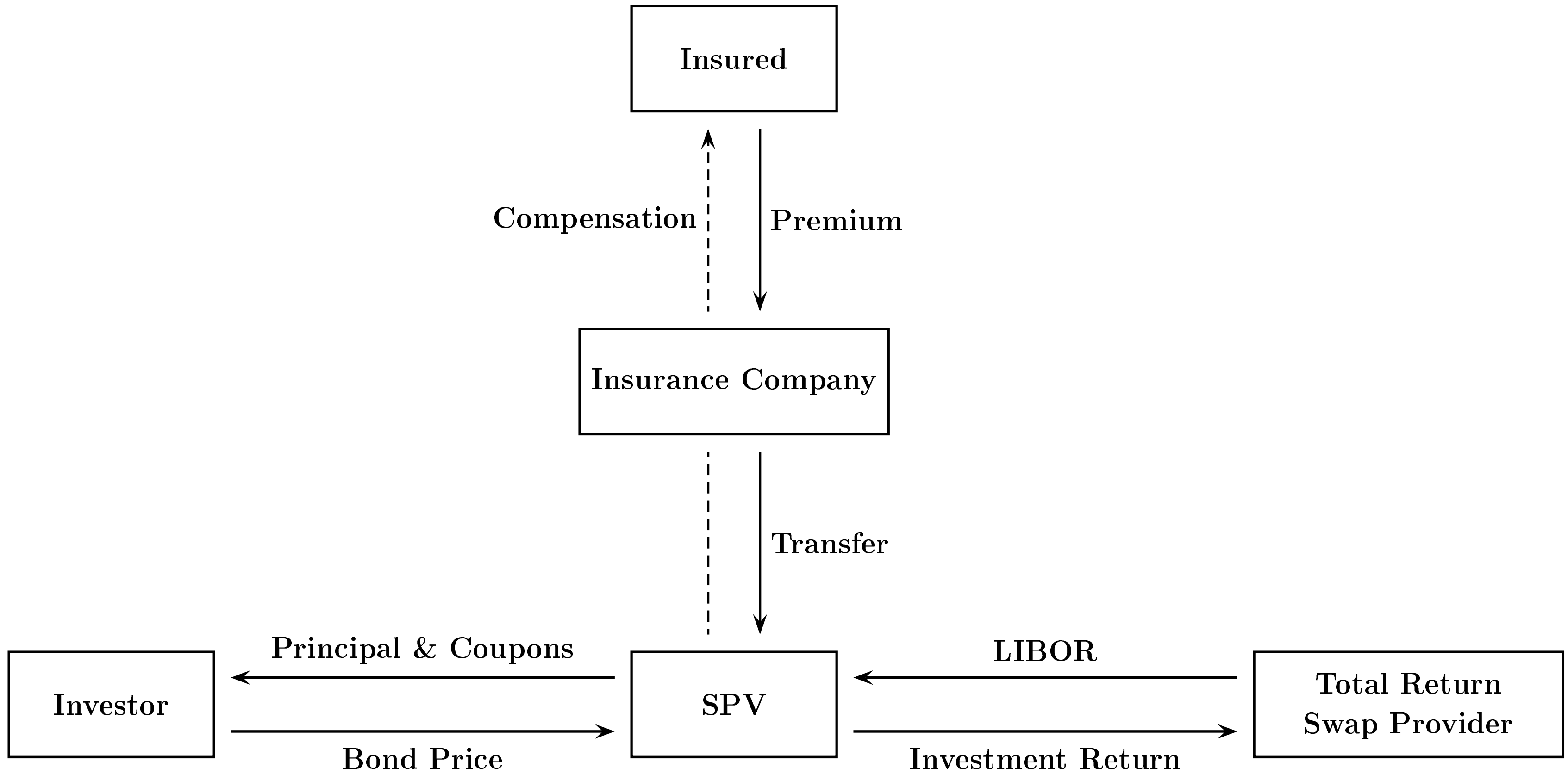
- Catastrophe (cat) bonds: popular securitization products linked to catastrophic insurance risks
- Successive disasters in 1990's (e.g., Hurricane Andrew) had threatened the capacity of the traditional reinsurance market.

$\Longrightarrow$ The issues of cat bonds started.

