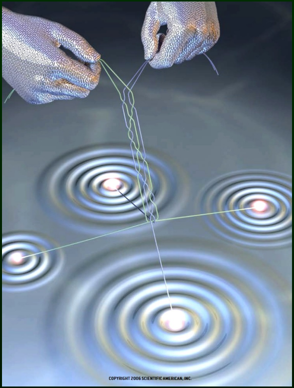
Introduction to TQC theory and spin networks

藤井淳一（J.I.Fujii）

大阪教育大学（Osaka Kyoiku Univ.）

anyon in TQC



Scientific American, 2006

TQC means Toplogical Quantum Computing

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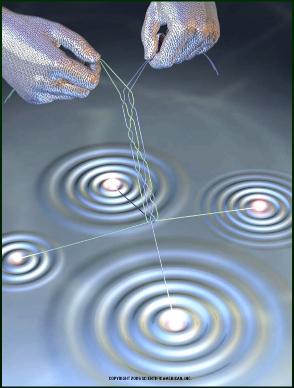

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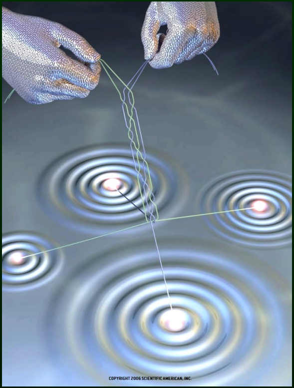

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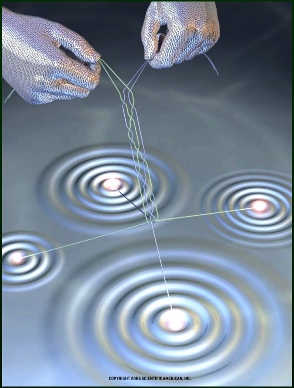

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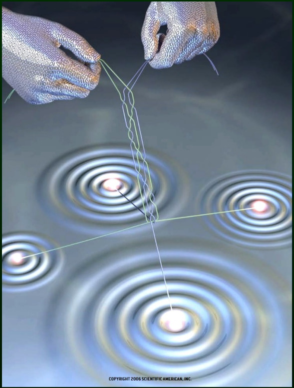

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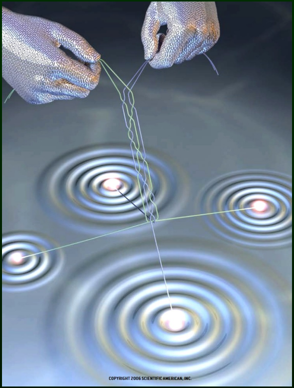

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- (abelian) anyon: $|y\rangle \otimes |x\rangle \sim e^{it}|x\rangle \otimes |y\rangle$
- (non-abelian) anyon:
  $$|y\rangle \otimes |x\rangle \sim e^{iH}(|x\rangle \otimes |y\rangle)$$

fusion in TQC

$\mathcal{F}$: fixed finite set of particles, $a, b \in \mathcal{F}$

$$V_a \otimes V_b \ni a \otimes b \to \sum_{x \in \mathcal{F}} N_{ab}^x \, x \in \bigoplus_{x \in \mathcal{F}} V_x, \quad N_{ab}^x \in \mathbb{N} \cup \{0\} \text{ (dimension of } x\text{)}$$

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Assume: vacuum $\mathbf{0} \in \mathcal{F}$, anti-particle $\bar{a} \in \mathcal{F}$ $(\forall a \in \mathcal{F})$

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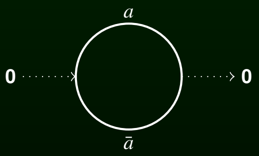
graphical expression



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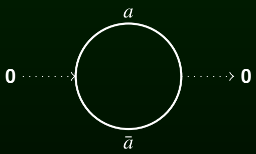

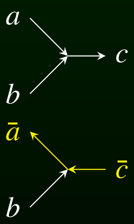

$$N_{ab}^c = N_{ba}^c = N_{b\bar{c}}^{\bar{a}}$$

Fibonacci anyon τ

$$\mathcal{F} = \{\mathbf{0}, \tau\}, \quad \tau \otimes \tau \to \mathbf{0} + \tau, \qquad N_{\tau\tau}^{\tau} = 1,\ N_{\tau\tau}^{\mathbf{0}} = 1,\ N_{\tau\mathbf{0}}^{\mathbf{0}} = 0,$$

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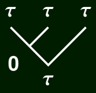
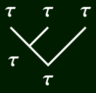
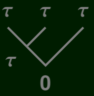
Definition

fusion vectors (fusion trees, conformal blocks)

$|0\rangle=$ 

$|1\rangle=$ 

$|N\rangle=$ 

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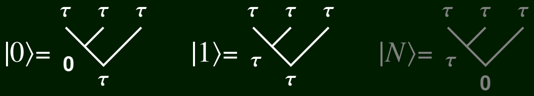

We want to obtain the braiding matrices $B \in \mathrm{B}(\mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle \oplus \mathbb{C}|N\rangle)$

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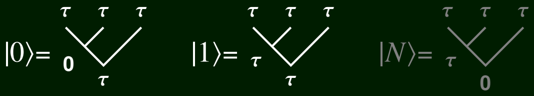
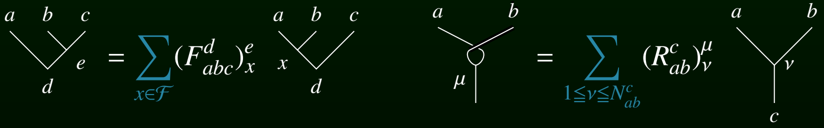
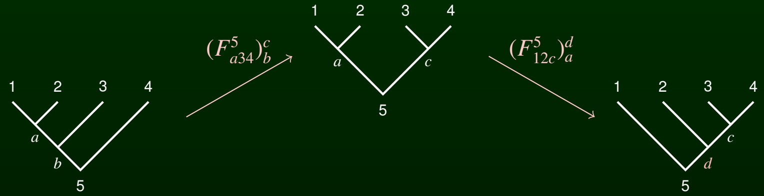


We want to obtain the braiding matrices $B \in \mathrm{B}(\mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle \oplus \mathbb{C}|N\rangle)$
by using fundamental matrices $F, R \in \mathrm{B}(V_x \oplus V_y)$:



Pentagon axiom

$$(F^5_{12c})^d_a (F^5_{a34})^c_b = \sum_x (F^d_{234})^c_x (F^5_{1x4})^d_b (F^b_{123})^x_a$$



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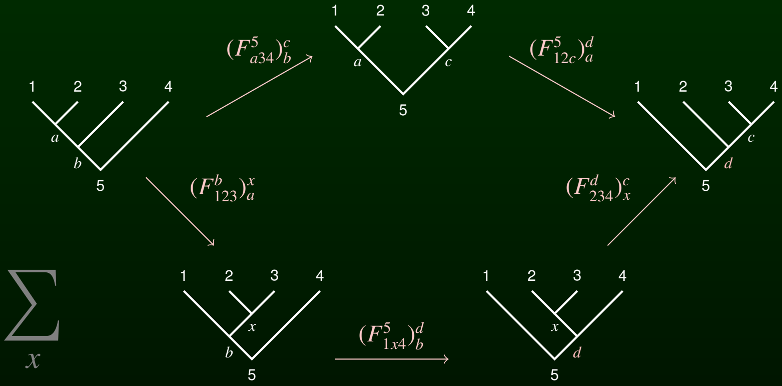

