On an extension of an algorithm of higher-order weak approximation to SDEs

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Outline

Abstract

Background

- The weak approximation problem
- Simulation – MC and QMC
- Two important remarks
- Our Higher-order scheme
- Free Lie Algebra

Main result

- $\mathcal{L}_{\mathbb{R}}((A))$-valued random variables
- Approximation theorem
- The Algorithms

The Runge–Kutta method

Numerical Example

Recent developments

Appendix/Backup

Background:

- A new higher-order weak approximation scheme based on [Kusuoka '01] and [Lyons and Victoir '04]

Objective:

- Construction of Concrete higher-order weak approximation algorithms that are:
  - Versatile, (applicable to a broad class of SDEs)
  - Easy to use, (Blackbox algorithm)

Current status:

- Two kinds of schemes of order 2 (say Alg 1 [Victoir &N '08] and Alg 2 [Ninomiya &N '09]) Both work in practice.
- General extrapolation method for Alg 1 [Oshima, Teichmann and Veluscek '09]
  Enables arbitraly order weak approximation

This talk is on:

- Construction of concrete Alg 2 algorithm of order > 2.
- The state is under construction.

References 1/2:

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- Lyons, T., Victoir, N.: *"Cubature on Wiener Space."* Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **460**, 169–198 (2004)

References 2/2:

- N. and Victoir, N.: *"Weak Approximation of Stochastic Differential Equations and Application to Derivative Pricing."* Applied Mathematical Finance **15**, 107–121 (2008)
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- Oshima, K., Teichmann, J. and Veluscek, D.: *"A new extrapolation method for weak approximation schemes with applications"* Preprint: `arXiv:0911.4380v1 [math.PR]` (2009)

The Problem :

Numerical calculation of $E[f(X(T,x))]$, where

$$
\begin{aligned}
X(t,x) &= x + \sum_{j=0}^{d} \int_0^t V_j\,(X(s,x)) \circ \, dB^j(s) \\
V_j &\in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N) \\
B(t) &= (B^0(t), B^1(t), \ldots, B^d(t)), \\
B^0(t) &= t, \quad (B^1(t), \ldots, B^d(t)) \quad : d\text{-dim Std. BM},
\end{aligned}
\tag{1}
$$

$\circ\, dB^j(s)$: Stratonovich integral.

Two approaches:

PDE method

$$\frac{\partial u}{\partial t}(t, x) = Lu, \quad u(0, x) = f(x).$$

where $L = V_0 + (1/2) \sum_{i=1}^{d} V_i^2$.

Probabilistic method — "Simulation"

Step 1. Discretize $X(t, x)$ and obtain $X^n(t, x)$.
$n = \sharp\{\text{partitions of } [0, T]\}$
Euler–Maruyama, Higher order (Milstein, Kusuoka,..)

Step 2. Integrate $f(X^n(T, x))$ over $D(n)$-dimensional domain $[0, 1)^{D(n)}$ by MC, QMC, etc.
$D(n)$ depends on the discretization scheme

Step. 2 of the simulation is the numerical integration:

$$E\left[f\left(X^n(T,x)\right)\right] = \int_{[0,1)^{D(n)}} F(a_1,\ldots,a_{D(n)})\, da_1 \cdots da_{D(n)}$$

Calc. RHS by using **MC or QMC.**

MC and QMC

$$W : \text{r. v.}/(\Omega, \mathcal{F}, P), \quad M \in \mathbb{Z}_{>0}$$

$$\mathrm{MC}(W, M) := \frac{1}{M} \sum_{k=1}^{M} W_k, \quad \text{where } \{W_i\}_{i=1}^{M} \text{: iid s. t. } W_1 \sim W$$

$$\mathrm{QMC}(W, M) := \frac{1}{M} \sum_{k=1}^{M} W(\omega_k), \quad \{\omega_i\}_{i=1}^{\infty} \text{: deterministic sequence (LDS)}$$

$$\mathrm{MC}(W, M) : \quad \text{r. v.}$$

$$\mathrm{QMC}(W, M) \in \mathbb{R}.$$

Two types of approximation errors in simulation:

1. Discritization error

$$\left| E\left[f(X(T,x))\right] - E\left[f\left(X^n(T,x)\right)\right] \right|$$

2. Integration error

$$\left| \mathrm{MC}\left(f\left(X^n(T,x)\right), M\right)(\omega) - E\left[f\left(X^n(T,x)\right)\right] \right|$$

or

$$\left| \mathrm{QMC}\left(f\left(X^n(T,x)\right), M\right) - E\left[f\left(X^n(T,x)\right)\right] \right|$$

Integration error and MC

By CLT,

$$\mathrm{MC}(f(X^n(T,x)), M) \sim N\left(E\left[f(X^n(T,x))\right], \frac{\mathrm{Var}\left[f(X^n(T,x))\right]}{M}\right).$$

When we proceed simulation

$$\mathrm{Var}\left[f(X(T,x))\right] \approx \mathrm{Var}\left[f(X^n(T,x))\right]$$

Remark 1

As long as we use MC, the number of sample points M needed to attain the given accuracy is independent of n and discretizing algorithm (Euler–Maruyama or Kusuoka etc.).