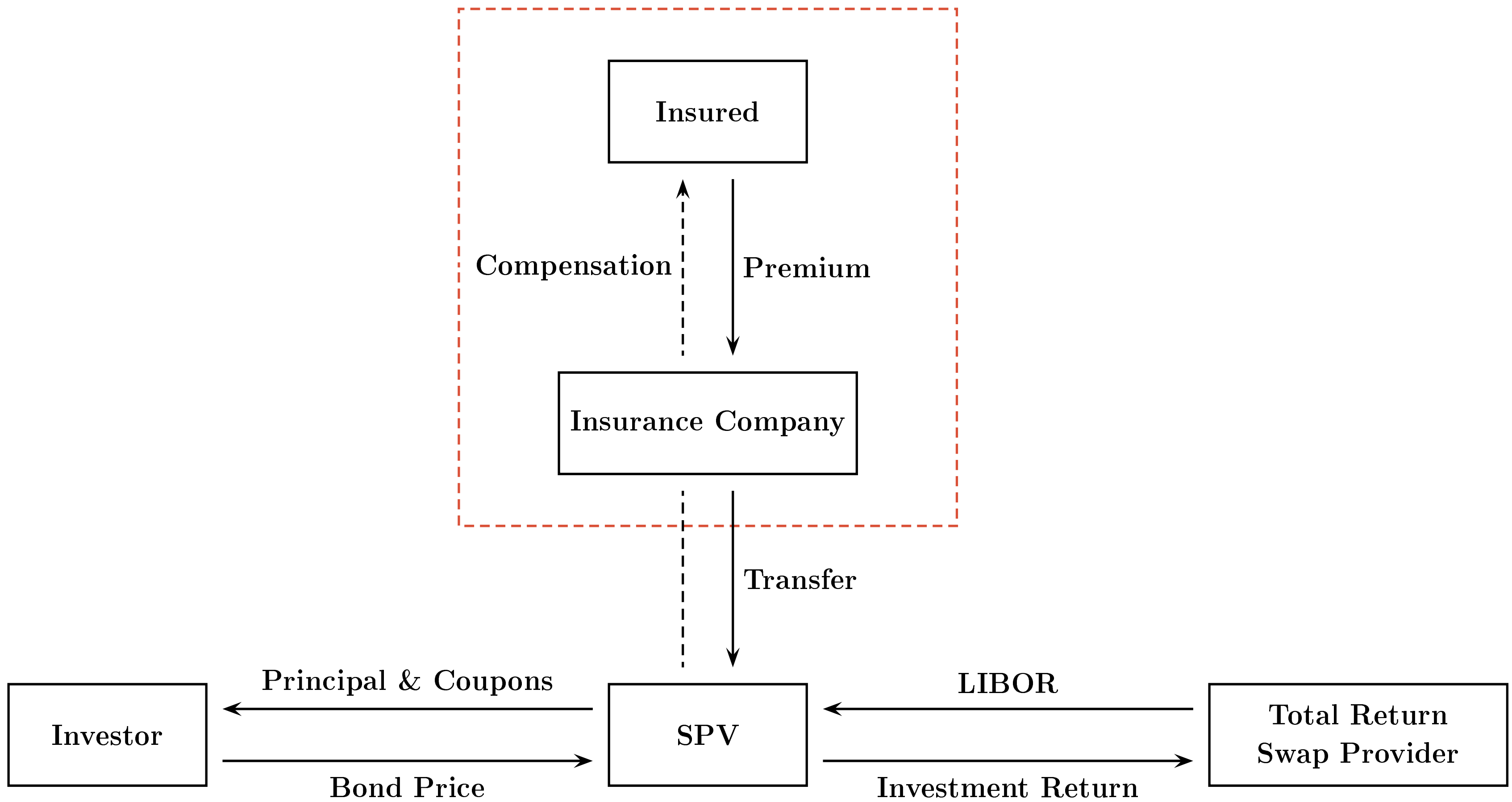
Cat bond structure:

- Principal is reduced if pre-defined catastrophic events occur before a maturity.
- Coupon rate: LIBOR + constant spread