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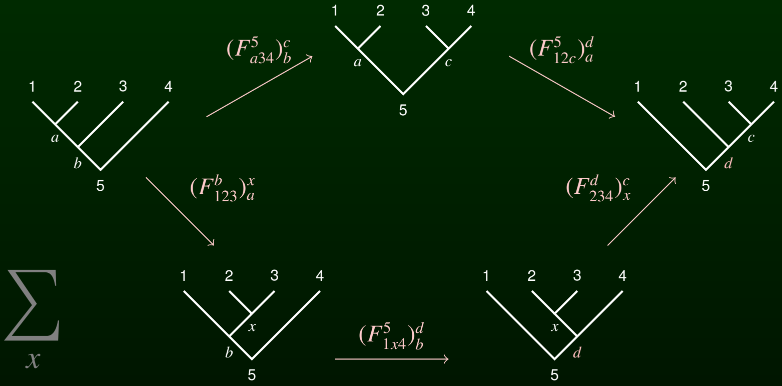

For the Fibonacci, $F^\tau_{\tau\tau\tau} = \frac{1}{g}\begin{pmatrix} 1 & \sqrt{g} \\ \sqrt{g} & -1 \end{pmatrix}$ where $g = \frac{1+\sqrt{5}}{2}$

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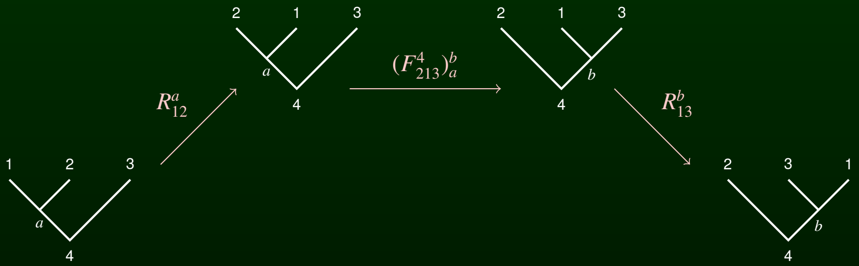
$$R^b_{13}(F^4_{213})^b_a R^a_{12} = \sum_x (F^4_{231})^b_x R^4_{1x}(F^4_{123})^x_a$$



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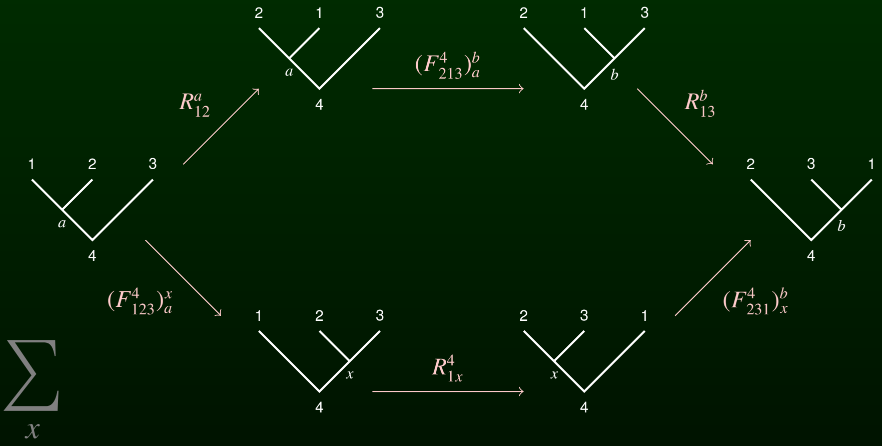

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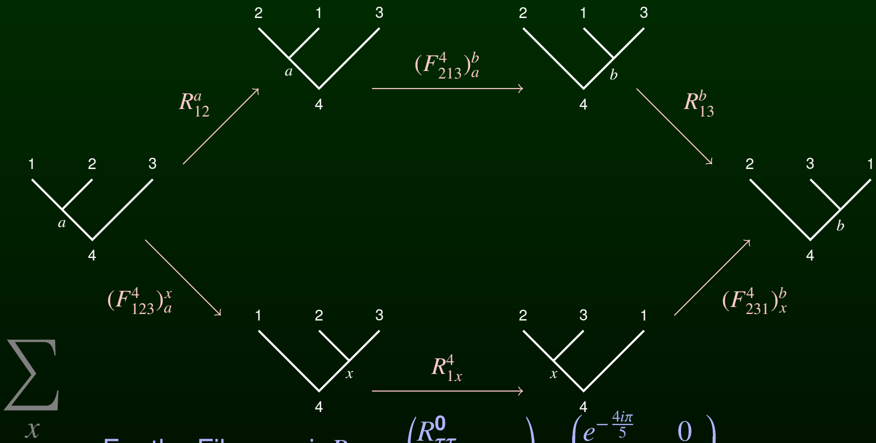

For the Fibonacci, $R_{\tau\tau} = \begin{pmatrix} R^{\mathbf{0}}_{\tau\tau} & \\ 0 & R^{\tau}_{\tau\tau} \end{pmatrix} = \begin{pmatrix} e^{-\frac{4i\pi}{5}} & 0 \\ 0 & e^{\frac{3i\pi}{5}} \end{pmatrix}$

Braiding matrices

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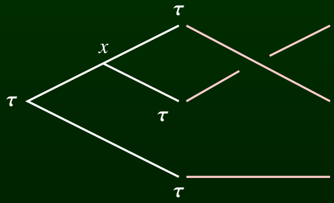

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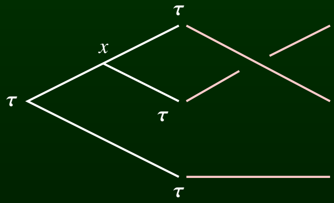

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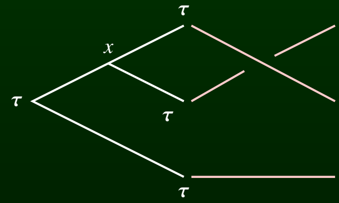

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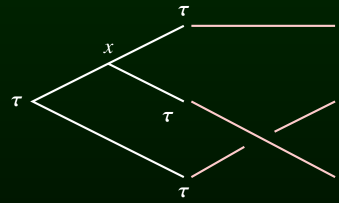
B_2:



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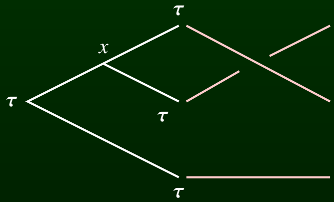

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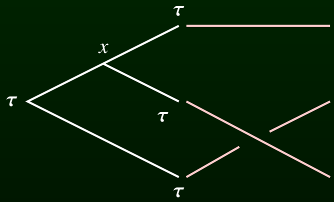

compute as $\sum_y (F^\tau_{\tau\tau\tau})^x_y R^y_{\tau\tau} (F^\tau_{\tau\tau\tau})^y_x$.