Integration error and QMC

There exist sequences which satisfy

$$^{\exists}C_{f,n} > 0 \ ^{\forall}M \in \mathbb{Z}_{>0}$$

$$\left|\mathrm{QMC}\left(f\left(X^n(T,x)\right), M\right) - E\left[f\left(X^n(T,x)\right)\right]\right| \leq C_{f,D(n)} \frac{(\log M)^{D(n)}}{M}.$$

Remark 2

In contrast to the MC case, the number of sample points M needed by the QMC to attain the given accuracy depends heavily on the dimension of integration $D(n)$. Smaller the dimension, smaller number of samples are needed.

$$D(n) = \begin{cases} n \times d & \text{Euler–Maruyama,} \\ n \times (d+1) & \text{N. \& Victoir,} \\ 2n \times d & \text{Ninomiya, \& N.} \end{cases}$$

Order 1: Euler–Maruyama scheme

$$X^{(\mathrm{EM}),n}(0,x) = x,$$

$$X^{(\mathrm{EM}),n}\left(\frac{k+1}{n},x\right) = X^{(\mathrm{EM}),n}\left(\frac{k}{n},x\right) + \sqrt{\frac{T}{n}}\sum_{i=0}^{d}\tilde{V}_i\left(X^{(\mathrm{EM}),n}\left(\frac{k}{n},x\right)\right)Z_{k+1}^{i},$$

where,

$$^{\forall}j\ Z_j^k = \begin{cases} \sqrt{T/n}, & \text{if } k=0, \\ \text{iid. r. v.} \sim N(0,1), & \text{if } k \in \{1,\dots,d\}, \end{cases}$$

$$\tilde{V}_k^i(y) = \begin{cases} V_0^i(y) + \frac{1}{2}\sum_j V_j V_j^i(y) & \text{if } k=0, \\ V_k^i & \text{if } k \in \{1,\dots,d\}. \end{cases}$$

Approx. Error of Euler–Maruyama scheme

$$\left|E[f(X(T,x))] - E\left[f\left(X^{(\mathrm{EM}),n}(T,x)\right)\right]\right| = O\left(n^{-1}\right)$$
$$= O\left(\Delta t\right)$$

when f : bdd.& measurable and $\{V_i\}_{i=0}^{d}$: Unif. Hörmander Cond. [Bally & Talay '96][Kohatsu-Higa '00]

Euler–Maruyama scheme is an order 1 scheme.

Intuitive explanation of the scheme (1/3)

$$\left(P_t^X f\right)(x) := E\left[f\left(X(t,x)\right)\right], \quad f \in C_b^\infty(\mathbb{R}^N)$$

$$L := V_0 + \frac{1}{2}\sum_{j=1}^{d} V_j^2.$$

Intuitive explanation of the scheme (2/3)

Applying Ito formula repeatedly, we obtain

$$
\begin{aligned}
E\left[f\left(X(t,x)\right)\right] = \left(P_t^X f\right)(x) &= f(x) + \int_0^t \left(P_{s_1}^X Lf\right)(x)\, ds_1 \\
&= f(x) + \int_0^t \left\{ (Lf)(x) + \int_0^{s_1} \left(P_{s_2}^X L^2 f\right)(x)\, ds_2 \right\} ds_1 \\
&\vdots \\
&= \sum_{i=0}^{n} \left(\frac{(tL)^i}{i!} f \right)(x) + \frac{1}{n!} \int_0^t (t-s)^n \left(P_s^X L^{n+1} f\right)(x)\, ds.
\end{aligned}
$$

Intuitive explanation of the scheme(3/3)

Observation: $\sum_{i=0}^{n}\left(\frac{(tL)^i}{i!}f\right)(x)$ gives a nth order approx. of $E\left[f(X(t,x))\right]$.