Securitization of insurance risk: general structure

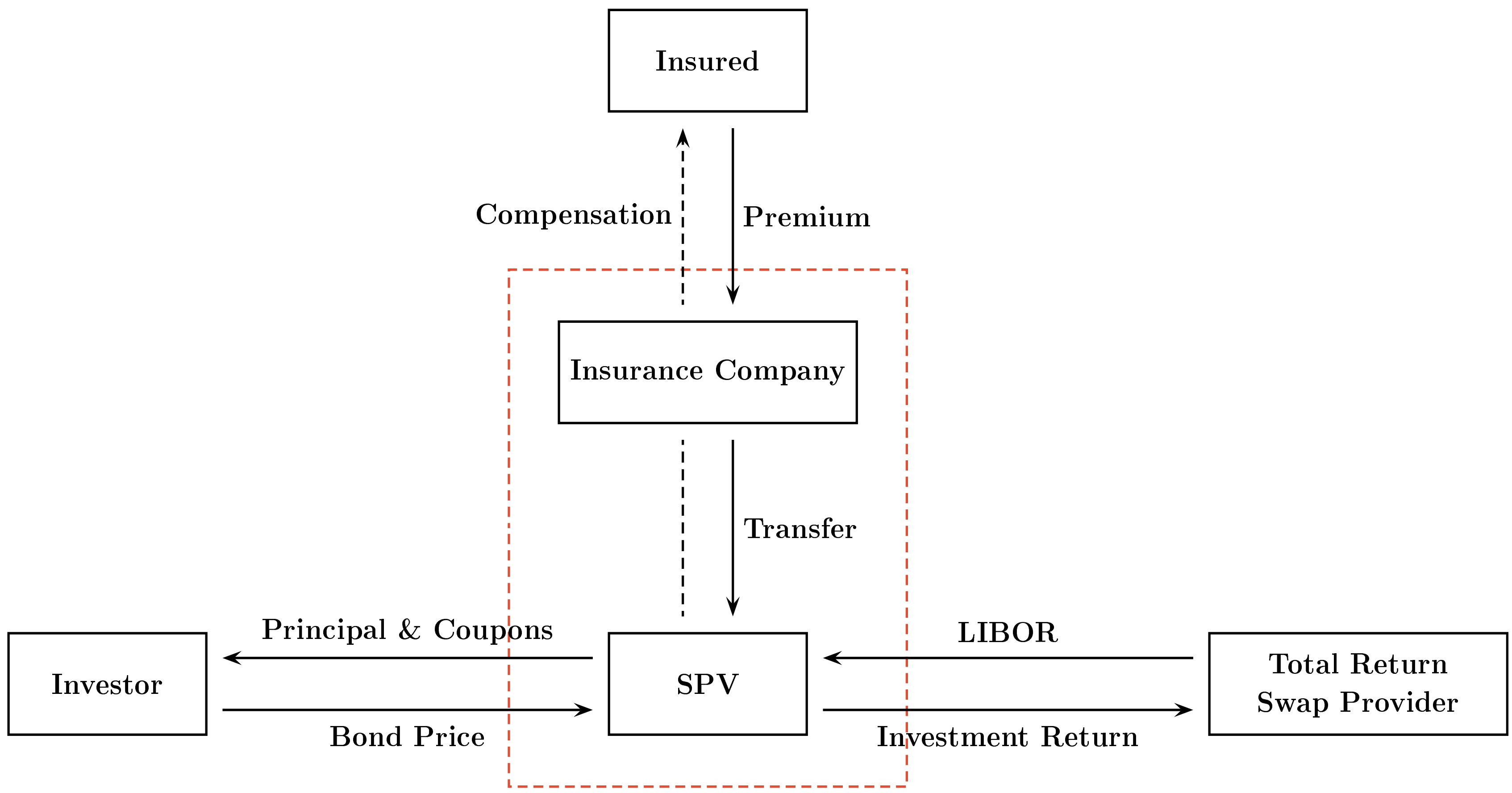


Securitization of insurance risk: general structure



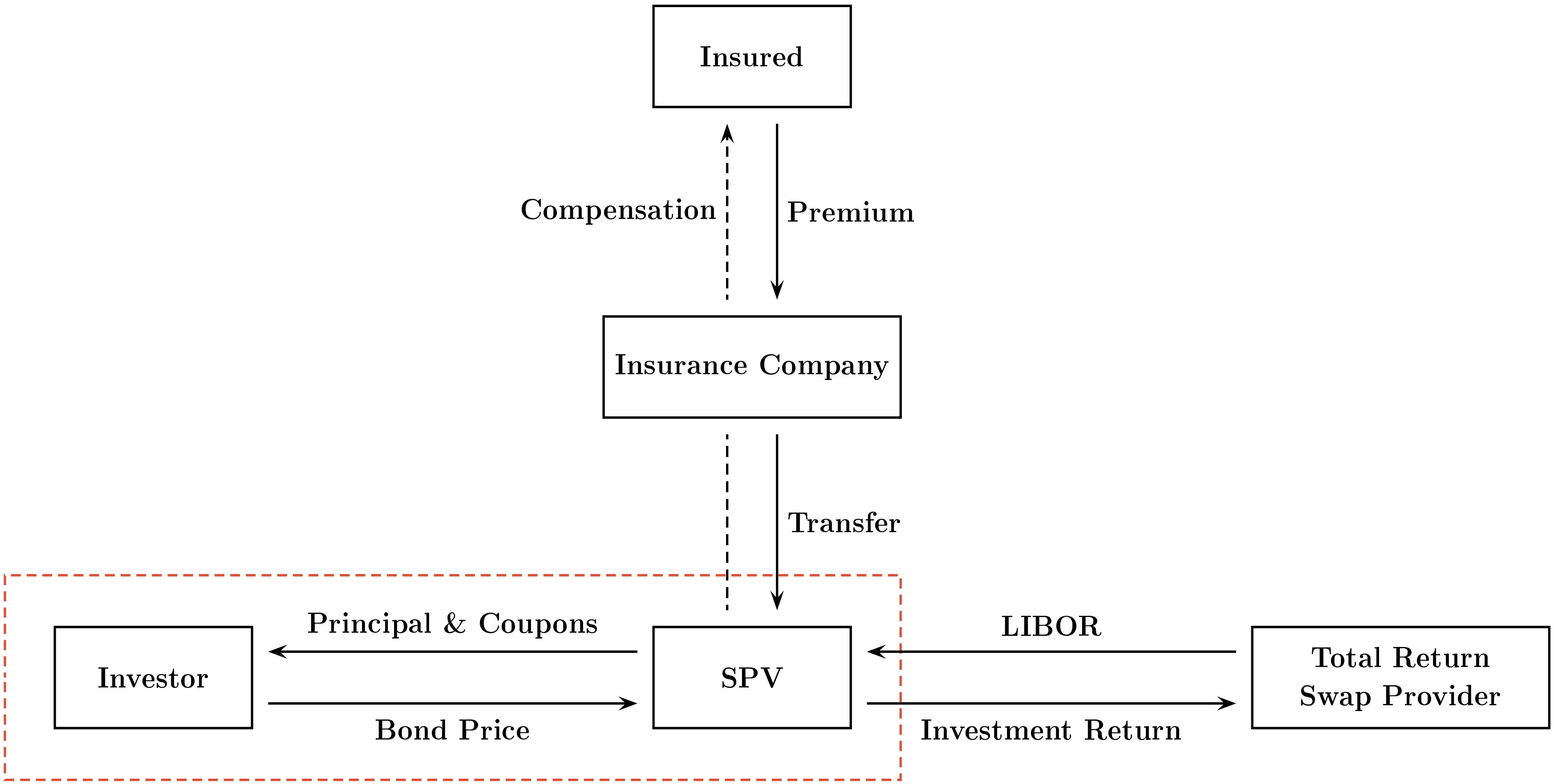
- Insurance contract between a firm (insured) and an insurance company (insurer)