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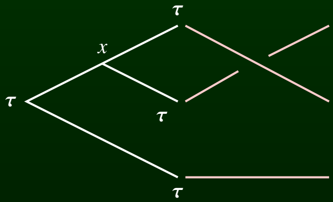

$R_{\tau\tau} = \text{diag}(R^x_{\tau\tau})$ itself.

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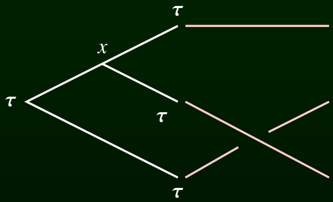

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$$B_1 = \begin{pmatrix} e^{-\frac{4i\pi}{5}} & 0 \\ 0 & e^{\frac{3i\pi}{5}} \end{pmatrix}, \quad B_2 = \frac{1}{g} \begin{pmatrix} e^{\frac{4i\pi}{5}} & \sqrt{g}e^{-\frac{3i\pi}{5}} \\ \sqrt{g}e^{-\frac{3i\pi}{5}} & -1 \end{pmatrix} \qquad g = \frac{1+\sqrt{5}}{2}$$

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Hadamard gate $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $\frac{\pi}{8}$ gate $\begin{pmatrix}e^{-\frac{i\pi}{8}} & 0\\ 0 & e^{\frac{i\pi}{8}}\end{pmatrix}$, CNOT gate $\begin{pmatrix}I & O\\ O & X\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$

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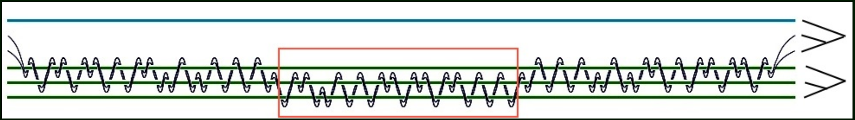
The Fibonacci set $\{B_1, B_2\}$ is approximately universal:

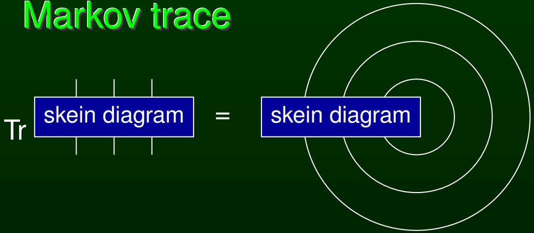
$$B_1^{-2}B_2^{-4}B_1^4B_2^{-2}B_1^2B_2^2B_1^{-2}B_2^4B_1^{-2}B_2^4B_1^2B_2^{-4}B_1^2B_2^{-2}B_1^2B_2^{-2}B_1^{-2} \approx iX$$

CNOT approximation



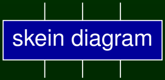
ejection part $\quad$ iX part $\quad$ injection part

Markov trace



Markov trace

Tr



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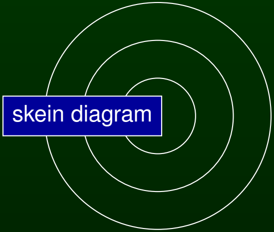
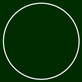


Tr $\Big|_a$ =



a

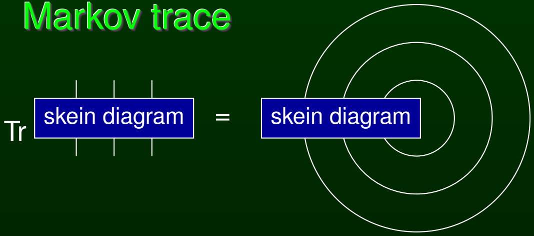
$= d_a$

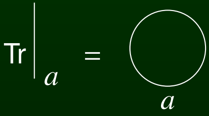
$d_0 = 1$

Trace property

$$\mathrm{Tr}\, X = \mathrm{Tr}\, Y \;\Leftrightarrow\; X \underset{\text{mod bracket poly.}}{=} Y\,, \quad d_a d_b = \sum_x N_{ab}^x d_x \;\Leftrightarrow\; a \otimes b = \sum_x N_{ab}^x x$$

Markov trace





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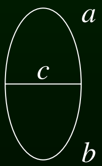
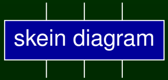
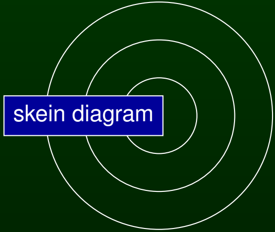
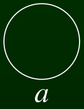
θ net: $\Theta(a, b, c)$ **(if exists)**



$=$

Markov trace

Tr  = 

Tr  =  $= d_a$

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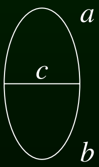
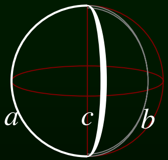
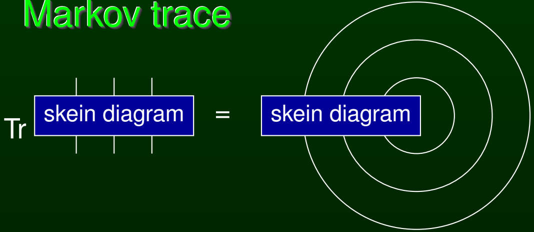
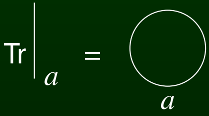
Trace property

$$\mathrm{Tr}\, X = \mathrm{Tr}\, Y \Leftrightarrow X \underset{\text{mod bracket poly.}}{=} Y\,, \quad d_a d_b = \sum_x N_{ab}^x d_x \Leftrightarrow a \otimes b = \sum_x N_{ab}^x x$$

θ net: $\Theta(a, b, c)$ **(if exists)**  = 

Markov trace





$= d_a$

$d_0 = 1$

Trace property

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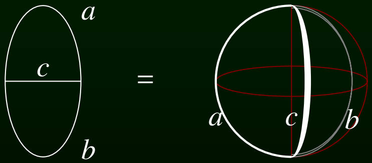
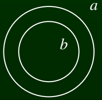


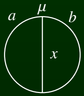
$= \sqrt{d_a d_b d_c}$

fusion and splitting formulae



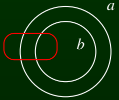
$\mu=1,2,\ldots,N_{ab}^x$

$$= d_a d_b = \sum_x N_{ab}^x d_x = \sum_x N_{ab}^x \mathbf{d}_{ab}^x \sqrt{d_a d_b d_x} = \sum_{x,\mu} \mathbf{d}_{ab}^x$$

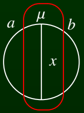


where $\mathbf{d}_{ab}^x = \sqrt{\dfrac{d_x}{d_a d_b}}$.

fusion and splitting formulae

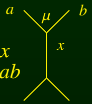


$$= d_a d_b = \sum_x N_{ab}^x d_x = \sum_x N_{ab}^x \mathbf{d}_{ab}^x \sqrt{d_a d_b d_x} = \sum_{x,\mu}^{\mu=1,2,\ldots,N_{ab}^x} \mathbf{d}_{ab}^x$$



