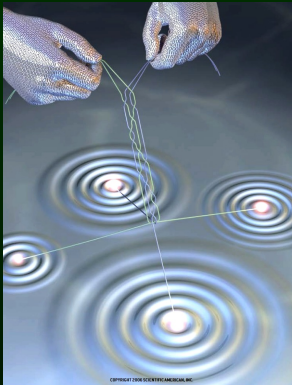


Introduction to TQC theory and spin networks

藤井淳一 (J.I.Fujii)

大阪教育大学 (Osaka Kyoiku Univ.)

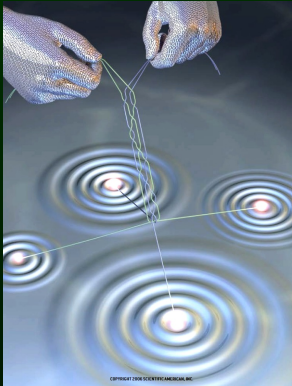
anyon in TQC



Scientific American, 2006

TQC means Topological Quantum Computing

anyon in TQC



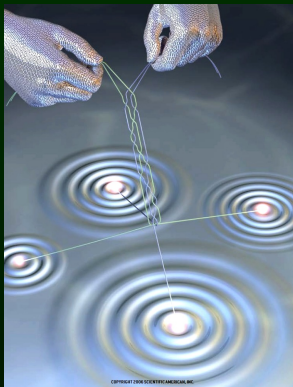
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An anyon is a quasi-particle

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$$|y\rangle \otimes |x\rangle \sim |x\rangle \otimes |y\rangle$$

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Scientific American, 2006

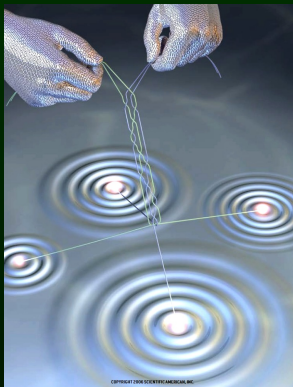
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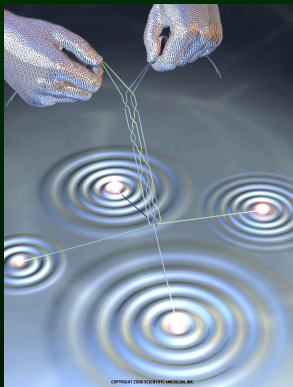
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- (non-abelian) anyon:
 $|y\rangle \otimes |x\rangle \sim e^{iH} (|x\rangle \otimes |y\rangle)$

fusion in TQC

\mathcal{F} : fixed finite set of particles, $a, b \in \mathcal{F}$

$$V_a \otimes V_b \ni a \otimes b \rightarrow \sum_{x \in \mathcal{F}} N_{ab}^x x \in \bigoplus_{x \in \mathcal{F}} V_x, \quad N_{ab}^x \in \mathbb{N} \cup \{0\} \text{ (dimension of } x)$$

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fusion $a \otimes \bar{a} \rightarrow \mathbf{0}$, splitting $\mathbf{0} \rightarrow a \otimes \bar{a}$, $N_{a\bar{a}}^x = \delta_{x,\mathbf{0}}$, $N_{a\mathbf{0}}^x = N_{\mathbf{0}a}^x = \delta_{a,x}$

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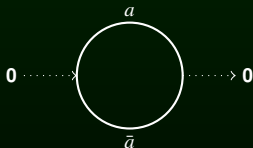
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graphical expression



fusion in TQC

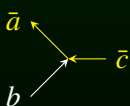
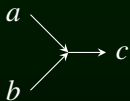
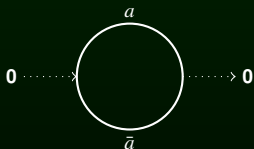
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graphical expression



$$N_{ab}^c = N_{ba}^c = N_{b\bar{c}}^{\bar{a}}$$

Fibonacci anyon τ

$$\mathcal{F} = \{\mathbf{0}, \tau\}, \quad \tau \otimes \tau \rightarrow \mathbf{0} + \tau, \quad N_{\tau\tau}^{\tau} = 1, N_{\tau\tau}^{\mathbf{0}} = 1, N_{\tau\mathbf{0}}^{\mathbf{0}} = 0,$$

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Definition

fusion vectors (fusion trees, conformal blocks)

$$|0\rangle = \begin{array}{c} \tau \quad \tau \quad \tau \\ \diagdown \quad \diagup \\ \tau \end{array}$$

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We want to obtain the braiding matrices $B \in \mathbf{B}(\mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle \oplus \mathbb{C}|N\rangle)$

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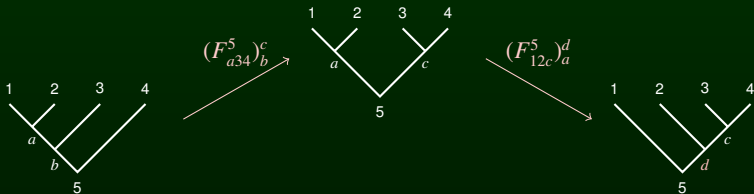
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We want to obtain the braiding matrices $B \in \mathbf{B}(\mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle \oplus \mathbb{C}|N\rangle)$
by using fundamental matrices $F, R \in \mathbf{B}(V_x \oplus V_y)$:

$$\begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \\ d \quad e \end{array} = \sum_{x \in \mathcal{F}} (F_{abc}^d)_x^e \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \\ x \quad d \end{array} \quad \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \mu \end{array} = \sum_{1 \leq \nu \leq N_{ab}^c} (R_{ab}^c)_\nu^\mu \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \nu \\ c \end{array}$$