Securitization of insurance risk: general structure



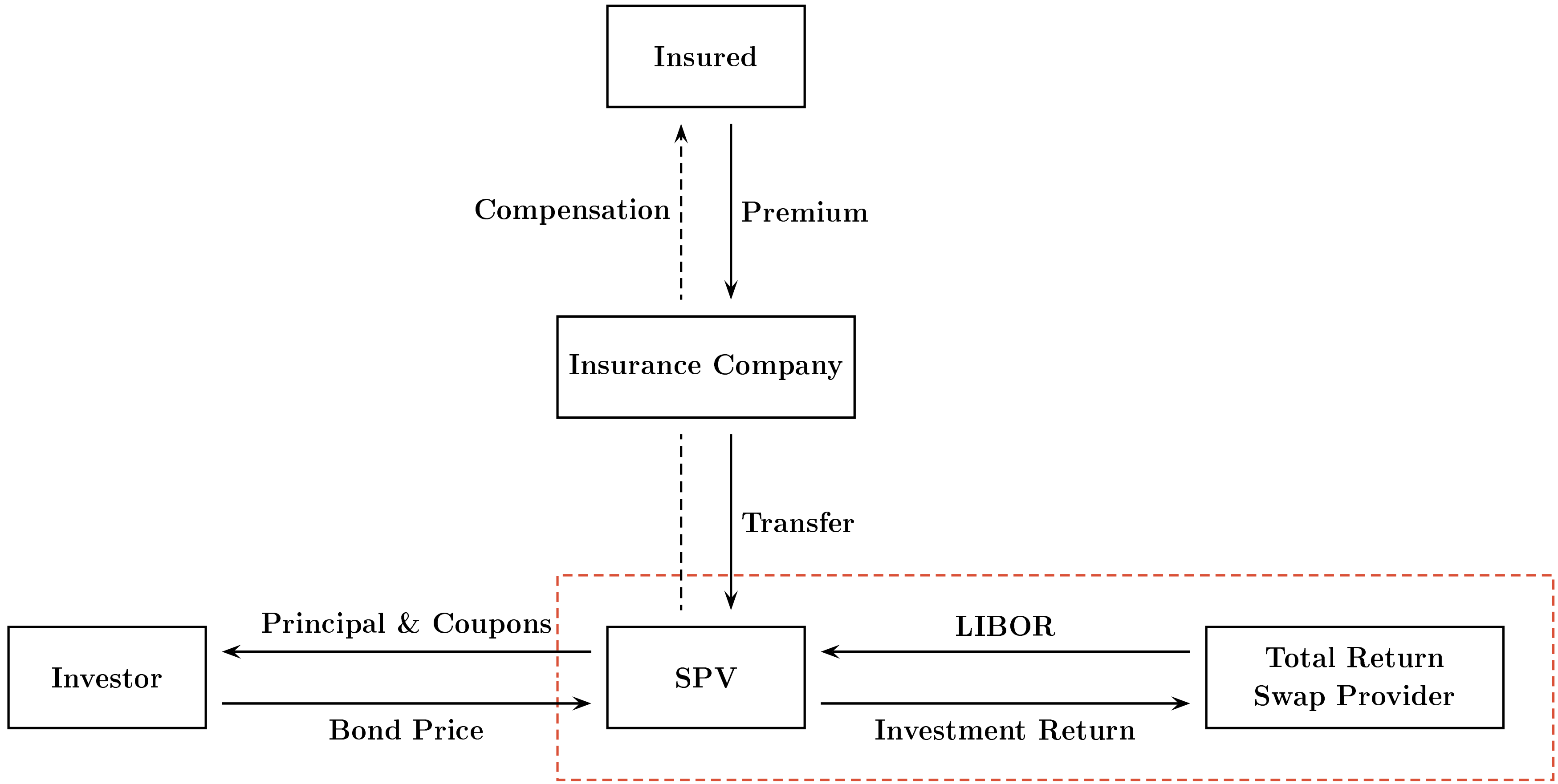
- The insurance company creates a special purpose vehicle (SPV) for the securitization of the insurance risk.

Securitization of insurance risk: general structure



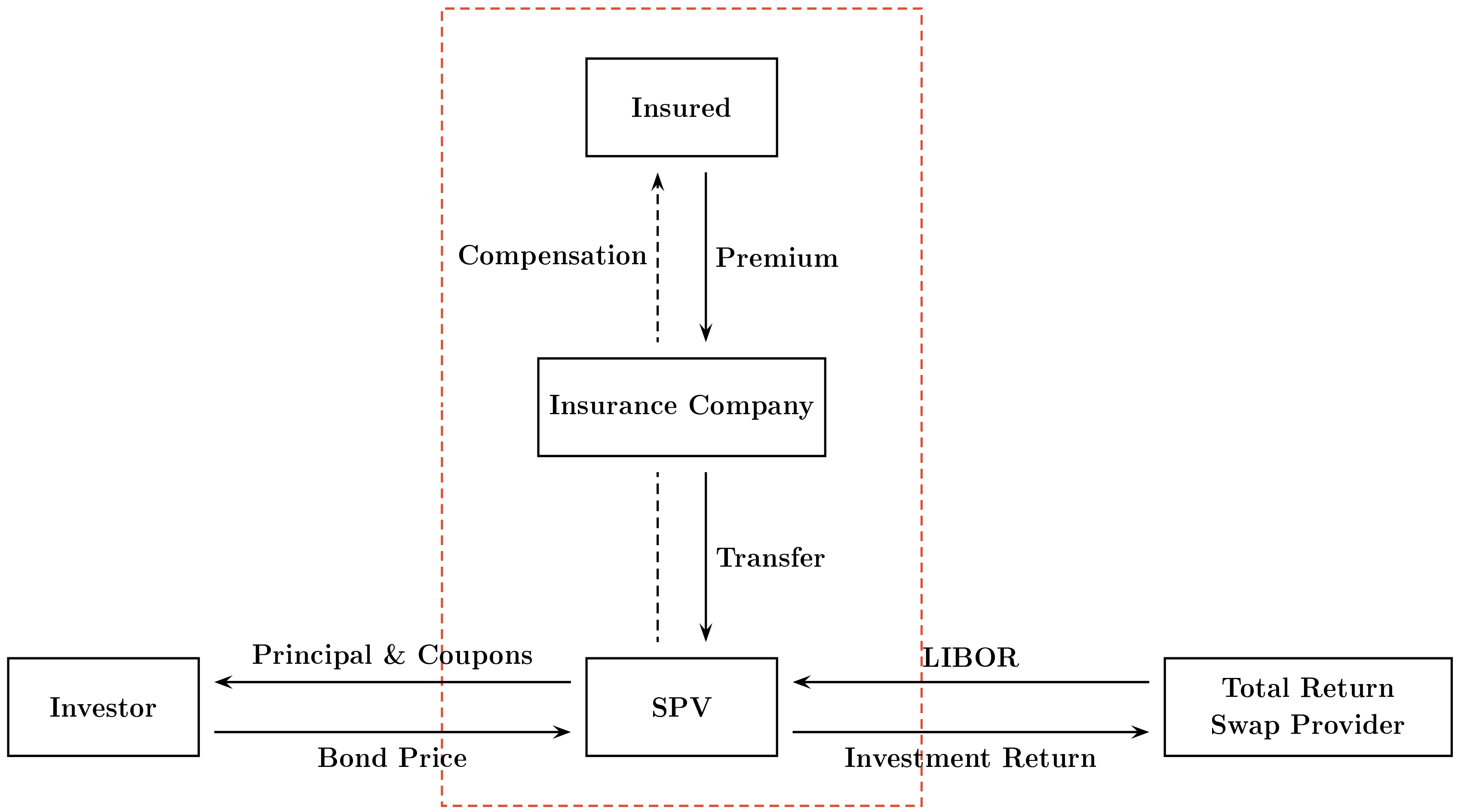
- SPV issues cat bonds to investors to cover the contingent payout.
- The bond makes coupon payments to investors of LIBOR plus constant spreads.