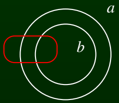
where $\mathbf{d}_{ab}^x = \sqrt{\dfrac{d_x}{d_a d_b}}$. Then

$$= \sum_{x,\mu} \mathbf{d}_{ab}^x$$

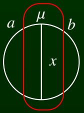


concource formula

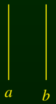
fusion and splitting formulae



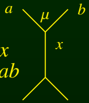
$$= d_a d_b = \sum_x N_{ab}^x d_x = \sum_x N_{ab}^x \mathbf{d}_{ab}^x \sqrt{d_a d_b d_x} = \sum_{x,\mu}^{\mu=1,2,\dots,N_{ab}^x} \mathbf{d}_{ab}^x$$



where $\mathbf{d}_{ab}^x = \sqrt{\dfrac{d_x}{d_a d_b}}$. Then

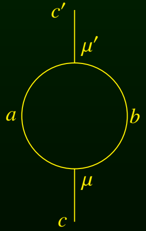


$$= \sum_{x,\mu} \mathbf{d}_{ab}^x$$



concource formula

Similarly, the trace computation for $\sqrt{d_a d_b d_x} = \frac{1}{\mathbf{d}_{ab}^x} d_x$, the symmetry says



$$= \frac{\delta_{c,c'}\delta_{\mu,\mu'}}{\mathbf{d}_{ab}^c}$$

popping bubble formula

F and indicator

formulae for $|F|$

$$d_a = \frac{1}{\left|(F^a_{a\bar{a}a})^{\mathbf{0}}_{\mathbf{0}}\right|} \quad \left(\begin{array}{c} \chi_a \equiv d_a (F^a_{a\bar{a}a})^{\mathbf{0}}_{\mathbf{0}}, \qquad |\chi_a| = 1 \\ \text{For } a = \bar{a},\ \chi_a \text{ is the Frobenius-Schur indicator } (\pm 1) \\ \text{For } a \neq \bar{a}, \text{ we may assume } \chi_a = 1. \end{array} \right), \quad \mathbf{d}^c_{ab} = |(F^b_{a\bar{a}b})^c_0|$$

F and indicator

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$= \chi_a$

$$\chi_{\bar{a}} = \bar{\chi}_a$$

$= \bar{\chi}_a$

$d_a (F^a_{a\bar{a}a})^{\mathbf{0}}_{\mathbf{0}}$

$= (F^a_{a\bar{a}a})^{\mathbf{0}}_{\mathbf{0}}$

$=$

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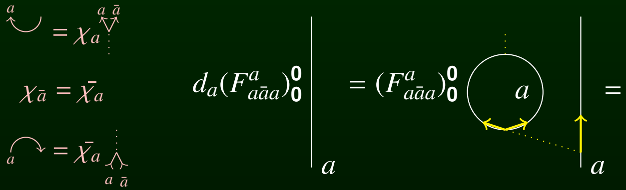

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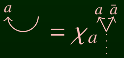
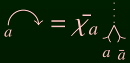
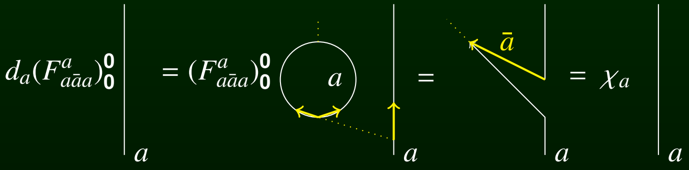


$\chi_{\bar{a}} = \bar{\chi}_a$





F and indicator

formulae for $|F|$

$$d_a = \frac{1}{\left|(F^{a}_{a\bar{a}a})^{\mathbf{0}}_{\mathbf{0}}\right|}$$

$\chi_a \equiv d_a (F^a_{a\bar{a}a})^{\mathbf{0}}_{\mathbf{0}}$, $\quad |\chi_a| = 1$

For $a = \bar{a}$, χ_a is the Frobenius-Schur indicator (± 1)

For $a \neq \bar{a}$, we may assume $\chi_a = 1$.

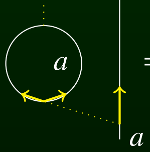
$$\mathbf{d}^c_{ab} = |(F^b_{a\bar{a}b})^c_0|$$

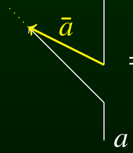
$= \chi_a$


$\chi_{\bar{a}} = \bar{\chi}_a$

$= \bar{\chi}_a$

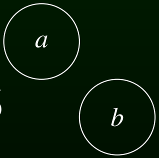

$d_a (F^a_{a\bar{a}a})^{\mathbf{0}}_{\mathbf{0}}$


$= (F^a_{a\bar{a}a})^{\mathbf{0}}_{\mathbf{0}}$


$=$


$= \chi_a$


$$(F^b_{a\bar{a}b})^c_0 d_a d_b = (F^b_{a\bar{a}b})^c_0$$



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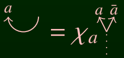

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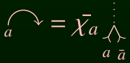
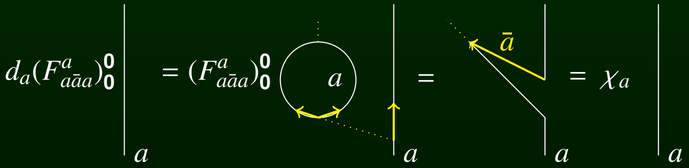
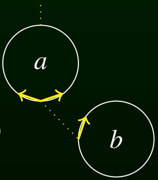




$$(F^b_{a\bar{a}b})^c_0 d_a d_b = (F^b_{a\bar{a}b})^c_0$$



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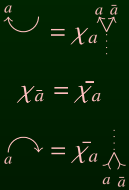
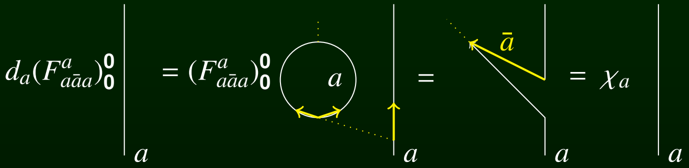
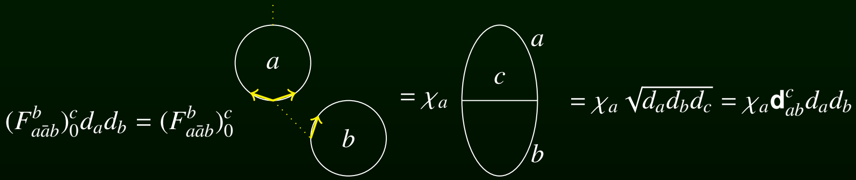
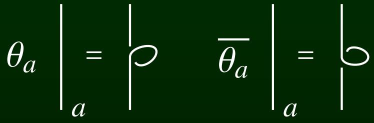






topological spin (Dehn twist) θ_a

This is a typical concept in TQC:



$$\overline{\theta_a} = \theta_a^{-1}$$

topological spin (Dehn twist) θ_a

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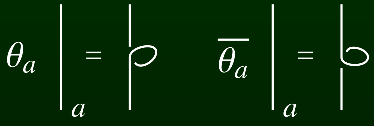
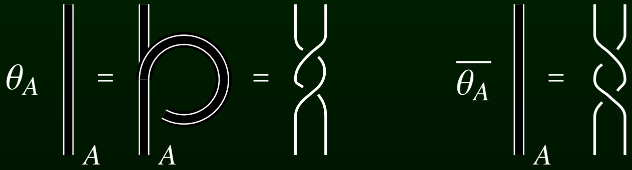
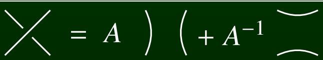


$$\overline{\theta_a} = \theta_a^{-1}$$

The ribbon figure



difference between knots & TQC

bracket polynomial: 