Slogan: Construct a random variable Ξ s. t.

$$E[\Xi] = \sum_{i=0}^{n}\left(\frac{(tL)^i}{i!}f\right)(x).$$

Non-commutative algebra (1)

$$A := \{v_0, \ldots, v_d\} \text{ : alphabet}$$

$$A^* := \left(\cup_{k=1}^{\infty} A^k\right) \cup \{1\} \text{ : free monoid on } A$$

For $w = w_1 \cdots w_k \in A^*$, $(w_i \in A)$

$$|w| := k, \, \|w\| := |w| + \mathrm{card}\left(\{1 \le i \le |w| \, ; \, w_i = v_0\}\right),$$

$$A^*_m := \{w \in A^* \mid |w| = m\},$$

$$A^*_{\le m} := \{w \in A^* \mid |w| \le m\}.$$

Concatenation product:

For $u = u_1 \cdots u_k$, $v = v_1 \cdots v_l \in A^*$,

$$uv := u_1 \cdots u_k v_1 \cdots v_l.$$

Non-commutative algebra (2)

$$\mathbb{R}\langle\langle A\rangle\rangle := \left\{ \sum_{w\in A^*} a_w w \,\middle|\, a_w \in \mathbb{R} \right\} : \mathbb{R}\text{-algebra of formal series with basis } A^*,$$

$$\mathbb{R}\langle A\rangle := \left\{ \sum_{w\in A^*} a_w w \in \mathbb{R}\langle\langle A\rangle\rangle \,\middle|\, {}^{\exists}k \in \mathbb{N} \text{ s.t. } a_w = 0 \text{ if } |w| \geq k \right\}$$

$: \text{free } \mathbb{R}\text{-algebra with basis } A^*,$

$$(P, w) := a_w, \quad \text{for } P = \sum_{w\in A^*} a_w w \in \mathbb{R}\langle\langle A\rangle\rangle,$$

$$(PQ, w) := \sum_{\substack{uv=w \\ u,v\in A^*}} (P, u)(Q, v), \text{ for } P, Q \in \mathbb{R}\langle\langle A\rangle\rangle, \ w \in A^*.$$

Non-commutative algebra (3)

- Projection, Truncation:

$$j_m(P) := \sum_{\|w\| \le m} (P, w)w$$

$$P|_k := \sum_{|w|=k} (P, w)w$$

- Homogeneous component:

$$\mathbb{R}\langle A \rangle_m := \{ P \in \mathbb{R}\langle A \rangle \mid (P, w) = 0 \text{ if } \|w\| \ne m \}$$

Free Lie algebra:

$$[P, Q] := PQ - QP \quad \text{for } P, Q \in \mathbb{R}\langle\langle A \rangle\rangle,$$

$$\mathcal{J}_A := \left\{ K \underset{\text{sub } \mathbb{R}\text{-module}}{\subset} \mathbb{R}\langle A \rangle \,\middle|\, A \subset K,\ [x, y] \in K\ {}^\forall x, y \in K, \right\}$$

$$\mathcal{L}_\mathbb{R}(A) := \bigcap_{K \in \mathcal{J}_A} K \quad : \mathbb{R}\text{-coefficients free Lie algebra on } A$$

$$\mathcal{L}_\mathbb{R}((A)) := \left\{ P \in \mathbb{R}\langle\langle A \rangle\rangle \quad \text{s.t.} \quad P|_k \in \mathcal{L}_\mathbb{R}(A),\ {}^\forall k \in \mathbb{N} \right\}.$$

Logarithm and exponential:

For $P \in \mathbb{R}\langle\langle A \rangle\rangle$ s.t. $(P, 1) = 0$

$$\exp(P) := 1 + \sum_{k=1}^{\infty} \frac{P^k}{k!}.$$

For $Q \in \mathbb{R}\langle\langle A \rangle\rangle$ s.t. $(Q, 1) = 1$

$$\log(Q) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(Q-1)^k}{k}.$$

$$\log(\exp(P)) = P \quad \text{and} \quad \exp(\log(Q)) = Q.$$

Hausdorff product:

For $z_1, z_2 \in \mathcal{L}_{\mathbb{R}}((A))$,

$$z_1 \mathrm{H} z_2 := \log(\exp(z_1)\exp(z_2)).$$

- $(z_1 \mathrm{H} z_2)\mathrm{H} z_3 = \log(\exp(z_1)\exp(z_2)\exp(z_3)) = z_1 \mathrm{H}(z_2 \mathrm{H} z_3)$ $=: z_1 \mathrm{H} z_2 \mathrm{H} z_3$
- $z_2, z_1 \in \mathcal{L}_{\mathbb{R}} \implies z_2 \mathrm{H} z_1 \in \mathcal{L}_{\mathbb{R}}((A))$ $\because$ *the Baker–Campbell–Hausdorff–Dynkin formula.*