Securitization of insurance risk: general structure



- SPV enters into a total-return swap with a highly rated counterparty to get LIBOR-based cash flows.

Securitization of insurance risk: general structure



- SPV makes the contingent payout to the insured after the triggering event.

- Insured: East Japan Railway Company
- Insurance company: Munich Re
- Triggering event: Earthquake in Tokyo
- Issue amount: USD 260 million
- Bond period: 2007/10 – 2012/10
- Specified area: A and $B \setminus A$, where

$$A = \{x \in \mathbb{R}^2 : |x - \{\text{Tokyo Station}\}| \leq 40\text{km}\},$$
$$B = \{x \in \mathbb{R}^2 : |x - \{\text{Tokyo Station}\}| \leq 70\text{km}\}.$$

- Coupon rate: 3 month LIBOR + 275bp
- Amount paid at maturity: principal$\times$ $(1 -$ reduction rate$)$

Reduction Rate

Magnitude	A	$B \setminus A$
≥ 7.7	1.000	1.000
7.6	1.000	0.750
7.5	1.000	0.500
7.4	1.000	0.375
7.3	1.000	0.250
7.2	0.750	0.125
7.1	0.500	0.000
7.0	0.250	0.000
≤ 6.9	0.000	0.000

Optimal design problem

- A natural and interesting question is how to determine the constant spread in the coupon payments of a cat bond.
- However, little attempts have been made to the mathematical analysis of this problem except Barrieu–El Karoui (2002).

Barrieu–El Karoui's approach

- The whole structure of the securitization is considered:
  - Optimal insurance constract: insured vs insurance company
  - Optimal price and coupon: insurance company vs investor

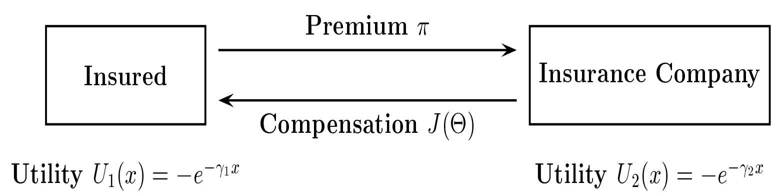
Insurance risk

- Loss: $\Theta = M \sum_{i=1}^{n} 1_{\varepsilon_i} \beta_i$
    - n (year): the maturity
    - β_i: a capitalization factor of year i to n. E.g., $\beta_i = (1+r)^{n-i}$
    - ε_i: a triggering event
    - M: loss upon the triggering events

Assumption

- β_i: deterministic
- No investment in a financial market

Problem 1: Optimal reinsurance agreement



- Agents: an insured (utility: $U_1(x) = -e^{-\gamma_1 x}$) and an insurance company (utility: $U_2(x) = -e^{-\gamma_2 x}$)
- Premium: π
- Compensation: $J(\Theta)$ such that $0 \leq J(\Theta) \leq \Theta$

Resulting wealths at maturity (year n)

- Insured: $-\pi\beta_0 - \Theta + J(\Theta)$
- Insurance company: $\pi\beta_0 - J(\Theta)$

Optimal π and J ?

- Insurance company will accept this deal if

$$\mathbb{E}[U_2(\pi\beta_0 - J(\Theta))] \geq \mathbb{E}[U_2(0)] = -1.$$

- Thus, the insured designs this contract so as to

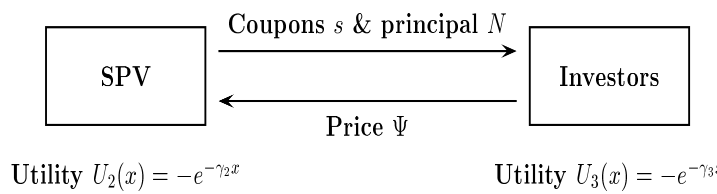
$$\begin{aligned} &\underset{\pi,\ J}{\text{maximize}} \quad \mathbb{E}[U_1(-\pi\beta_0 - \Theta + J(\Theta))] \\ &\text{under} \quad \mathbb{E}[U_2(\pi\beta_0 - J(\Theta))] \geq -1. \end{aligned}$$

- Optimal π and J can be obtained by the classical variational method.

Barrieu–El Karoui's approach

Problem 2: Optimal design of a cat bond

- Insurance company decides to issue a cat bond to manage the risk w.r.t. the reinsurance contract.



- Actual principal: $N - (\alpha/M)\Theta$
- Price: Φ
- Coupon: s

Resulting wealths at year n

- SPV (insurance company): $\pi\beta_0 - J(\Theta) + \Phi\beta_0 - s\sum_{i=1}^{n}\beta_i - N + \dfrac{\alpha}{M}\Theta$
- Investor: $-\Phi\beta_0 + s\sum_{i=1}^{n}\beta_i + N - \dfrac{\alpha}{M}\Theta$

Optimal Φ, α and s ?