$d = -A^2 - A^{-2}$

difference between knots & TQC

bracket polynomial:  $= A$  $+ A^{-1}$  $\quad d = -A^2 - A^{-2}$

spin:  $= A$  $+ A^{-1}$  $= (Ad + A^{-1})d = -A^3 d$

difference between knots & TQC

bracket polynomial:  $= A$  $+ A^{-1}$  $d = -A^2 - A^{-2}$

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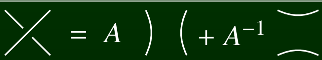
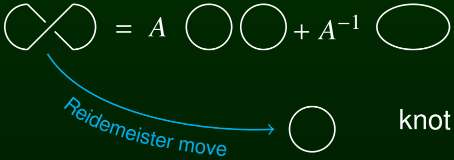
knot invariant: Jones polynomial

difference between knots & TQC

bracket polynomial: 

$d = -A^2 - A^{-2}$

spin: 

$= (Ad + A^{-1})d = -A^3 d$

knot invariant: Jones polynomial

Considering the topological spin:

skein	fitting polynomial
knot theory ($SU(2)$ type)	Jones polynomial
$SU(n)$ type	HOMFLY polynomial
TQC theory	bracket polynomial

skein for R

monodromy equation

$$R^c_{ba} R^c_{ab} = \frac{\theta_c}{\theta_a \theta_b} I$$

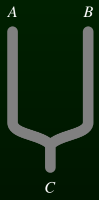
skein for R

monodromy equation

$$R^c_{ba}R^c_{ab} = \frac{\theta_c}{\theta_a\theta_b} I$$

Ribbon expression says:

$R^C_{BA}R^C_{AB}$



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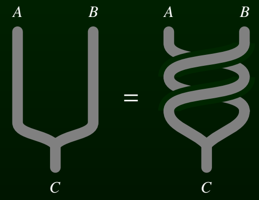

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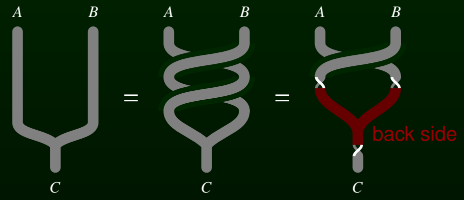

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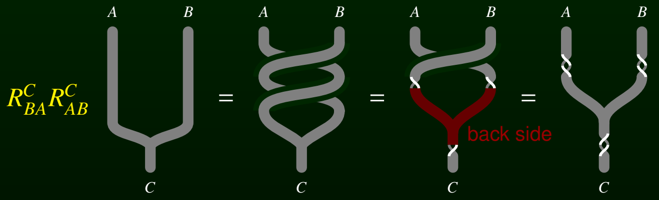

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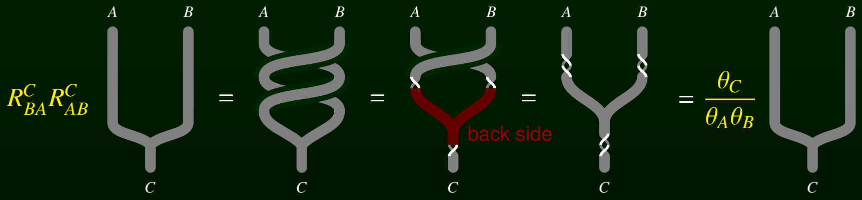
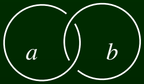
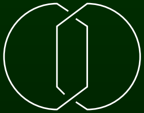


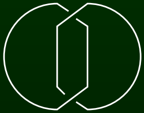
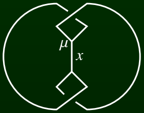
S in modularity

$s_{ab} =$ 

$=$ 

S in modularity

$s_{ab} =$ 

$=$ 
$= \sum_{x,\mu} \mathbf{d}_{ab}^{x}$ 

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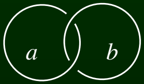
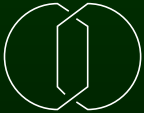
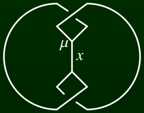
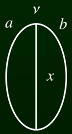
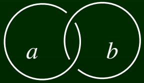
$s_{ab} =$ 

$=$ 
$= \sum_{x,\mu} \mathbf{d}^x_{ab}$ 

$= \sum_{x,\mu,\nu} \mathbf{d}^x_{ab}(R^x_{ba}R^x_{ab})^\mu_\nu$ 

S in modularity

$s_{ab} =$



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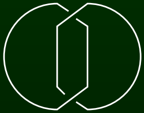

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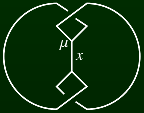

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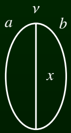
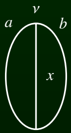


$= \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}^x_{ab}$



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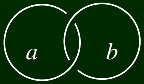

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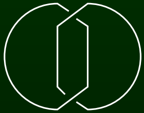
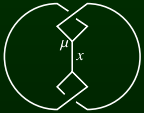
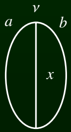


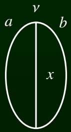
$= \sum_{x,\mu} \mathbf{d}_{ab}^{x}$



$= \sum_{x,\mu,\nu} \mathbf{d}_{ab}^{x} (R_{ba}^{x} R_{ab}^{x})_{\nu}^{\mu}$



$= \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}_{ab}^{x}$



$$= \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}_{ab}^{x} \sqrt{d_a d_b d_x}$$

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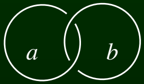

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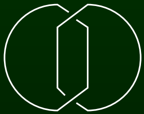

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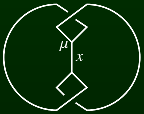

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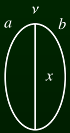

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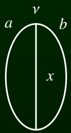

$$= \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}_{ab}^{x} \sqrt{d_a d_b d_x} = \sum_{x} N_{ab}^{x} \frac{\theta_x}{\theta_a \theta_b} d_x .$$

S in modularity

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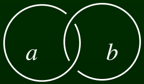

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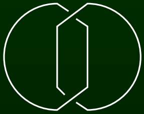
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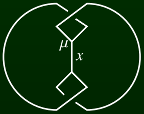

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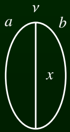

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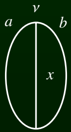

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S is a unitary matrix $\left(\frac{s_{ab}}{\sqrt{\sum_x d_x^2}} \right)$ on $\bigoplus_{x \in \mathcal{F}} V_x$.

TQC models

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- modular tensor category

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- spin network based on Jones-Wenzl projections

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In the remainder, we introduce the spin network model.