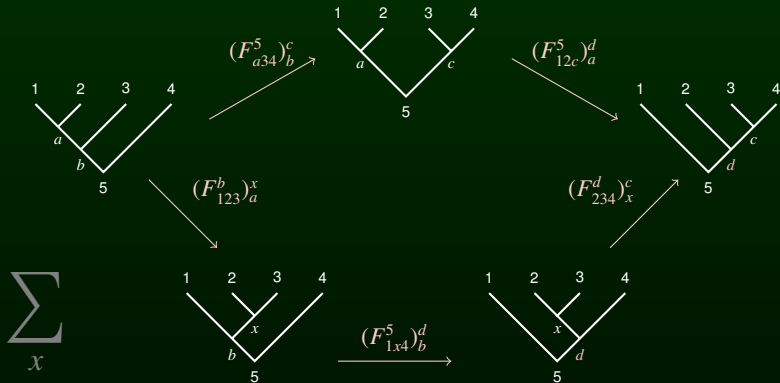
Pentagon axiom

$$(F_{12c}^5)_a^d (F_{a34}^5)_b^c = \sum_x (F_{234}^d)_x^c (F_{1x4}^5)_b^d (F_{123}^b)_a^x$$



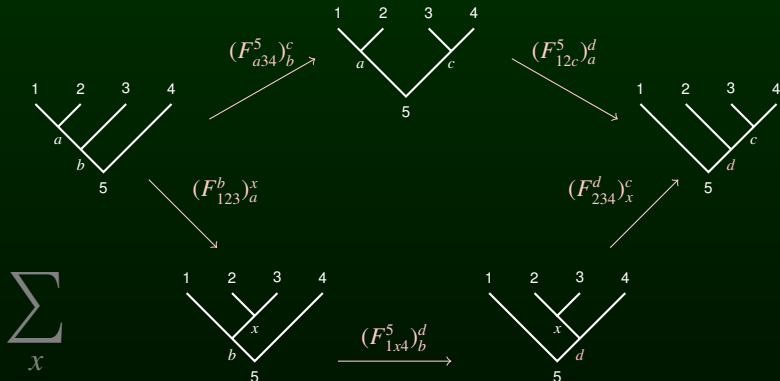
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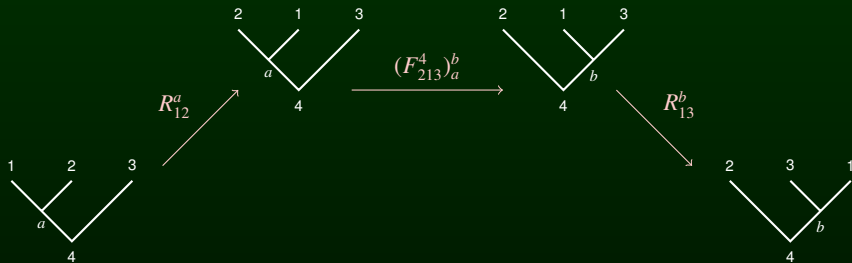


For the Fibonacci, $F_{\tau\tau\tau}^\tau = \frac{1}{g} \begin{pmatrix} 1 & \sqrt{g} \\ \sqrt{g} & -1 \end{pmatrix}$

where $g = \frac{1+\sqrt{5}}{2}$

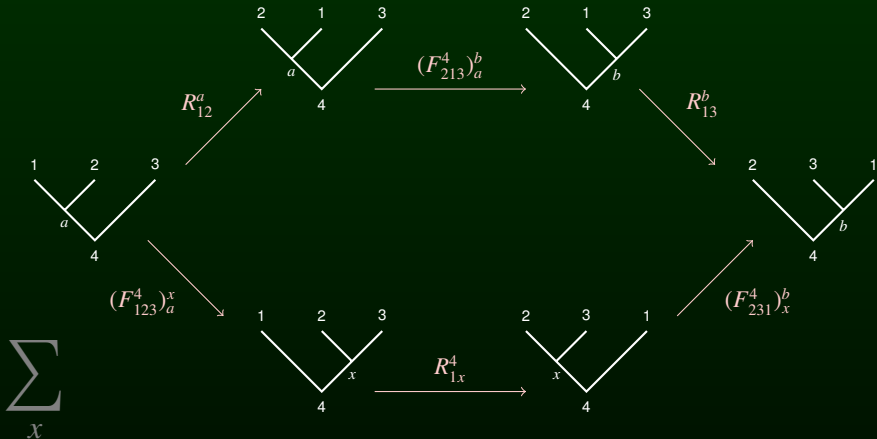
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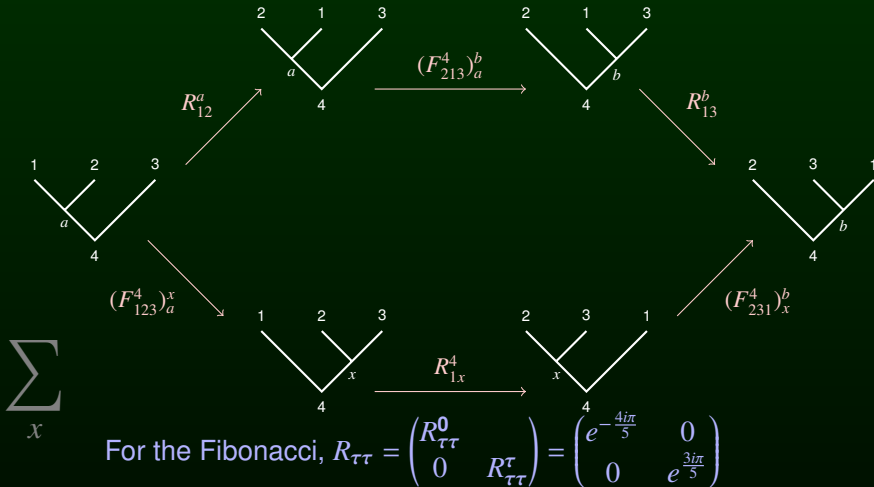
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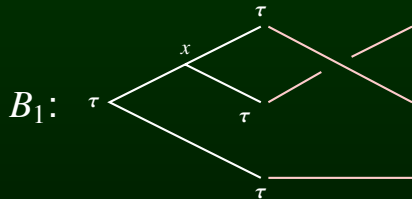


Braiding matrices

What happens in (Fibonacci) vectors $|0\rangle, |1\rangle$ by particle-braiding ?