- Investor will accept this deal if

$$\mathbb{E}U_3\left(-\Phi\beta_0 + s\sum_{i=1}^{n}\beta_i + N - \frac{\alpha}{M}\Theta\right) \geq \mathbb{E}[U_3(0)] = -1$$

where $U_3(x) = -e^{-\gamma_3 x}$ is the utility function of the investor.

- Thus, SPV designs this bond so as to

$$\begin{aligned}
\underset{\Phi,\,\alpha,\,s}{\text{maximize}} \quad & \mathbb{E}U_2\left(\pi\beta_0 - J(\Theta) + \Phi\beta_0 - s\sum_{i=1}^{n}\beta_i - N + \frac{\alpha}{M}\Theta\right) \\
\text{under} \quad & \mathbb{E}U_3\left(-\Phi\beta_0 + s\sum_{i=1}^{n}\beta_i + N - \frac{\alpha}{M}\Theta\right) \geq -1.
\end{aligned}$$

- Optimal Φ, α and s can be obtained by the classical variational method.

Our formulation allows

- both agents to invest in a dynamic financial market;
- stochastic interest rates.

Fixed principal: $H := F1_{\{\tau>T\}} + Ff(Z)1_{\{\tau\leq T\}}$.

Here,

- $(\Omega, \mathcal{G}, \mathbb{P})$: prob.sp.
- $T \in (0, \infty)$: the maturity
- τ: the random time of the occurrence of a predefined cat event
- Z: an index related to the cat event, with values in a Polish space K.
- $f : K \to [0, 1]$: a reduction rule of the principal, Borel measurable
- τ and Z are assumed to be mutually independent under $\mathbb{P}$.

Example: the case of a single earthquake disaster.

- The trigger $Z(\omega)$ can be captured by a 3-dimensional random variable $Z = (Z_1, Z_2, Z_3)$.
- (Z_1, Z_2): the focus of the targeted earthquake
- Z_3: the corresponding magnitude
- $K = \mathbb{R}^3$
- $f(Z) = \sum_{j=1}^{m} 1_{C_j}(Z_1, Z_2) g_j(Z_3)$
  - g_j: a nonincreasing function for each j
  - $C_1, \ldots, C_m \subset \mathbb{R}^2$: a partition of the predefined area.

Formulation

Available information for agents:

$$\text{(Bond) market information} + \text{cat information}$$

Market setup:

- $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$: a filtration with usual conditions
  - the filtration $\mathbb{F}$ is the information structure used for investing interest rate instruments.
- $\{r_t\}_{t\geq 0}$: a short rate process, $\mathbb{F}$-adapted
- T: the terminal time of the market.
- $\{B_t\}_{0\leq t\leq T}$: the money market account process, $B_t = \exp(\int_0^t r_s ds)$
- $X_t = (X_t^1, \ldots, X_t^d)$, $0 \leq t \leq T$: the price of the d-risky assets, $\mathbb{F}$-semimartingale with continuous paths

- $\{\beta_t\}_{0\le t\le T}$: the discount process, $\beta_t = 1/B_t$
- $\tilde{X}_t^i := \beta_t X_t^i$, $0 \le t \le T$: the discounted price process
- Assume that there exists a unique $\mathbb{Q} \sim \mathbb{P}$ such that $\{\tilde{X}_t^i\}_{0\le t\le T}$ is a $(\mathbb{Q}, \mathbb{F})$-martingale for $i = 1, \dots, d$.
- Assume that $\mathbb{E}_{\mathbb{Q}}[(\beta_T)^2] < \infty$ and $\sum_{i=1}^{d} \mathbb{E}_{\mathbb{Q}}[(\tilde{X}_T^i)^2] < \infty$.
- Then our market is complete.
    - This statement is equivalent to that the process $\tilde{X}_t = (\tilde{X}_1, \dots, \tilde{X}_t^d)$, $0 \le t \le T$, has the martingale representation property.

Setup for cat information:

- $\mathbb{H} = \{\mathcal{H}_t\}_{t\geq 0}$: the catastrophe information
    - $\mathcal{H}_t := \sigma(N_u^\Lambda : u \leq t,\ \Lambda \in \mathcal{B}(K))$, $t \geq 0$
    - $N_t^\Lambda := 1_{\{Z\in\Lambda\}}1_{\{\tau\leq t\}}$, $t \geq 0$, $\Lambda \in \mathcal{B}(K)$.
- $\mathbb{G}$: the full filtration, $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$
- Assume that $\mathbb{F}$ is independent of τ and Z.
- Note that the $\{\tilde{X}_t\}_{0\leq t\leq T}$ is a $(\mathbb{Q}, \mathbb{G})$-martingale.

On trading strategies:

- $\mathbb{G}$: the available information for the market participants
- $(\phi_t^0, \phi_t^1, \dots, \phi_t^d)$, $0 \le t \le T$: trading strategy, $\mathbb{G}$-predictable process s.t.

$$\int_0^T |\phi_t^0| dB_t < \infty, \quad \int_0^T (\phi_t^i)^2 d[X^i, X^i]_t < \infty, \text{ a.s.,} \quad i = 1, \dots, d.$$

- ϕ_t^0: the number of shares of the money market account held by an agent at time t.
- ϕ_t^i: the number of shares of i-th risky asset held by the agent at time t for $i = 1, \dots, d$.