Jones projection $\mathbf{e}_k$ in finite algebra

$\mathcal{M}_1 \subset \mathcal{M}_2$: finite dim. C*-alg.

$$\tau = [\mathcal{M}_2 : \mathcal{M}_1]^{-1} = \mathrm{Tr}(\mathbf{e}_1)\ (\mathbf{e}_1 \equiv \mathbf{e}_{\mathcal{M}_1}),\ \ \mathrm{Tr}x\mathbf{e}_1 = \tau \mathrm{Tr}x \quad (x \in \mathcal{M}_2).$$

Consider the tower $\quad (\mathbf{e}_k \equiv \mathbf{e}_{\mathcal{M}_k})$

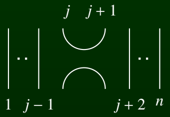
$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \equiv \langle \mathcal{M}_2, \mathbf{e}_1 \rangle \subset \mathcal{M}_4 \equiv \langle \mathcal{M}_3, \mathbf{e}_2 \rangle \subset \cdots$$

Then

$$\mathbf{e}_j^2 = \mathbf{e}_j, \qquad \mathbf{e}_j\mathbf{e}_k = \mathbf{e}_k\mathbf{e}_j \quad (|j-k| \geqq 2), \qquad \mathbf{e}_j\mathbf{e}_k\mathbf{e}_j = \tau\mathbf{e}_j \quad (|j-k| = 1)$$

$\mathbf{f}_k = 1 - \mathbf{e}_1 \vee \cdots \vee \mathbf{e}_{k-1}$: minimal proj. $\quad 0 \leqq \tau \leqq \frac{1}{4}, \tau = \frac{1}{4}\sec^2\frac{\pi}{n}\ (n \geqq 3).$

Temperley-Lieb algebra $T_n(d)$

$T_n(d)$ is the skein algebra generated by the base U_j: 

with the pile product: $\boxed{A} \times \boxed{B} = \begin{array}{|c|}\hline A \\ \hline B \\ \hline\end{array}$ replaced by $d =$ 

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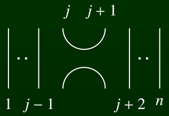
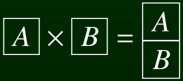
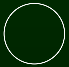
$T_n(d)$ is the skein algebra generated by the base U_j: 

with the pile product: 

replaced by $d =$ 

Then, the generators satisfy

$U_jU_k = U_kU_j \quad (|j-k| \geqq 2),$

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as in the right figure:

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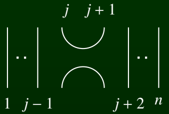
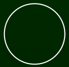
$T_n(d)$ is the skein algebra generated by the base U_j: 

with the pile product: $\boxed{A} \times \boxed{B} = \begin{array}{|c|}\hline A \\ \hline B \\ \hline \end{array}$ replaced by $d =$ 

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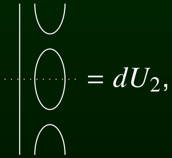
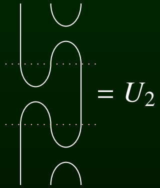


U_2^2



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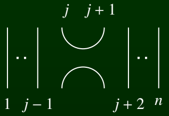
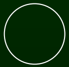
$T_n(d)$ is the skein algebra generated by the base U_j: 

with the pile product: $\boxed{A} \times \boxed{B} = \begin{array}{|c|}\hline A \\ \hline B \\ \hline\end{array}$ replaced by $d =$ 

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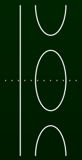

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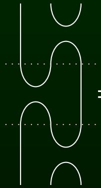


$= U_2$

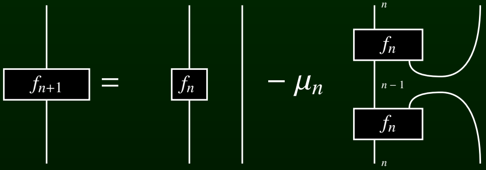
$U_2U_1U_2$

Thus $e_n = \frac{1}{d}U_n$ satisfy the properties of Jones projections for $\tau = \frac{1}{d^2}$.

Jones-Wenzl projection f_n

$$f_1 = \mathbf{1}, \quad f_{n+1} = f_n - \mu_n f_n U_n f_n \qquad \mu_n = -\frac{A^{2n} - A^{-2n}}{A^{2(n+1)} - A^{-2(n+1)}}, \; d = -A^2 - A^2$$

correspond to Jones minimal projections: $\mathbf{f}_k = 1 - \mathbf{e}_1 \vee \cdots \vee \mathbf{e}_{k-1}$



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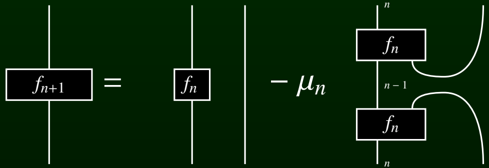
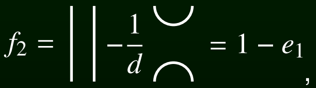
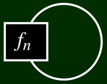




properties of f_n



$$= \Delta_n = (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}, \quad \mu_n = \frac{\Delta_{n-1}}{\Delta_n} = \frac{1}{d - \mu_{n-1}}$$

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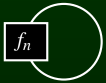

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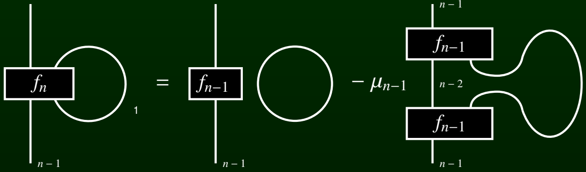

$$= (d - \mu_{n-1}) f_{n-1} \quad (*)$$

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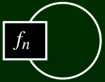

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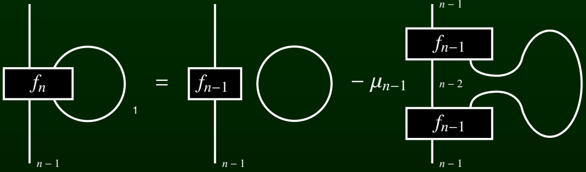

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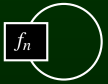

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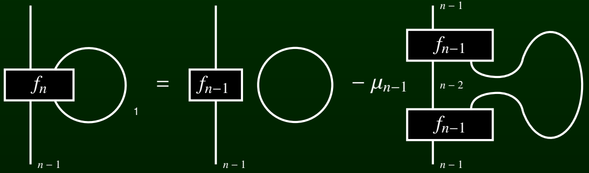

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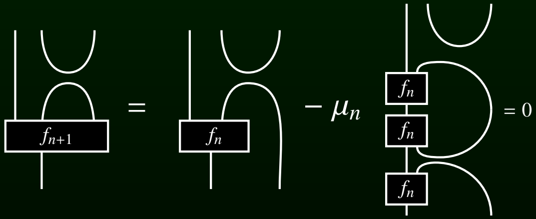

admissible 3-valent graph

$\{a, b, c\}$: integers

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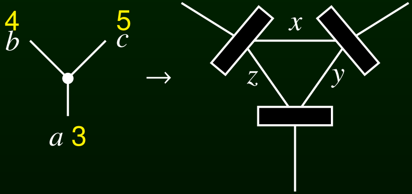
Vertices are expressed via f_n

f_n is often omitted

(expressed by only a box)

$$x = \frac{b+c-a}{2},\ y = \frac{a+c-b}{2},\ z = \frac{a+b-c}{2}$$

are nonnegative



which is equivalent to admissibility.

admissible 3-valent graph

$\{a, b, c\}$: integers

$$a + b + c : \text{even} \iff a + b - c,\ b + c - a,\ c + a - b : \text{even}$$

In addition, all numbers are nonnegative, then they are called admissible

$\{1, 2, 3\}$, $\{3, 4, 5\}$ are admissible, but $\{2, 3, 4\}$, $\{4, 5, 6\}$ are not.

