Bures distance for completely positive maps

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- Minimality: $\mathcal{H} = \overline{\text{span}}\{\pi(a)z : a \in \mathcal{A}\}$.

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- Question: If ϕ_1, ϕ_2 are close can we make GNS representations close?
- Idea: Look at common representations: $(\mathcal{H}, \pi, z_1), (\mathcal{H}, \pi, z_2)$.
- Example: Consider direct sum:

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \pi = \pi_1 \oplus \pi_2,$$

$$z_1 \oplus 0, 0 \oplus z_2.$$

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Completely positive (CP) maps

- A linear map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is said to be completely positive (CP) if,

$$\sum_{i,j} b_i^* \phi(a_i^* a_j) b_j \geq 0$$

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- $*$-homomorphisms, positive linear functionals are (CP).
- Compositions, sums, convex combinations of CP maps are CP.
- CP maps are very important for understanding C^*-algebras and from applications point of view.

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- The infimum is attained and one has lower and upper bounds for β.

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- β is a metric.
- Theorem (D. Kretschmann, D. Schlingemann, R. F. Werner): Let $\phi_i : \mathcal{A} \to \mathcal{B}$ be normal CP maps, then

$$\frac{\|\phi_1 - \phi_2\|_{cb}}{\sqrt{\|\phi_1\|_{cb}} + \sqrt{\|\phi_2\|_{cb}}} \leq \beta(\phi_1, \phi_2) \leq \sqrt{\|\phi_1 - \phi_2\|_{cb}}$$

Crucial Lemma

- Define:

$$\mathcal{N}_E(\phi_1, \phi_2) = \{\langle z_1, z_2 \rangle : \phi_i(a) = \langle z_i, a.z_i \rangle, \forall a \in \mathcal{A}, z_i \in E, i = 1, 2\}$$

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$$\mathcal{M}(\phi_1, \phi_2) = \mathcal{N}(\phi_1, \phi_2) = \mathcal{N}_{\hat{E}_1 \oplus \hat{E}_2}(\phi_1, \phi_2)$$

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- Corollary $\beta(\phi_1, \phi_2)$ is attained in $\hat{E}_1 \oplus \hat{E}_2$.

Counter Examples- I

- $\beta(\phi_1, \phi_2)$ is not attained in all common representations.
- There is an example where

$$\sqrt{\|\phi_1 - \phi_2\|} < \beta(\phi_1, \phi_2) < \sqrt{\|\phi_1 - \phi_2\|_{cb}}.$$

So it is crucial to have the cb-norm.

Counter Example -II

- Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Consider the unital C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$:

$$\mathcal{A} := C^*\left\{\mathcal{K}(\mathcal{H} \oplus \mathcal{H}) \cup \left\{ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right\}\right\}$$

$$= \left\{ \begin{bmatrix} \lambda_1 I + a_{11} & a_{12} \\ a_{21} & \lambda_2 I + a_{22} \end{bmatrix} : \lambda_i \in \mathbb{C},\ a_{ij} \in \mathcal{K}(H) \right\}$$

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- Suppose $u \in \mathcal{B}(\mathcal{H})$ is a unitary and $1 < r \in \mathbb{R}$. Set

$$z_1 = \begin{bmatrix} 0 & u \\ 0 & rI \end{bmatrix},\ z_2 = \begin{bmatrix} 0 & 0 \\ 0 & rI \end{bmatrix} \text{ and } z_3 = \begin{bmatrix} 0 & I \\ 0 & rI \end{bmatrix}$$

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- Define CP-maps $\varphi_i : \mathcal{A} \to \mathcal{A}$ by $\varphi_i(a) := z_i^* a z_i$, $i = 1, 2, 3$.

Counter Examples III

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  is a Hilbert $\mathcal{A}$-$\mathcal{A}$ -module with a natural inner product and bimodule structure. Note that $z_i \in S(E_{12}, \varphi_i)$, $i = 1, 2$, and hence $\beta(\varphi_1, \varphi_2) \leq \|z_1 - z_2\| = 1$.

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- For any common representation module E of φ_1, φ_3, we prove that $\langle x_1, x_3 \rangle = 0$ for all $x_i \in S(E, \varphi_i)$.
- Then $\beta(\varphi_1, \varphi_3) > 2 \geq \beta(\varphi_1, \varphi_2) + \beta(\varphi_2, \varphi_3)$. Hence β fails to satisfy triangle inequality.

Bures distance for homomorphisms

Here we make some explicit computations of Bures distance. Let $\varphi_1, \varphi_2 : \mathcal{A} \to \mathcal{B}$ be two unital $*$-homomorphisms. In the following $\Re$ denotes real part.

- $\beta(\varphi_1, \varphi_2) = \sqrt{2} \inf \{ \|1 - \Re(b)\|^{\frac{1}{2}} : b \in \mathcal{B}, \|b\| \leq 1, \varphi_1(a)b = b\varphi_2(a) \; \forall a \in \mathcal{A} \}$.

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- If $\mathcal{A} = \mathcal{B}$ and $\varphi_2(a) = u^*\varphi_1(a)u$ for some unitary $u \in \mathcal{B}$, then

$$\beta(\varphi_1, \varphi_2) = \sqrt{2} \inf \left\{ \left\|1 - \Re(b'u)\right\|^{\frac{1}{2}} : b' \in \varphi_1(\mathcal{A})', \left\|b'\right\| \leq 1 \right\}.$$

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- If $u \in M_n(\mathbb{C})$ is a unitary and $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is the $*$-homomorphism $\varphi(a) = u^*au$, then

$$\beta(id, \varphi) = \sqrt{2} \inf \{ \|1 - \Re(\lambda u)\|^{\frac{1}{2}} : \lambda \in [-1, 1] \}.$$

A rigidity theorem

- Theorem: Suppose $\mathcal{B}$ is a von Neumann algebra and $\phi : \mathcal{B} \to \mathcal{B}$ is a completely positive map such that $\beta(\phi, id.) < 1,$ where $id.$ is the identity map. Then any Stinespring module E of ϕ has the form

$$E = \mathcal{B} \oplus E'$$

for some module E'.

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THANKS