The resulting wealth V_t of the agent at time t is then given by

$$V_t = \phi_t^0 B_t + \sum_{i=1}^d \phi_t^i X_t^i.$$

- $\Gamma = \{\Gamma_t\}_{0 \le t \le T}$: the cumulative wealth received or consumed by the agent on $(0, t]$, a finite variation, càdlàg, $\mathbb{G}$-adapted process with $\Gamma_0 = 0$

If the agent is financed by only the initial wealth V_0 and Γ, then the process V_t is formally described by

$$dV_t = d\Gamma_t + \phi_t^0 dB_t + \sum_{i=1}^{d} \phi_t^i dX_t^i.$$

- $\tilde{V}_t := \beta_t V_t$, the discounted wealth

By the product formula,

$$\tilde{V}_t = V_0 + \sum_{i=1}^{d} \int_0^t \phi_s^i d\tilde{X}_s^i + \int_0^t \beta_s d\Gamma_s, \quad 0 \le t \le T.$$

Formulation

The trading strategies $\phi = \{(\phi_t^1, \ldots, \phi_t^d)\}_{0 \le t \le T}$ are restricted to the class of processes such that

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T (\phi_t^i)^2 d[\tilde{X}^i, \tilde{X}^i]_t\right] < \infty, \quad i = 1, \ldots, d.$$

- $\Gamma = \{\Gamma_t\}_{0 \le t \le T}$: cumulative income process, a finite variation, càdlàg, $\mathbb{G}$-adapted process with $\Gamma_0 = 0$ such that $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \beta_t d|\Gamma|_t\right] < \infty$, where $\{|\Gamma|_t\}_{0 \le t \le T}$ is the total variation process of Γ.
- $V_t = V_t^{v,\phi,\Gamma}$, $0 \le t \le T$, denotes the wealth process for an initial wealth v, a trading strategy ϕ and a cumulative income process Γ.

- We identify cumulative coupon payments on $(0, t]$ with Γ_t such that Γ is a cumulative income process with increasing paths.
- The originator issues the bond with price p and cumulative coupon process Γ.

Resulting discounted wealths at T

- the issuer: $p + \tilde{V}_T^{v,\phi,-\Gamma} - \beta_T H$
  - v: a given initial wealth of the issuer
  - ϕ: a trading strategy of the issuer
- the investor: $-p + \tilde{V}_T^{v',\phi',\Gamma} + \beta_T H$
  - v': a given initial wealth of the investor
  - ϕ': a trading strategy of the investor

- The issuer is willing to sell the bond if $p + \tilde{V}_T^{v,\phi,-\Gamma} - \beta_T H \geq v$ a.s.
- The investor will be interested in this deal if s/he can find ϕ' such that $-p + \tilde{V}_T^{v',\phi',\Gamma} + \beta_T H \geq v'$ a.s.

Problem

Minimize the bond price p subject to

- the issuer's constraint: $p + \tilde{V}_T^{v,\phi,-\Gamma} - \beta_T H \geq v$, a.s. for some trading strategy ϕ and cumulative coupon process Γ;
- the investor's constraint: $-p + \tilde{V}_T^{v',\phi',\Gamma} + \beta_T H \geq v'$, a.s. for some trading strategy ϕ'.

- Our formulation does not rely on utility functions.

Reduction to super-hedging problem

If p and Γ satisfy the constraints of the transaction, then

$$-\int_0^T \phi_t d\tilde{X}_t \leq p - \int_0^T \beta_t d\Gamma_t - \beta_T H \leq \int_0^T \phi'_t d\tilde{X}_t \text{ a.s.,}$$

so that $\int_0^T (\phi_t + \phi'_t) d\tilde{X}_t \geq 0$ a.s. $\quad \therefore \phi' = -\phi$.

Thus our original problem is reduced to the following minimization problem:

$$\hat{p} := \inf \left\{ p \in \mathbb{R} : p + \int_0^T \phi_t d\tilde{X}_t - \int_0^T \beta_t d\Gamma_t = \beta_T H \text{ for some } \phi, \Gamma \right\}.$$

Here ϕ and Γ range over all trading strategies and cumulative coupon processes respectively.

- Assume that $\mathbb{P}(\tau > t) > 0$ for $t \geq 0$ and that $t \mapsto \log \mathbb{P}(\tau > t)$ is differentiable.
- Denote by γ the hazard rate function of τ: $\mathbb{P}(\tau > t) = e^{-\int_0^t \gamma(s)ds}$.
- Let $\mu(dt \times dz)$ be the random measure on $((0,\infty) \times K, \mathcal{B}((0,\infty) \times K))$ determined uniquely by $\mu((0,t] \times \Lambda) = N_t^\Lambda$.
- The $\mathbb{G}$-predictable process

$$\lambda(t,\Lambda) = 1_{\{t \leq \tau\}} \gamma(t) \mathbb{P}(Z \in \Lambda), \quad 0 \leq t \leq T, \quad \Lambda \in \mathcal{B}(K),$$

gives the intensity kernel of μ with respect to $\mathbb{Q}$ and $\mathbb{G}$.

- Denote by $\mathcal{D}$ the set of all bounded $\mathbb{G}$-predictable process $\{\kappa_t\}_{0\le t\le T}$ such that $\kappa_t > -1$, $0 \le t \le T$, a.s.
- For $\kappa \in \mathcal{D}$, the process

$$Z_t^\kappa = (1 + \kappa_\tau 1_{\{\tau \le t\}}) \exp\left(-\int_0^t 1_{\{s\le\tau\}}\kappa_s\gamma(s)ds\right), \quad 0 \le t \le T,$$

  is a positive $(\mathbb{Q}, \mathbb{G})$-martingale.
- For $\kappa \in \mathcal{D}$, define the probability measure $\mathbb{Q}^\kappa$ by $d\mathbb{Q}^\kappa/d\mathbb{Q} = Z_T^\kappa$.
- Under $\mathbb{Q}^\kappa$ the intensity kernel for $\mu(dt \times dz)$ is given by $(1+\kappa_t)\lambda(t, dz)$.
- Since $[\tilde{X}, Z^\kappa] = 0$, $\mathbb{E}_{\mathbb{Q}}[Z_t^\kappa \tilde{X}_t|\mathcal{G}_s] = Z_s^\kappa \tilde{X}_s$ for $s \le t$.
- Hence $\{\tilde{X}_t\}$ is a $(\mathbb{Q}^\kappa, \mathbb{G})$-martingale.
- Thus, $\{\mathbb{Q}^\kappa : \kappa \in \mathcal{D}\}$ is a class of the equivalent martingale measures.