Vertices are expressed via f_n

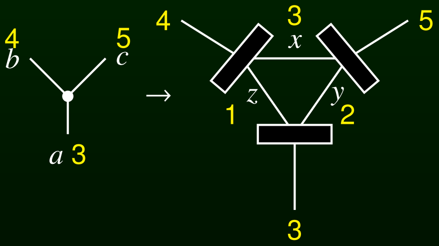
f_n is often omitted

(expressed by only a box)

$$x = \frac{b+c-a}{2},\ y = \frac{a+c-b}{2},\ z = \frac{a+b-c}{2}$$

are nonnegative

which is equivalent to admissibility.



the vertex for $\{3, 4, 5\}$ is admissible

spin network

Spin network is a graph whose vertices • are all admissible.

Every edge is labelled by a number that is just a number of lines in it ,

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Fibonacci model Put $\mathbf{0} = 0$, $\boldsymbol{\tau} = 2$, $A = \exp(\frac{3i\pi}{5})$.

$$\Delta_1 = d = -(A^2 + A^{-2}) = \frac{1+\sqrt{5}}{2} \equiv g = g^2 - 1 = A^4 + 1 + A^{-4} = \Delta_2,$$

$$\Delta_3 = -A^6 - A^{-6} + g = 1 = \Delta_0, \quad \Delta_4 = (-1)^4 \frac{A^{10} - A^{-10}}{A^2 - A^{-2}} = 0.$$

The last fact implies $f_4 = 0$ and hence the admissible $\{2, 2, 4\}$ is vanished.

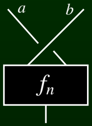
Thus $\mathcal{F} = \{0, 2\}$ and possible numbers are $\underset{\tau\otimes\tau\to\mathbf{0},\ \tau\otimes\mathbf{0}\to\tau}{\{2, 2, 0\}}$, $\underset{\tau\otimes\tau\to\tau}{\{2, 2, 2\}}$,

which represents the Fibonacci particle set.

cross and topological spin

$\sigma_i \equiv$ 

$= A\mathbf{1}_n + A^{-1}U_i$ and $\sigma_i f_n = A f_n + A^{-1}0 = A f_n = f_n \sigma_i$,



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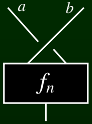 $= A^{ab} f_n$

bracket poly. comp.:  $= (A^{-1} + Ad) \,\big|\, = (A^{-1} - A^3 - A^{-1}) \,\big|\, = -A^3 \,\big|$

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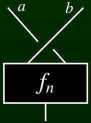

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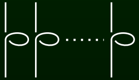
Then putting $R[n]$:  $S[n]$: 

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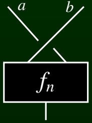

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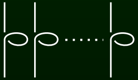
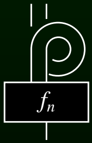
Then putting $R[n]$:  $S[n]$: 

$=$



$= S[n](R[n])^n f_n = (-A^3)^n A^{n(n-1)} f_n = (-1)^n A^{n(n+2)} f_n.$

R matrix

$\widetilde{R^c_{ab}} = (-1)^z A^{-z(z+2)+xy-yz-xz} = (-1)^{\frac{a+b-c}{2}} A^{\frac{c(c+2)-a(a+2)-b(b+2)}{2}}$ by

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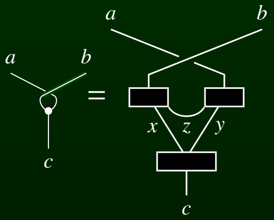

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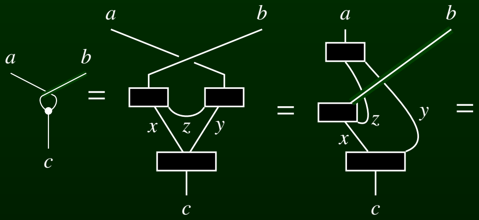

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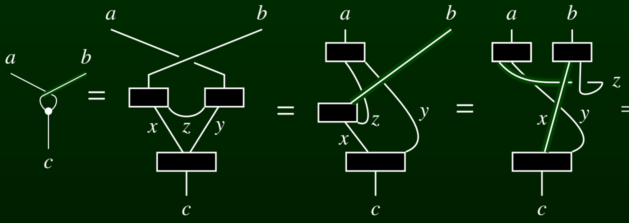

$= (-1)^z A^{-z(z+2)} A^{xy-yz-xz}$

R matrix

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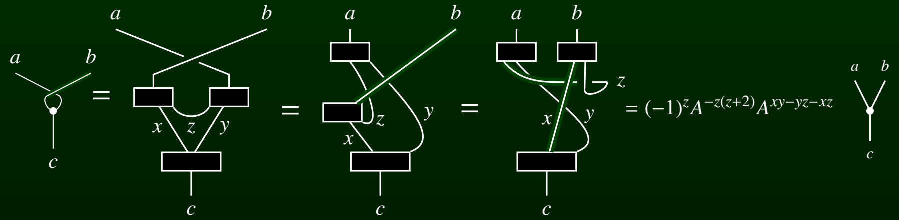

Fibonacci model

$$A = \exp(\tfrac{3i\pi}{5})$$

$$\widetilde{R^0_{22}} = (-1)^2 A^{-8} = \exp\left(\frac{-24i\pi}{5}\right) = \exp\left(\frac{-4i\pi}{5}\right)$$

$$\widetilde{R^2_{22}} = (-1)^1 A^{-4} = -\exp\left(\frac{-12i\pi}{5}\right) = -\exp\left(\frac{-2i\pi}{5}\right) = -\exp\left(\frac{3i\pi}{5}\right)$$

θ net and tetra net

Though the computations are complicated, the following results are known:

θ net and tetra net

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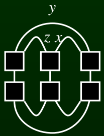
$\tilde{\Theta}(a,b,c) =$

$=$



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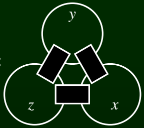
triangle net



$$= \frac{\Delta_{x+y+z}!\,\Delta_{x-1}!\,\Delta_{y-1}!\,\Delta_{z-1}!}{\Delta_{y+z-1}!\,\Delta_{z+x-1}!\,\Delta_{x+y-1}!}$$

θ net and tetra net

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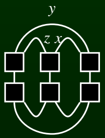
$=$ 

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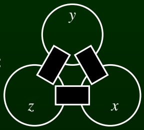

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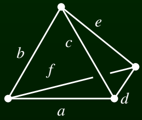
tetra net