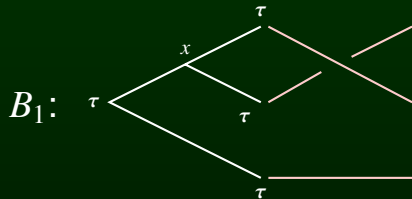
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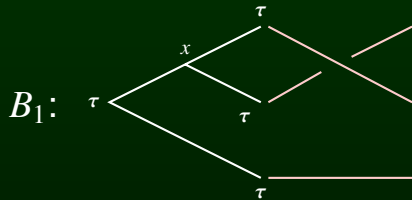
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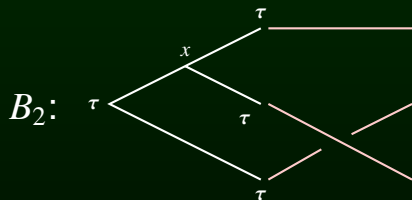
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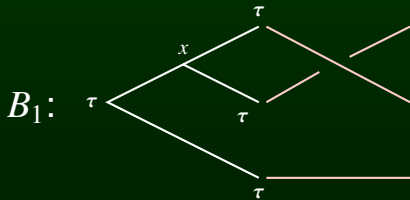


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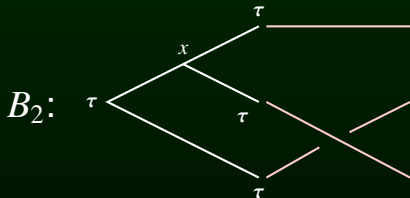


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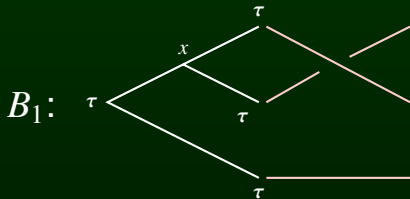


$$\text{compute as } \sum_y (F_{\tau\tau\tau}^\tau)_y^x R_{\tau\tau}^y (F_{\tau\tau\tau}^\tau)_x^y.$$

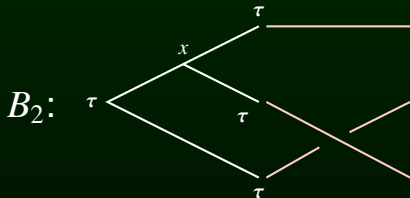
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B_2 is not diagonal

$$B_1 = \begin{pmatrix} e^{-\frac{4i\pi}{5}} & 0 \\ 0 & e^{\frac{3i\pi}{5}} \end{pmatrix}, \quad B_2 = \frac{1}{g} \begin{pmatrix} e^{\frac{4i\pi}{5}} & \sqrt{g}e^{-\frac{3i\pi}{5}} \\ \sqrt{g}e^{-\frac{3i\pi}{5}} & -1 \end{pmatrix} \quad g = \frac{1 + \sqrt{5}}{2}$$

Quantum gate and its universality

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$$\begin{array}{ccc} \text{Hadamard gate} & \frac{\pi}{8} \text{ gate} & \text{CNOT gate} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, & \begin{pmatrix} e^{-\frac{i\pi}{8}} & 0 \\ 0 & e^{\frac{i\pi}{8}} \end{pmatrix}, & \begin{pmatrix} I & O \\ O & X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Quantum gate and its universality

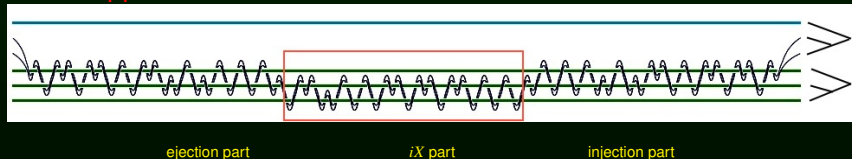
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The Fibonacci set $\{B_1, B_2\}$ is approximately universal:

$$B_1^{-2} B_2^{-4} B_1^4 B_2^{-2} B_1^2 B_2^2 B_1^{-2} B_2^4 B_1^{-2} B_2^4 B_1^2 B_2^{-4} B_1^2 B_2^{-2} B_1^2 B_2^{-2} B_1^{-2} \approx iX$$

CNOT approximation



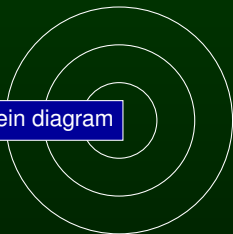
Markov trace

$$\text{Tr} \left[\begin{array}{c} | \\ | \\ | \\ \hline \text{skein diagram} \\ \hline | \\ | \\ | \end{array} \right] = \begin{array}{c} \bigcirc \\ \bigcirc \\ \bigcirc \\ \bigcirc \\ \hline \text{skein diagram} \end{array}$$

Markov trace

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skein diagram



$$\text{Tr} \left| \begin{array}{c} | \\ | \\ | \\ | \\ | \\ | \\ | \\ | \\ | \\ | \end{array} a \right. = \bigcirc_a = d_a$$

$d_0 = 1$

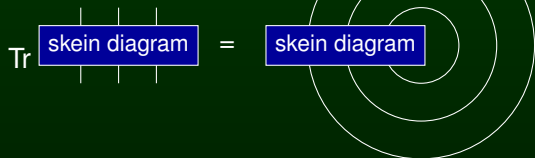
Trace property

$$\text{Tr } X = \text{Tr } Y \Leftrightarrow X \stackrel{\text{mod bracket poly.}}{=} Y, \quad d_a d_b = \sum_x N_{ab}^x d_x \Leftrightarrow a \otimes b = \sum_x N_{ab}^x x$$