Let $\{U_t\}_{0\le t\le T}$ be the right-continuous version of

$$U_t = \operatorname*{ess\,sup}_{\kappa\in\mathcal{D}} \mathbb{E}_{\mathbb{Q}^\kappa}[\beta_T H \,|\, \mathcal{G}_t], \quad 0 \le t \le T.$$

Proposition

If there exist ϕ and Γ such

$$U_t = U_0 + \int_0^t \phi_s d\tilde{X}_s - \int_0^t \beta_s d\Gamma_s, \quad 0 \le t \le T,$$

then U_0 is an optimal price and Γ is an optimal cumulative coupon process.

- This representation is called an optional decomposition of $\{U_t\}_{0\le t\le T}$.
- El Karoui–Quentz (1995): Brownian models
- Kramkov (1996) and Föllmer–Kabanov (1998): general semimartingale models

Notation

- $\{Y_t\}_{0\leq t\leq T}$: the continuous version of $Y_t = \mathbb{E}_{\mathbb{Q}}[\beta_T|\mathcal{G}_t] = \mathbb{E}_{\mathbb{Q}}[\beta_T|\mathcal{F}_t]$.
- $\{D_t^T\}_{0\leq t\leq T}$: the price process of the zero-coupon bond with maturity T, given by
$$D_t^T = B_t Y_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \,\middle|\, \mathcal{F}_t \right].$$
- Since the market without cat information is assumed to be complete, there exists a trading strategy $\{\varphi_t\}_{0\leq t\leq T}$ such that
$$Y_t = Y_0 + \int_0^t \varphi_s d\tilde{X}_s, \quad 0 \leq t \leq T.$$
- Define the trading strategy $\{\hat{\varphi}_t\}$ by
$$\hat{\varphi}_t = F\varphi_t \left(1_{\{t\leq\tau\}} + f(Z)1_{\{t>\tau\}} \right), \quad 0 \leq t \leq T.$$

Solution

Theorem

The process $\{U_t\}_{0\leq t\leq T}$ is represented as

$$U_t = FY_t\left(1_{\{\tau>t\}} + f(Z)1_{\{\tau\leq t\}}\right), \quad 0\leq t\leq T.$$

Moreover, the optional decomposition of $\{U_t\}_{0\leq t\leq T}$ is given by

$$U_t = U_0 + \int_0^t \hat{\varphi}_s d\tilde{X}_s - F\int_0^t\int_K (1-f(z))Y_s\mu(ds\times dz), \quad 0\leq t\leq T.$$

Thus the optimal price $\hat{p}$ and the cumulative coupon process $\{\hat{\Gamma}_t\}_{0\leq t\leq T}$ are given respectively by

$$\hat{p} = FD_0^T, \quad \hat{\Gamma}_t = F\int_0^t\int_K (1-f(z))D_s^T\mu(ds\times dz).$$

Outline of the proof

- $L_t := (1 - f(Z))1_{\{\tau \le t\}}$.
- $U_t = FY_t - F \operatorname{ess\,inf}_{\kappa \in \mathcal{D}} \mathbb{E}_{\mathbb{Q}^\kappa}[Y_T L_T | \mathcal{G}_t]$.

Then,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^\kappa}[Y_T L_T | \mathcal{G}_t] &= Y_t L_t + \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_t^T L_{s-} dY_s + \int_t^T Y_s dL_s \,\middle|\, \mathcal{G}_t\right] \\ &\to Y_t L_t,\end{aligned}$$

as $\kappa_t = \kappa \searrow -1$. From this, we have $U_t = FY_t(1 - L_t)$.

- Consider a LIBOR-based coupon payments
- The discounted value of this cash flow stream:

$$\sum_{j=1}^{n} \delta F \left(L(T_{j-1}) + s \right) \beta_{T_j} 1_{\{T_j < \tau\}},$$

  - $0 = T_0 < T_1 < \cdots < T_n = T$ are dates of coupon payments with constant fraction $\delta = T_i - T_{i-1}$
  - $L(T_{j-1}) = \frac{1 - D_{T_{j-1}}^{T_j}}{\delta D_{T_{j-1}}^{T_j}}$, $j = 1, \ldots, n$, is the discretely compounded LIBOR rate prevailing at T_{j-1} over the period from T_{j-1} to T_j.

- The issuer provides this cash flow stream to the investor by entering into a swap contract.

The spread s must satisfy

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \beta_t d\hat{\Gamma}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\sum_{j=1}^{n} \delta F\left(L(T_{j-1}) + s\right) \beta_{T_j} 1_{\{T_j < \tau\}}\right]$$

if $\mathbb{Q}$ is the valuation measure.

It is straightforward to see that the spread s is given by

$$s = \frac{(1 - \mathbb{E}[f(Z)])\mathbb{E}_{\mathbb{Q}}[\beta_T]\mathbb{P}(\tau \leq T) - \sum_{j=1}^{n} \mathbb{E}_{\mathbb{Q}}[\beta_{T_{j-1}} - \beta_{T_j}]\mathbb{P}(T_j < \tau)}{\delta \sum_{j=1}^{n} \mathbb{E}_{\mathbb{Q}}[\beta_{T_j}]\mathbb{P}(T_j < \tau)}.$$