$\mathcal{L}_{\mathbb{R}}(A)$ and $\mathbb{R}\langle A\rangle$

$$\text{Element of } \mathcal{L}_{\mathbb{R}} \Longleftrightarrow \text{Vector field generated by } V_0, \ldots, V_d$$
$$\text{Elements of } \mathbb{R}\langle A\rangle \Longleftrightarrow \text{Differential operator generated by } V_0, \ldots, V_d$$

Resurgence and rescaling

- Resurgence operator Φ:

$$\Phi(1) := \mathrm{id}, \quad \Phi(v_{i_1} \cdots v_{i_n}) := V_{i_1} \cdots V_{i_n}$$

- Rescaling operator: For $s > 0$, $\Psi_s : \mathbb{R}\langle\langle A \rangle\rangle \to \mathbb{R}\langle\langle A \rangle\rangle$ is defined as:

$$\Psi_s \left(\sum_{m=0}^{\infty} P_m \right) = \sum_{m=0}^{\infty} s^{m/2} P_m \quad \text{where } P_m \in \mathbb{R}\langle A \rangle_m.$$

Example:

$$\Phi\left(\Psi_s\left(v_0 + \frac{1}{2}[v_1, v_2]\right)\right) = sV_0 + \frac{s}{2}[V_1, V_2]$$

Notation: $\exp(V)(x)$

For a smooth vector field V, (i.e. $V \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$)

$$\exp(V)(x) := y(1)$$

where $y(t)$ is a solution to the ODE:

$$y(0) = x, \quad \frac{\mathrm{d}y(t)}{\mathrm{d}t} = V(y(t))$$

Order m Integration scheme: $\mathcal{IS}(m)$

$$g \in \mathcal{IS}(m) \underset{\text{def}}{\Longleftrightarrow} \begin{cases} g: \ C_b^\infty(\mathbb{R}^N;\mathbb{R}^N) \longrightarrow \left(\mathbb{R}^N \to \mathbb{R}^N\right), \\ {}^\exists C_m > 0 \quad \text{s.t. } {}^\forall W \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N) \\ \sup_{x\in\mathbb{R}^N} |g(W)(x) - \exp(W)(x)| \le C_m \left(\|W\|_{C^{m+1}}\right)^{m+1} \end{cases}$$

- $\mathcal{IS}(m) \fallingdotseq$ "Set of order m ODE solver"
- We need to integrate, for example:

$$sV_0 + \frac{\sqrt{s}}{2}(V_1 + \cdots + V_d).$$

Not necessarily s-linear.

$\mathcal{R}\langle\langle A\rangle\rangle$, $\mathcal{L}_{\mathbb{R}}((A))$-valued random variables

- Topology: $\mathcal{R}\langle\langle A\rangle\rangle \approx \mathcal{R}^{\infty}$ – Direct product topology
- $\mathcal{R}\langle\langle A\rangle\rangle$-valued and $\mathcal{L}_{\mathbb{R}}((A))$-valued probability theory can be considered.
  (Ito formula, etc.)

Approximation theorem (1/2)

$m \geq 1$, $M \geq 2$,
$Z_1, \ldots, Z_M$: $\mathcal{L}_{\mathbb{R}}((A))$-valued random variables s. t.

$$Z_i = j_m Z_i \quad \text{for } i = 1, \ldots, M,$$

$$E\left[\|j_m Z_i\|_2\right] < \infty \quad \text{for } i = 1, \ldots, M,$$

$$E\left[\exp\left(a \sum_{j=1}^{M} \left\|\Phi\left(\Psi_s\left(Z_j\right)\right)\right\|_{C^{m+1}}\right)\right] < \infty \quad \text{for any } a > 0.$$

Approximation theorem (2/2)

$\Longrightarrow$

$^{\forall}p \in [1,\infty)$, $^{\forall}g_1,\ldots,{}^{\forall}g_M \in \mathcal{IS}(m)$, $^{\exists}C_{m,M} > 0$ s. t.

$$\left\| \sup_{x\in\mathbb{R}^N} \left| g_1\left(\Phi\left(\Psi_s\left(Z_1\right)\right)\right) \circ \cdots \circ g_M\left(\Phi\left(\Psi_s\left(Z_M\right)\right)\right)(x) - \exp\left(\Phi\left(\Psi_s\left(j_m\left(Z_M \sqcup \cdots \sqcup Z_1\right)\right)\right)\right)(x) \right| \right\|_{L^p} \leq C_{m,M} s^{(m+1)/2}$$

for $^{\forall}s \in (0,1]$ where $C_{m,M}$ depends only on m and M.
$f \circ g(x) := f\left(g(x)\right)$

Cor. to the Approximation theorem

"$Q_{(s)}$ gives $(m-1)/2$-order weak approximation."