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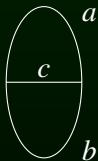
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θ net: $\Theta(a, b, c)$ (if exists)



$$=$$

Markov trace

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$$\text{Tr} \left|_a = \begin{array}{c} \bigcirc \\ a \end{array} = d_a$$

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Markov trace

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fusion and splitting formulae

$$\begin{array}{c} \text{a} \\ \text{b} \end{array} \text{ (concentric circles) } = d_a d_b = \sum_x N_{ab}^x d_x = \sum_x N_{ab}^x \mathbf{d}_{ab}^x \sqrt{d_a d_b d_x} = \sum_{x, \mu}^{\mu=1, 2, \dots, N_{ab}^x} \mathbf{d}_{ab}^x \begin{array}{c} \text{a} \quad \mu \quad \text{b} \\ \text{x} \end{array} \text{ (circle with vertical line) }$$

where $\mathbf{d}_{ab}^x = \sqrt{\frac{d_x}{d_a d_b}}$.

fusion and splitting formulae

$$\begin{array}{c} a \\ \circlearrowleft \\ \circlearrowright \\ \circlearrowleft \\ b \end{array} = d_a d_b = \sum_x N_{ab}^x d_x = \sum_x N_{ab}^x \mathbf{d}_{ab}^x \sqrt{d_a d_b d_x} = \sum_{x,\mu}^{\mu=1,2,\dots,N_{ab}^x} \mathbf{d}_{ab}^x \begin{array}{c} \mu \\ \circlearrowleft \\ \circlearrowright \\ \circlearrowleft \\ x \end{array}$$

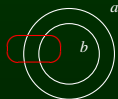
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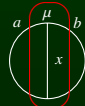
Then

$$\begin{array}{c} | \\ a \end{array} \begin{array}{c} | \\ b \end{array} = \sum_{x,\mu} \mathbf{d}_{ab}^x \begin{array}{c} a \quad \mu \quad b \\ \diagdown \quad | \quad / \\ \quad \quad x \quad \quad \\ / \quad \quad \backslash \end{array}$$

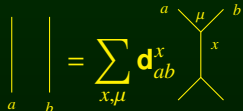
concourse formula

fusion and splitting formulae



$$= d_a d_b = \sum_x N_{ab}^x d_x = \sum_x N_{ab}^x \mathbf{d}_{ab}^x \sqrt{d_a d_b d_x} = \sum_{x, \mu}^{\mu=1, 2, \dots, N_{ab}^x} \mathbf{d}_{ab}^x$$


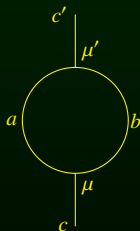
where $\mathbf{d}_{ab}^x = \sqrt{\frac{d_x}{d_a d_b}}$. Then



$$= \sum_{x, \mu} \mathbf{d}_{ab}^x$$

concourse formula

Similarly, the trace computation for $\sqrt{d_a d_b d_x} = \frac{1}{\mathbf{d}_{ab}^x} d_x$, the symmetry says



$$= \frac{\delta_{c, c'} \delta_{\mu, \mu'}}{\mathbf{d}_{ab}^c} \Big|_c$$

popping bubble formula

F and indicator

formulae for $|F\rangle$

$$d_a = \frac{1}{|(F_{a\bar{a}a}^a)_0^0|} \left(\begin{array}{l} \chi_a \equiv d_a (F_{a\bar{a}a}^a)_0^0, \quad |\chi_a| = 1 \\ \text{For } a = \bar{a}, \chi_a \text{ is the Frobenius-Schur indicator } (\pm 1) \\ \text{For } a \neq \bar{a}, \text{ we may assume } \chi_a = 1. \end{array} \right), \quad \mathbf{d}_{ab}^c = |(F_{a\bar{a}b}^c)_0^c|$$

F and indicator

formulae for $|F|$

$$d_a = \frac{1}{|(F_{a\bar{a}a}^a)_0^0|} \left(\begin{array}{l} \chi_a \equiv d_a(F_{a\bar{a}a}^a)_0^0, \quad |\chi_a| = 1 \\ \text{For } a = \bar{a}, \chi_a \text{ is the Frobenius-Schur indicator } (\pm 1) \\ \text{For } a \neq \bar{a}, \text{ we may assume } \chi_a = 1. \end{array} \right), \quad \mathbf{d}_{ab}^c = |(F_{a\bar{a}b}^c)_0^c|$$

$$\overset{a}{\curvearrowright} = \chi_a \overset{a}{\downarrow} \overset{\bar{a}}{\uparrow}$$

$$\chi_{\bar{a}} = \bar{\chi}_a$$

$$\overset{a}{\curvearrowleft} = \bar{\chi}_a \overset{a}{\downarrow} \overset{\bar{a}}{\uparrow}$$

$$d_a(F_{a\bar{a}a}^a)_0^0 \Big|_a = (F_{a\bar{a}a}^a)_0^0 \bigcirc a \Big|_a =$$

F and indicator

formulae for $|F|$

$$d_a = \frac{1}{|(F_{a\bar{a}a}^a)_0^0|} \left(\begin{array}{l} \chi_a \equiv d_a(F_{a\bar{a}a}^a)_0^0, \quad |\chi_a| = 1 \\ \text{For } a = \bar{a}, \chi_a \text{ is the Frobenius-Schur indicator } (\pm 1), \\ \text{For } a \neq \bar{a}, \text{ we may assume } \chi_a = 1. \end{array} \right), \quad \mathbf{d}_{ab}^c = |(F_{a\bar{a}b}^c)_0^c|$$

$$\begin{array}{c} a \\ \curvearrowright \\ = \chi_a \end{array} \begin{array}{c} a \\ \nearrow \\ \bar{a} \\ \downarrow \end{array}$$

$$\chi_{\bar{a}} = \bar{\chi}_a$$

$$\begin{array}{c} \curvearrowleft \\ a \\ = \bar{\chi}_a \end{array} \begin{array}{c} \downarrow \\ a \\ \bar{a} \end{array}$$

$$d_a(F_{a\bar{a}a}^a)_0^0 \left| \begin{array}{c} \\ \\ \\ a \end{array} \right. = (F_{a\bar{a}a}^a)_0^0 \begin{array}{c} \circlearrowleft \\ a \\ \curvearrowright \\ a \end{array} \left| \begin{array}{c} \\ \\ \\ a \end{array} \right. =$$