$\mathrm{Tet}^{abc}_{efg} =$ 

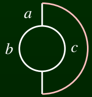
$$= \frac{\prod_{i,j}[b_j - a_i]!}{[a]![b]![c]![d]![e]![f]!} \sum_{m \leqq k \leqq M} \frac{(-1)^k [k+1]!}{\prod_i [k-a_i]! \prod_j [b_j - k]!}$$

where a, b, c, d, e, f are positive and $[n] = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}$ (the quantum integer),

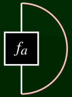
$$a_1 = \frac{a+d+e}{2},\ a_2 = \frac{b+c+e}{2},\ a_3 = \frac{a+b+f}{2},\ a_4 = \frac{c+d+f}{2},\ m = \max\{a_i\},$$

$$b_1 = \frac{b+d+e+f}{2},\ b_2 = \frac{a+c+e+f}{2},\ b_3 = \frac{a+b+c+d}{2},\ M = \min\{b_j\}$$

quantum 6-j symbol



$$= \frac{\tilde{\theta}(a, b, c)}{\Delta_a}$$



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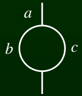

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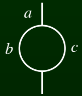

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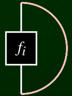
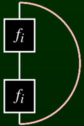
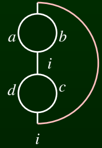


$$=$$



$$= \frac{\Delta_i^2}{\tilde{\theta}(d,a,i)\tilde{\theta}(b,c,i)}$$



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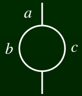
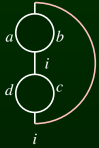


$$= \frac{\tilde{\theta}(a,b,c)}{\Delta_a}$$

$$I = \frac{\Delta_i}{\tilde{\theta}(a,b,i)\tilde{\theta}(c,d,i)}$$



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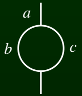

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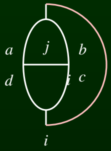
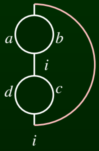


$$= \frac{\mathrm{Tet}^{dai}_{bcj}\,\Delta_i}{\tilde{\theta}(a,b,i)\tilde{\theta}(c,d,i)}$$



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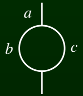

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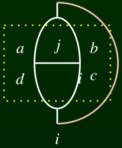

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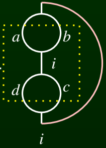
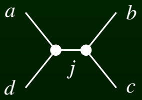
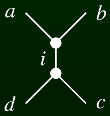




$$= \sum_i \begin{Bmatrix} d & a & i \\ b & c & j \end{Bmatrix}_A$$



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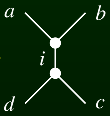

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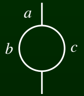

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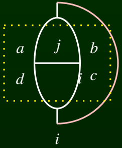

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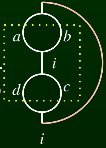

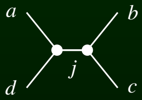

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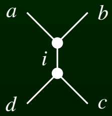

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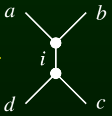
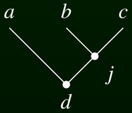
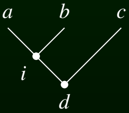




$$= \sum_i (\widetilde{F^d_{abc}})^j_i$$



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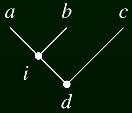
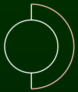
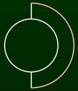


normalization to TQC

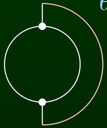
$\sqrt{d_a d_b d_c}$



normalization to TQC



$$\sim \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\tilde{\theta}(a,b,c)}$$



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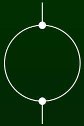

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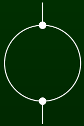
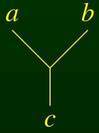
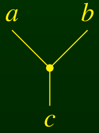
$$\sim \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\tilde{\theta}(a,b,c)}$$



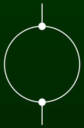


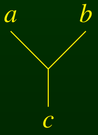
$$\sim \frac{\sqrt[4]{\Delta_a \Delta_b \Delta_c}}{\sqrt{\tilde{\theta}(a,b,c)}}$$



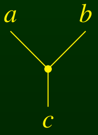
normalization to TQC

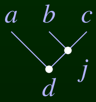
$$\sim \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\tilde{\theta}(a,b,c)}$$



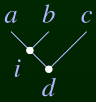


$$\sim \frac{\sqrt[4]{\Delta_a \Delta_b \Delta_c}}{\sqrt{\tilde{\theta}(a,b,c)}}$$



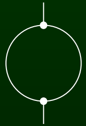


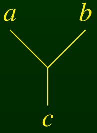
$$= \quad (\widetilde{F^d_{abc}})^j_i$$



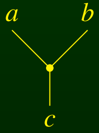
normalization to TQC

$$\sim \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\tilde{\theta}(a,b,c)}$$

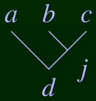




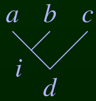
$$\sim \frac{\sqrt[4]{\Delta_a \Delta_b \Delta_c}}{\sqrt{\tilde{\theta}(a,b,c)}}$$



$$\frac{\sqrt{\tilde{\theta}(a,d,j)\tilde{\theta}(b,c,j)}}{\sqrt[4]{\Delta_a \Delta_j \Delta_d \Delta_b \Delta_c \Delta_j}}$$

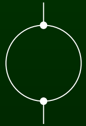


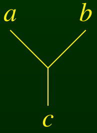
$$= \frac{\sqrt{\tilde{\theta}(a,b,i)\tilde{\theta}(c,d,i)}}{\sqrt[4]{\Delta_a \Delta_b \Delta_i \Delta_c \Delta_d \Delta_i}} \quad (\widetilde{F^d_{abc}})^j_i$$



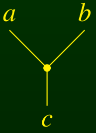
normalization to TQC

$$\sim \frac{\sqrt{\Delta_a\Delta_b\Delta_c}}{\tilde{\theta}(a,b,c)}$$

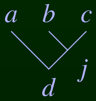




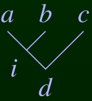
$$\sim \frac{\sqrt[4]{\Delta_a\Delta_b\Delta_c}}{\sqrt{\tilde{\theta}(a,b,c)}}$$



$$\frac{\sqrt{\tilde{\theta}(a,d,j)\tilde{\theta}(b,c,j)}}{\sqrt[4]{\Delta_a\Delta_j\Delta_d\Delta_b\Delta_c\Delta_j}}$$



$$= \frac{\sqrt{\tilde{\theta}(a,b,i)\tilde{\theta}(c,d,i)}}{\sqrt[4]{\Delta_a\Delta_b\Delta_i\Delta_c\Delta_d\Delta_i}} \quad (\widetilde{F^d_{abc}})^j_i$$



Thus, we can obtain F also from the spin network:

$$(F^d_{abc})^j_i = \sqrt{\frac{\Delta_i\Delta_j}{\tilde{\theta}(a,b,i)\tilde{\theta}(c,d,i)\tilde{\theta}(a,d,j)\tilde{\theta}(b,c,i)}}\ \mathrm{Tet}^{dai}_{bcj}$$

In case of Fibonacci model, two types of F coincide.