Let $Q_{(s)}$ for $s \in (0,1]$ be

$$(Q_{(s)}f)(x) := E\left[f\left(g\left(\Phi\left(\Psi_s\left(Z_1\right)\right)\right)\circ\cdots\circ g\left(\Phi\left(\Psi_s\left(Z_M\right)\right)\right)(x)\right)\right],$$

where $f \in C_b^\infty(\mathbb{R}^N;\mathbb{R})$ and $g \in \mathcal{IS}(m)$ then ${}^\exists C > 0$,

$$\left\|P_s f - Q_{(s)} f\right\|_\infty \leq C s^{(m+1)/2} \|\mathrm{grad}(f)\|_\infty$$

$P_s(f) := E\left[f(X(t,x))\right]$

(**Remark:** When $\{V_0,\ldots,V_d\}$ finitely generated.)

Algorithm 1 [Victoir–N.(2004)]

$(\Lambda_i, Z_i)_{i\in\{1,\cdots,n\}}$: $2n$ indep. r. v.,

${}^{\forall} i\ P(\Lambda_i = \pm 1) = \frac{1}{2}$, $Z_i \sim N(0, I_d)$.

$\{X_k^{(\mathrm{Alg.1}),n}\}_{k=0,\ldots,n}$: a family of r. v. defined as:

$$X_0^{(\mathrm{Alg.1}),n} := x,$$

$$X_{(k+1)/n}^{(\mathrm{Alg.1}),n} :=$$

$$\begin{cases} \exp\left(\frac{V_0}{2n}\right)\exp\left(\frac{Z_k^1 V_1}{\sqrt{n}}\right)\cdots\exp\left(\frac{Z_k^d V_d}{\sqrt{n}}\right)\exp\left(\frac{V_0}{2n}\right)X_k^{(\mathrm{Alg.1}),n} & \text{if } \Lambda_k = +1, \\ \exp\left(\frac{V_0}{2n}\right)\exp\left(\frac{Z_k^d V_d}{\sqrt{n}}\right)\cdots\exp\left(\frac{Z_k^1 V_1}{\sqrt{n}}\right)\exp\left(\frac{V_0}{2n}\right)X_k^{(\mathrm{Alg.1}),n} & \text{if } \Lambda_k = -1. \end{cases}$$

Extrapolations of Algorithm 1

- To an arbitrary order [Oshima, Teichmann and Veluscek '09]
- To the 6th order [Fujiwara '06]

General framework of Algorithm 2 (1)

- Remind the Slogan
- Find the Ξ written as:

$$\Xi = Z_1 \mathrm{H} Z_2 \mathrm{H} \cdots \mathrm{H} Z_M$$

$$Z_j = c_j v_0 + \sum_{i=1}^{d} S_j^i v_i, \quad j \in \{1, \ldots, M\}$$

where

$$c_j \in \mathbb{R} \text{ s.t. } c_1 + \cdots + c_M = 1$$
$$E\left[S_j^i S_{j'}^{i'}\right] = R_{jj'} \delta_{ii'}$$
$$S_j^i \sim \mathrm{N}(0, R_{jj}) \ (i \in \{1, \ldots, d\},\ j \in \{1, \ldots, M\})$$

General framework of Algorithm 2 (2)

The slogan is equivalent to:

$$E\left[j_m\left(\exp(Z_1)\cdots\exp(Z_M)\right)\right] = j_m\left(\exp\left(v_0 + \frac{1}{2}\sum_{i=1}^{d} v_i^2\right)\right) \tag{2}$$

The problem:

Find real numbers $\{c_j\}_{j=1}^{M}$, $\left\{R_{ij}\right\}_{1\leq i\leq j\leq M}$ that satisfy (2).

$$\#\text{unknown vars} = \frac{1}{2}M(M+3)$$

Main tools (Theorem 1):

For the LHS of (2)

If $n^w(i)$ is odd for some $i \in \{1, \dots, d\}$, then $C(w) = 0$.
If $n^w(i)$ is even for every $i \in \{1, \dots, d\}$, then

$$C(w) = \sum_{\vec{k}=(k_1,\dots,k_M)\in\mathcal{K}_\ell(M)} \frac{1}{k_1!\cdots k_M!} \prod_{j=1}^{M} \left(c_j\right)^{N^w\left(0,j,\vec{k}\right)} \times \prod_{p=1}^{d} \left(\sum_{\substack{\{d_{ij}\}_{1\le i\le j\le M}\in \\ e\left(N^w\left(p,1,\vec{k}\right),\dots,N^w\left(p,M,\vec{k}\right)\right)}} 2^{-\sum_{i=1}^{M} d_{ii}} \frac{\prod_{j=1}^{M}\left(N^w\left(p,j,\vec{k}\right)!\right)}{\prod_{1\le i\le j\le M}\left(d_{ij}!\right)} \prod_{1\le i\le j\le M} R_{ij}^{d_{ij}} \right) \tag{3}$$