F and indicator

formulae for $|F|$

$$d_a = \frac{1}{|(F_{a\bar{a}a}^a)_0^0|} \left(\begin{array}{l} \chi_a \equiv d_a(F_{a\bar{a}a}^a)_0^0, \quad |\chi_a| = 1 \\ \text{For } a = \bar{a}, \chi_a \text{ is the Frobenius-Schur indicator } (\pm 1), \\ \text{For } a \neq \bar{a}, \text{ we may assume } \chi_a = 1. \end{array} \right), \quad \mathbf{d}_{ab}^c = |(F_{a\bar{a}b}^c)_0^c|$$

$$\begin{array}{c} a \\ \curvearrowright \\ \vdots \\ a \end{array} = \chi_a \begin{array}{c} a \\ \swarrow \\ \bar{a} \\ \downarrow \\ \vdots \\ a \end{array}$$

$$\chi_{\bar{a}} = \bar{\chi}_a$$

$$\begin{array}{c} \curvearrowright \\ a \\ \vdots \\ a \end{array} = \bar{\chi}_a \begin{array}{c} \swarrow \\ \bar{a} \\ \downarrow \\ \vdots \\ a \end{array}$$

$$d_a(F_{a\bar{a}a}^a)_0^0 \left| \begin{array}{c} \vdots \\ \vdots \\ a \end{array} \right. = (F_{a\bar{a}a}^a)_0^0 \left| \begin{array}{c} \vdots \\ \vdots \\ a \end{array} \right. = \left| \begin{array}{c} \bar{a} \\ \swarrow \\ a \end{array} \right| = \chi_a \left| \begin{array}{c} \vdots \\ \vdots \\ a \end{array} \right.$$

F and indicator

formulae for $|F|$

$$d_a = \frac{1}{|(F_{a\bar{a}a}^a)_0|} \left(\begin{array}{l} \chi_a \equiv d_a(F_{a\bar{a}a}^a)_0, \quad |\chi_a| = 1 \\ \text{For } a = \bar{a}, \chi_a \text{ is the Frobenius-Schur indicator } (\pm 1), \\ \text{For } a \neq \bar{a}, \text{ we may assume } \chi_a = 1. \end{array} \right), \quad \mathbf{d}_{ab}^c = |(F_{a\bar{a}b}^c)_0|$$

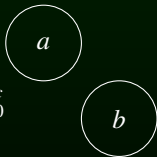
$$\begin{array}{c} a \\ \curvearrowright \\ = \chi_a \end{array} \begin{array}{c} a \\ \nearrow \\ \bar{a} \end{array}$$

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$$\begin{array}{c} \curvearrowright \\ a \\ \bar{a} \end{array} = \bar{\chi}_a \begin{array}{c} \searrow \\ a \\ \bar{a} \end{array}$$

$$d_a(F_{a\bar{a}a}^a)_0 \left| \begin{array}{c} \vdots \\ \vdots \\ a \\ \vdots \\ \vdots \end{array} \right|_a = (F_{a\bar{a}a}^a)_0 \left| \begin{array}{c} \vdots \\ \vdots \\ \textcircled{a} \\ \vdots \\ \vdots \end{array} \right|_a = \left| \begin{array}{c} \vdots \\ \vdots \\ \bar{a} \\ \vdots \\ \vdots \end{array} \right|_a = \chi_a \left| \begin{array}{c} \vdots \\ \vdots \\ a \\ \vdots \\ \vdots \end{array} \right|_a$$

$$(F_{a\bar{a}b}^b)_0 d_a d_b = (F_{a\bar{a}b}^b)_0$$



F and indicator

formulae for $|F|$

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$$\begin{array}{c} a \\ \curvearrowleft \\ \vdots \\ a \end{array} = \chi_a \begin{array}{c} a \bar{a} \\ \nearrow \searrow \\ \vdots \\ a \end{array}$$

$$\chi_{\bar{a}} = \bar{\chi}_a$$

$$\begin{array}{c} \curvearrowright \\ a \\ \vdots \\ a \bar{a} \end{array} = \bar{\chi}_a \begin{array}{c} \vdots \\ \searrow \nearrow \\ a \end{array}$$

$$d_a(F_{a\bar{a}a}^a)_0 \left| \begin{array}{c} \vdots \\ \vdots \\ a \end{array} \right| = (F_{a\bar{a}a}^a)_0 \left| \begin{array}{c} \vdots \\ \textcircled{a} \\ \vdots \\ a \end{array} \right| = \left| \begin{array}{c} \vdots \\ \bar{a} \\ \vdots \\ a \end{array} \right| = \chi_a \left| \begin{array}{c} \vdots \\ a \\ \vdots \\ a \end{array} \right|$$

$$(F_{a\bar{a}b}^b)_0 d_a d_b = (F_{a\bar{a}b}^b)_0 \left| \begin{array}{c} \textcircled{a} \\ \curvearrowright \\ \textcircled{b} \end{array} \right|$$

F and indicator

formulae for $|F|$

$$d_a = \frac{1}{|(F_{a\bar{a}a}^a)_0^0|} \left(\begin{array}{l} \chi_a \equiv d_a(F_{a\bar{a}a}^a)_0^0, \quad |\chi_a| = 1 \\ \text{For } a = \bar{a}, \chi_a \text{ is the Frobenius-Schur indicator } (\pm 1) \\ \text{For } a \neq \bar{a}, \text{ we may assume } \chi_a = 1. \end{array} \right), \quad \mathbf{d}_{ab}^c = |(F_{a\bar{a}b}^b)_0^c|$$

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$$d_a(F_{a\bar{a}a}^a)_0^0 = (F_{a\bar{a}a}^a)_0^0 \begin{array}{c} \text{---} \\ \circlearrowleft \\ a \end{array} = \begin{array}{c} \text{---} \\ \nearrow \bar{a} \\ \searrow \\ a \end{array} = \chi_a \begin{array}{c} | \\ a \end{array}$$

$$(F_{a\bar{a}b}^b)_0^c d_a d_b = (F_{a\bar{a}b}^b)_0^c \begin{array}{c} \text{---} \\ \circlearrowleft \\ a \end{array} \begin{array}{c} \text{---} \\ \nearrow \\ b \end{array} = \chi_a \begin{array}{c} a \\ \text{---} \\ c \\ \text{---} \\ b \end{array} = \chi_a \sqrt{d_a d_b d_c} = \chi_a \mathbf{d}_{ab}^c d_a d_b$$

topological spin (Dehn twist) θ_a

This is a typical concept in TQC:

$$\theta_a \left| \begin{array}{c} | \\ a \end{array} \right. = \left| \begin{array}{c} | \\ \text{loop} \end{array} \right. \quad \overline{\theta}_a \left| \begin{array}{c} | \\ a \end{array} \right. = \left| \begin{array}{c} | \\ \text{loop} \end{array} \right. \quad \overline{\theta}_a = \theta_a^{-1}$$

topological spin (Dehn twist) θ_a

This is a typical concept in TQC:

$$\theta_a \Big|_a = \text{loop} \quad \overline{\theta}_a \Big|_a = \text{loop} \quad \overline{\theta}_a = \theta_a^{-1}$$

The ribbon figure

$$\theta_A \Big|_A = \text{ribbon loop} = \text{crossing} \quad \overline{\theta}_A \Big|_A = \text{crossing}$$

difference between knots & TQC

$$\text{bracket polynomial: } \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = A \quad \left(\begin{array}{c} + \\ - \end{array} \right) + A^{-1} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad d = -A^2 - A^{-2}$$

difference between knots & TQC

bracket polynomial: $\diagdown = A \diagup$ $\left(+ A^{-1} \begin{array}{c} \frown \\ \smile \end{array} \right) d = -A^2 - A^{-2}$

spin: $\bigcirc \bigcirc = A \bigcirc \bigcirc + A^{-1} \bigcirc \bigcirc = (Ad + A^{-1})d = -A^3 d$

difference between knots & TQC

bracket polynomial: $\begin{array}{c} \diagup \\ \diagdown \end{array} = A \quad \left(\begin{array}{c} + \\ - \end{array} \right) + A^{-1} \begin{array}{c} \diagdown \\ \diagup \end{array} \quad d = -A^2 - A^{-2}$

spin: $\begin{array}{c} \diagup \\ \diagdown \end{array} = A \begin{array}{c} \circ \\ \circ \end{array} + A^{-1} \begin{array}{c} \circ \\ \circ \end{array} = (Ad + A^{-1})d = -A^3d$

Reidemeister move



knot invariant: Jones polynomial

difference between knots & TQC

bracket polynomial: $\begin{array}{c} \diagup \\ \diagdown \end{array} = A \left(\begin{array}{c} \diagup \\ \diagup \end{array} + A^{-1} \begin{array}{c} \diagdown \\ \diagdown \end{array} \right) \quad d = -A^2 - A^{-2}$

spin: $\begin{array}{c} \diagup \\ \diagdown \end{array} = A \begin{array}{c} \circ \\ \circ \end{array} + A^{-1} \begin{array}{c} \circ \\ \circ \end{array} = (Ad + A^{-1})d = -A^3d$

Reidemeister move



knot invariant: Jones polynomial

Considering the topological spin:

skein	fitting polynomial
knot theory <i>SU</i> (2) type	Jones polynomial
<i>SU</i> (<i>n</i>) type	HOMFLY polynomial
TQC theory	bracket polynomial

skein for \mathcal{R}

monodromy equation

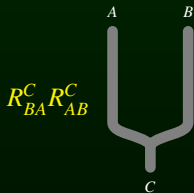
$$R_{ba}^c R_{ab}^c = \frac{\theta_c}{\theta_a \theta_b} I$$

skein for \mathcal{R}

monodromy equation

$$R_{ba}^c R_{ab}^c = \frac{\theta_c}{\theta_a \theta_b} I$$

Ribbon expression says:

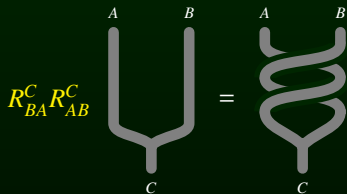


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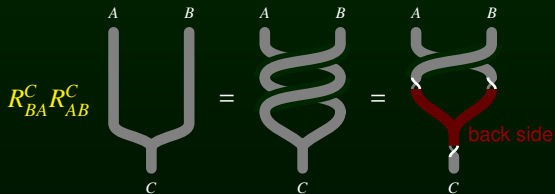


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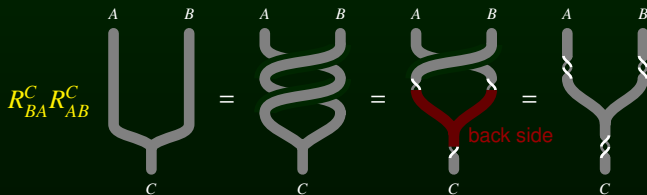


skein for \mathcal{R}

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skein for \mathcal{R}

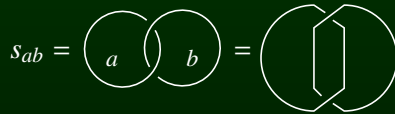
monodromy equation

$$R_{ba}^c R_{ab}^c = \frac{\theta_c}{\theta_a \theta_b} I$$

Ribbon expression says:

The diagram illustrates the monodromy equation in ribbon theory. It shows a sequence of five diagrams connected by equals signs, representing the equation $R_{BA}^C R_{AB}^C = \frac{\theta_C}{\theta_A \theta_B} I$.
1. The first diagram is the product of two braiding operators, R_{BA}^C and R_{AB}^C , shown as two strands (A and B) crossing each other twice. The label $R_{BA}^C R_{AB}^C$ is written in yellow to the left.
2. The second diagram shows the strands A and B crossing each other twice in a different configuration, forming a full twist.
3. The third diagram shows the strands A and B crossing each other once, with the bottom crossing highlighted in red and labeled "back side".
4. The fourth diagram shows the strands A and B crossing each other once, with the top crossing highlighted in white with a black outline.
5. The fifth diagram is the identity operator I , where the strands A and B do not cross. The label $\frac{\theta_C}{\theta_A \theta_B} I$ is written in yellow to the left of this diagram.
The strands are labeled A, B, and C at their respective ends.

S in modularity

$$s_{ab} = \begin{array}{c} \text{---} \\ \circ \quad \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circ \quad \text{---} \\ \text{---} \end{array}$$


S in modularity

$$s_{ab} = \begin{array}{c} \text{---} \\ \circ \quad \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array} = \sum_{x,\mu} \mathbf{d}_{ab}^x \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array}$$

The diagram shows the decomposition of the product of two circles, labeled a and b , into a sum over indices x and μ . The first circle is labeled a and the second is labeled b . The second diagram shows a single circle containing a vertical rectangle with a horizontal line through its center. The third diagram shows a single circle containing a vertical line with a horizontal line through its center, and a small square with a horizontal line through its center, both connected to the vertical line. The indices x and μ are labeled near the horizontal lines in the third diagram.

S in modularity

$$\begin{aligned}
 S_{ab} &= \text{diagram of two overlapping circles } a \text{ and } b = \text{diagram of two overlapping circles with a vertical line} = \sum_{x,\mu} \mathbf{d}_{ab}^x \text{diagram of two overlapping circles with a vertical line and labels } \mu, x \\
 &= \sum_{x,\mu,\nu} \mathbf{d}_{ab}^x (R_{ba}^x R_{ab}^x)^\mu \text{diagram of an oval with a vertical line and labels } a, \nu, b, x
 \end{aligned}$$

S in modularity

$$\begin{aligned}
 S_{ab} &= \text{Diagram of two overlapping circles } a \text{ and } b = \text{Diagram of two overlapping circles with a vertical line } x \text{ through the intersection} \\
 &= \sum_{x,\mu} \mathbf{d}_{ab}^x \text{Diagram of two overlapping circles with a vertical line } x \text{ through the intersection and a zigzag line } \mu \text{ connecting the top and bottom of } x \\
 &= \sum_{x,\mu,\nu} \mathbf{d}_{ab}^x (R_{ba}^x R_{ab}^x)^\mu_\nu \text{Diagram of a vertical oval with top labels } a, \nu, b \text{ and bottom label } x \\
 &= \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}_{ab}^x \text{Diagram of a vertical oval with top labels } a, \nu, b \text{ and bottom label } x
 \end{aligned}$$

S in modularity

$$\begin{aligned}
 S_{ab} &= \text{Diagram of two overlapping circles } a \text{ and } b = \text{Diagram of a circle with a vertical line } x \text{ through it} = \sum_{x,\mu} \mathbf{d}_{ab}^x \text{Diagram of a circle with a vertical line } x \text{ and a zigzag line } \mu \\
 &= \sum_{x,\mu,\nu} \mathbf{d}_{ab}^x (R_{ba}^x R_{ab}^x)_{\nu}^{\mu} \text{Diagram of a circle with a vertical line } x \text{ and labels } a, \nu, b \text{ at the top} = \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}_{ab}^x \text{Diagram of a circle with a vertical line } x \text{ and labels } a, \nu, b \text{ at the top} \\
 &= \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}_{ab}^x \sqrt{d_a d_b d_x}
 \end{aligned}$$

S in modularity

$$\begin{aligned}
 S_{ab} &= \text{Diagram of two overlapping circles } a \text{ and } b = \text{Diagram of a circle with a vertical line } x \text{ through it} = \sum_{x,\mu} \mathbf{d}_{ab}^x \text{Diagram of a circle with a vertical line } x \text{ and a horizontal line } \mu \text{ through it} \\
 &= \sum_{x,\mu,\nu} \mathbf{d}_{ab}^x (R_{ba}^x R_{ab}^x)^\mu \text{Diagram of a circle with a vertical line } x \text{ and a horizontal line } \nu \text{ through it} = \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}_{ab}^x \text{Diagram of a circle with a vertical line } x \text{ and a horizontal line } \nu \text{ through it} \\
 &= \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}_{ab}^x \sqrt{d_a d_b d_x} = \sum_x N_{ab}^x \frac{\theta_x}{\theta_a \theta_b} d_x.
 \end{aligned}$$

S in modularity

$$\begin{aligned}
 s_{ab} &= \text{Diagram of two overlapping circles } a \text{ and } b = \text{Diagram of two overlapping circles with a vertical line } x \text{ through the intersection} = \sum_{x,\mu} \mathbf{d}_{ab}^x \text{Diagram of two overlapping circles with a vertical line } x \text{ and a crossing } \mu \\
 &= \sum_{x,\mu,\nu} \mathbf{d}_{ab}^x (R_{ba}^x R_{ab}^x)^\mu \text{Diagram of a circle with a vertical line } x \text{ and a crossing } \nu = \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}_{ab}^x \text{Diagram of a circle with a vertical line } x \\
 &= \sum_{x,\nu} \frac{\theta_x}{\theta_a \theta_b} \mathbf{d}_{ab}^x \sqrt{d_a d_b d_x} = \sum_x N_{ab}^x \frac{\theta_x}{\theta_a \theta_b} d_x.
 \end{aligned}$$

S is a unitary matrix $\left(\frac{s_{ab}}{\sqrt{\sum_x d_x^2}} \right)$ on $\bigoplus_{x \in \mathcal{F}} V_x$.