Main tools (Theorem 2):

For the RHS of (2)

Let $A^0 = \{v_0, v_1 v_1, v_2 v_2, \ldots, v_d v_d\} \subset A^*$. Then

$$\exp\left(v_0 + \frac{1}{2}\sum_{i=1}^{d} v_i^2\right) = \sum_{\substack{w=w_1\cdots w_l \\ w_1,\ldots,w_l\in A^0}} \frac{1}{2^{|w|-l}\, l!} w,$$

that is,

$$C(w) = \begin{cases} \frac{1}{2^{|w|-l}\, l!} & \text{if } w \in A^0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Algebraic relations between $\{c_j\}_{j=1}^{M}$, $\{R_{ij}\}_{1\leq i\leq j\leq M}$

Using following 3 results, the algebraic relations are obtained.

An example of Algorithm 2 [Ninomiya–N. (2009)]

When $(m, M) = (5, 2)$,

$$c_1 = \frac{\mp\sqrt{2\,(2u-1)}}{2}, \quad c_2 = 1 \pm \frac{\sqrt{2\,(2u-1)}}{2}$$

$$R_{22} = 1 + u \pm \sqrt{2\,(2u-1)},\ R_{12} = -u \mp \frac{\sqrt{2\,(2u-1)}}{2}$$

$$R_{11} = u \quad \text{for some } u \geq 1/2.$$

Becomes 1-dimensional ideal.

Partial results (1):

The case $m \geq 6$

- $(m, M) = (7, 2)$: ideal becomes trivial
  (i.e. $= \mathbb{C}[c_1, c_2, R_{11}, R_{12}, R_{13}]$)
- $(m, M, d) = (7, 3, 1)$: ideal becomes trivial.

Partial results (2):

The case $(m, M, d) = (7, 3, 2)$ and v_0 free:

The algebraic relation:

$$14R_{22} + 24R_{13} - 13, \quad 4R_{33} + 14R_{22} + 4R_{11} - 15, \quad 36{R_{33}}^2 - 60R_{22} + 25,$$
$$-72R_{22}R_{33} + 68R_{33} + 24R_{23} + 22R_{22} - 17,$$
$$-288R_{23}R_{33} - 8R_{33} + 264R_{23} - 46R_{22} + 77,$$
$$12R_{23} - 22R_{22} + 12R_{12} + 23, \quad 8R_{33} + 216R_{22}R_{23} - 156R_{23} + 34R_{22} - 27,$$
$$-22R_{33} - 324{R_{23}}^2 - 192R_{23} + 46R_{22} - 57,$$
$$144{R_{33}}^2 - 64R_{33} + 168R_{23} + 46R_{22} + 7$$

Unfortunately, the solution is:

$$R_{33} = \frac{5 \pm \sqrt{-11}}{12} \notin \mathbb{R}$$

Partial results (3):

The case $(m, M, d) = (7, 4, 3)$ and v_0 free:

Over 1-month symbolical calculation cannot find the answer.....

We have some more results (Theorem 3): $A^* = \bigcup_{i=0}^{\infty} A^i$,
$\pitchfork : A^* \otimes A^* \to A^*$: Shuffle product.
$(A^*, \pitchfork)$ becomes an algebra (shuffle algebra).

$C(\cdot)$

$$C : (A^*, \pitchfork) \longrightarrow \mathbb{R}\left[c_j, R_{ij}, (1 \le i \le j \le M)\right]$$

is ring homomorphism.

Cor.

$$C\left((0)^{n_0} \pitchfork (11) \pitchfork \cdots \pitchfork (\ell\ell)\right) = C(0)^{n_0} C(11)^{\ell}$$

Def: shuffle product.

$A = \{v_1, \cdots v_d\}$: alphabet

$A^* = \bigcup_{i=0}^{\infty} A^i$,

$$\begin{cases} I_1 \sqcup \cdots \sqcup I_p = \{1, \cdots, n\} \\ u_i \in A^*, \quad |u_i| = \# I_i \quad (i = 1, \cdots, p) \end{cases}$$

$$A^* \ni \begin{pmatrix} I_1, I_2, \cdots\cdots I_p \\ u_1 \ u_2 \ \cdots \ u_p \end{pmatrix} = w \overset{\text{def}}{\iff} w|_{I_i} = u_i \quad 1 \le {}^\forall i \le p.$$

[where for $w = a_1 \cdots a_n$, $I = \{i_1 < \cdots < i_k\} \subset \{1, \cdots, n\}$

$w|_I := a_{i_1} a_{i_2} \cdots a_{i_k}$]

$$I(u_1, \cdots u_p) := \left\{ \{I_i\}_{i=1}^{p} \;\middle|\; \bigsqcup_{i=1}^{p} I_i = \{1, \cdots, n\}, \ \# I_i = u_i \right\}.$$

Then the shuffle product of $u_1, \cdots u_p$:

$$u_1 ⧢ u_2 ⧢ \cdots ⧢ u_p := \sum_{\{I_i\}_{i=1}^{p} \in I(u_1, \cdots, u_p)} \begin{pmatrix} I_1 \cdots I_p \\ u_1, \cdots, u_p \end{pmatrix}$$

Ex. $(12) ⧢ (13) = (1213) + (1123) + (1132) + (1123) + (1132) + (1312)$

$= (1213) + 2(1123) + 2(1132) + (1312)$

Some remarks

① $V(m.M.d) := \{(C_1 \cdots C_M, R_{11}, R_{12}, \cdots R_{MM} \mid (2) \text{ holds}\}$

- $V(m.M.d) \supset V(m+2.M.d)$
- $V(m.M.d) \supset V(m.M.d+1)$ hold.
- $V(m.M.d) \supset V(m.M+1.d)$
- $V(m.M.d) = (m.M.\frac{m-1}{2})$ for $\forall d \geq \frac{m-1}{2}$

What we know

(m.M.d)	# unknown vars	# relations	ideal
(5.2.2)	5	10	1-dim
(6.2.*)	5	68	trivial (even when d=2)
(6.3.1)	9	20	when $C_1 - C_3 = 0 \Rightarrow$ trivial o/w $\Rightarrow$ unknown
(6.3.2) (6.3.3)	9	53 68 $\Rightarrow$ 15	0-dim when v_e-free but $\notin \mathbb{R}$
(6.4.2) (6.4.3)	14	53 68 $\Rightarrow$ 15	?

An example of Algorithm 2 [Ninomiya–N. (2005)]

$\left(Z_{i,k}^j\right)_{\substack{i\in\{1,\dots,d\},j\in\{1,2\},\\ k\in\{0,\dots,n-1\}}}$,

$$\begin{pmatrix} Z_{i,k}^1 \\ Z_{i,k}^2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \eta_{i,k}^1 \\ \eta_{i,k}^2 \end{pmatrix}, \quad \text{where } \eta_{i,k}^j \overset{i.i.d.}{\sim} N(0,1).$$

$\{X_k^{(\mathrm{Alg.2}),n}\}_{k=0,\dots,n}$: a family of r. v. defined as:

$$X_0^{(\mathrm{Alg.2}),n} := x,$$
$$X_{(k+1)/n}^{(\mathrm{Alg.2}),n} :=$$
$$\exp\left(\frac{1}{2n}V_0 + \sum_{i=1}^d \frac{Z_{i,k}^1}{\sqrt{n}}V_i\right)\exp\left(\frac{1}{2n}V_0 + \sum_{i=1}^d \frac{Z_{i,k}^M}{\sqrt{n}}V_i\right)X_{k/n}^{(\mathrm{Alg.2}),n}$$

Theorem

Both $X^{(\mathrm{Alg.1}),n}$ and $X^{(\mathrm{Alg.2}),n}$ are of order 2.

Recent result by Kusuoka

If $Q_{(s)}$ is constructed by $X^{(\mathrm{Alg.1}),n}$ or $X^{(\mathrm{Alg.2}),n}$ then $^{\exists}C > 0$ s.t.

$$(P_s f)(x) - (Q_{(s)} f)(x) = Cs^{(m+1)/2} + O\left(s^{(m+3)/2}\right)$$

$D(n)$

$D(n)$: dimension of integration domain

$$D(n) = \begin{cases} n \times d & \text{Euler–Maruyama,} \\ n \times (d+1) & \text{Algorithm 1} \\ n \times 2 \times d & \text{Algorithm 2} \end{cases}$$

How to calc. $\exp(Z_i)(x)$?

- Lucky case: We can get exact form of $\exp(Z_i)(x)$.
  Often the case with Algorithm 1.
- Otherwise: Forced to proceed with nuerical approximation.

Good news:

Theorem [Ninomiya–N.]

Classical order m Runge–Kutta methods belongs to $\mathcal{IS}(m)$

MC or QMC with Algorithms 1 and 2

- W: "a set of ODEs"-valued r. v. by Algorithm 1 or 2
- MC or QMC:
  - Draw a set of ODEs "$W(\omega)$" from W and obtain (something like) $\exp(W(\omega))(x)$ numerically
  - Iterate the step above and calculate the average:

$$\frac{1}{M}\sum_{i=1}^{M}\exp(W(\omega_i))(x)$$

Advantages of the approache

- Free from symbolical calculation.
  - Calc. in group is easy.
    - Numerical ODE solver works (by the 2nd th'm)
  - Calc. in algebra is difficult.
    - Huge symbolical calc.
    - Simultaneous distributions of multiple integrations of BMs are not knowm. (except for the 2nd order.)
- Universal (applicable to non-commutative $\{V_i\}_{i=0}^d$)
  Naïve "Ito-Taylor expansion with the Runge–Kutta" suffers from the non-commutativity.

Numerical Example:

Pricing of Asian option under the Heston model:

Heston model:

$$Y_1(t,x) = x_1 + \int_0^t \mu Y_1(s,x)\,ds + \int_0^t Y_1(s,x)\sqrt{Y_2(s,x)}\,dW^1(s),$$

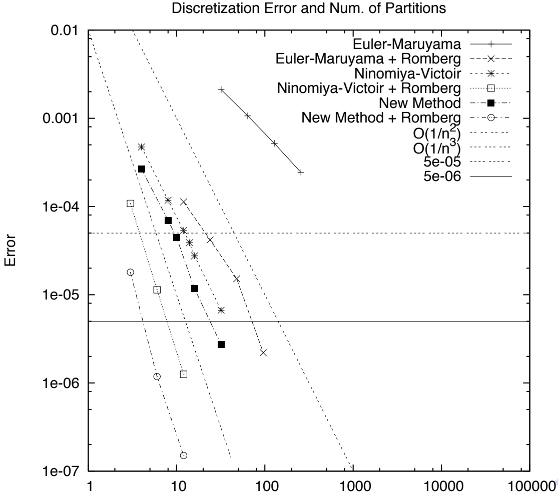
$$Y_2(t,x) = x_2 + \int_0^t \alpha\left(\theta - Y_2(s,x)\right)\,ds + \int_0^t \beta\sqrt{Y_2(s,x)}\,dW^2(s),$$

$$\left\langle W^1, W^2 \right\rangle_t = \rho t$$

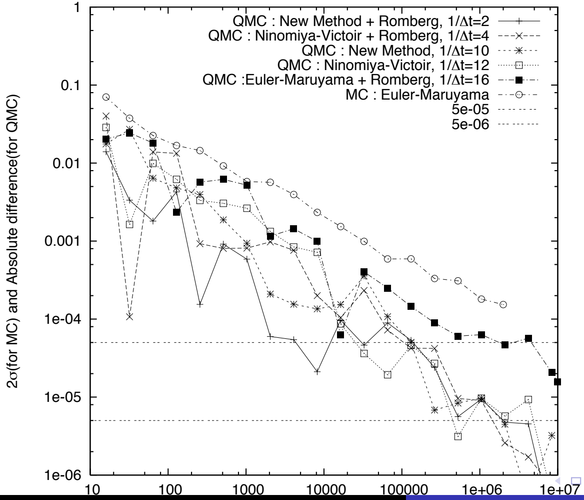
Asian Option:

$$Y_3(t,x) = \int_0^t Y_1(s,x)\,ds, \quad \text{Payoff} = \max\left(\frac{Y_3(T,x)}{T} - K, 0\right).$$

Discretization Error



Convergfence Error from quasi-Monte Carlo and Monte Carlo



Overall performance comparison:

#Partition, #Dim, #Sample, and CPU time required for 10^{-4} accuracy.

Method	#Partition	#Dim	#Sample	CPU time (sec)
E-M + MC	2000	4000	10^8	1.72×10^5
E-M + Extrpltn + QMC	$16 + 8$	48	5×10^6	1.27×10^2
N-V + QMC	12	36	2×10^5	3.24
N-V + Extrpltn + QMC	$4 + 2$	18	2×10^5	1.76
KNN + QMC	10	40	2×10^5	3.4
KNN + Extrpltn + QMC	$2 + 1$	12	2×10^5	1.2

Recent developments:

- Existence of the asymptotic expansion of the errors of Algorithms 1 and 2 [Kusuoka]
- Order 6 algorithm [Fujiwara]
- SPDE case [Teichmann]

Both approaches are necessary because:

- PDE approach works only when
  - L is elliptic,
  - dimesion is small.
- Simulation is the last resort but (people believes) very time consuming.

This presentation focuses on "Simulation."