TQC models

TQC models

- modular tensor category

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- modular tensor category
- spin network based on Jones-Wenzl projections

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- quantum group model, $(\mathfrak{su}(2)_k$ WZW model)

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In the remainder, we introduce the spin network model.

Jones projection \mathbf{e}_k in finite algebra

$\mathcal{M}_1 \subset \mathcal{M}_2$: finite dim. C^* -alg.

$$\tau = [\mathcal{M}_2 : \mathcal{M}_1]^{-1} = \text{Tr}(\mathbf{e}_1) \quad (\mathbf{e}_1 \equiv \mathbf{e}_{\mathcal{M}_1}), \quad \text{Tr}x\mathbf{e}_1 = \tau \text{Tr}x \quad (x \in \mathcal{M}_2).$$

Consider the tower $(\mathbf{e}_k \equiv \mathbf{e}_{\mathcal{M}_k})$

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \equiv \langle \mathcal{M}_2, \mathbf{e}_1 \rangle \subset \mathcal{M}_4 \equiv \langle \mathcal{M}_3, \mathbf{e}_2 \rangle \subset \cdots$$

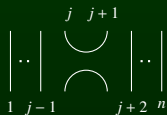
Then

$$\mathbf{e}_j^2 = \mathbf{e}_j, \quad \mathbf{e}_j\mathbf{e}_k = \mathbf{e}_k\mathbf{e}_j \quad (|j - k| \geq 2), \quad \mathbf{e}_j\mathbf{e}_k\mathbf{e}_j = \tau\mathbf{e}_j \quad (|j - k| = 1)$$

$$\mathbf{f}_k = 1 - \mathbf{e}_1 \vee \cdots \vee \mathbf{e}_{k-1}: \text{minimal proj.} \quad 0 \leq \tau \leq \frac{1}{4}, \tau = \frac{1}{4} \sec^2 \frac{\pi}{n} \quad (n \geq 3).$$

Temperley-Lieb algebra $T_n(d)$

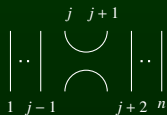
$T_n(d)$ is the skein algebra generated by the base U_j :



with the pile product: $\boxed{A} \times \boxed{B} = \frac{\boxed{A}}{\boxed{B}}$ replaced by $d = \bigcirc$

Temperley-Lieb algebra $T_n(d)$

$T_n(d)$ is the skein algebra generated by the base U_j :



with the pile product: $\boxed{A} \times \boxed{B} = \frac{A}{B}$ replaced by $d = \bigcirc$

Then, the generators satisfy

$$U_j U_k = U_k U_j \quad (|j - k| \geq 2),$$

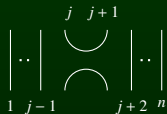
$$U_j^2 = d U_j \quad \text{and}$$

$$U_j U_k U_j = U_j \quad (|j - k| = 1)$$

as in the right figure:

Temperley-Lieb algebra $T_n(d)$

$T_n(d)$ is the skein algebra generated by the base U_j :



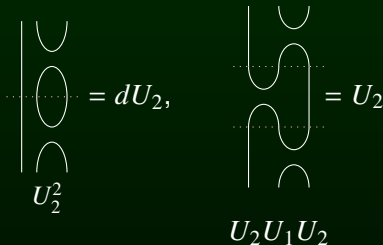
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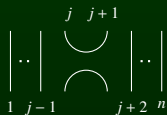
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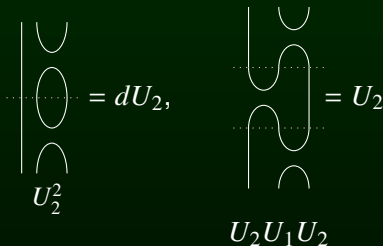
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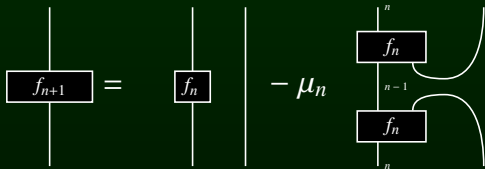
as in the right figure:

Thus $e_n = \frac{1}{d} U_n$ satisfy the properties of Jones projections for $\tau = \frac{1}{d^2}$.

Jones-Wenzl projection f_n

$$f_1 = \mathbf{1}, \quad f_{n+1} = f_n - \mu_n f_n U_n f_n \quad \mu_n = -\frac{A^{2n} - A^{-2n}}{A^{2(n+1)} - A^{-2(n+1)}}, \quad d = -A^2 - A^{-2}$$

correspond to Jones minimal projections: $\mathbf{f}_k = \mathbf{1} - \mathbf{e}_1 \vee \cdots \vee \mathbf{e}_{k-1}$



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$$f_{n+1} = f_n - \mu_n \left(\begin{array}{c} \text{---}^n \\ | \\ \boxed{f_n} \\ | \\ \text{---}^{n-1} \\ | \\ \boxed{f_n} \\ | \\ \text{---}^n \end{array} \right)$$

$$f_2 = \left| \left| -\frac{1}{d} \begin{array}{c} \cup \\ \cup \end{array} \right. \right| = 1 - e_1,$$

$$f_3 = \left| \left| \left| -\frac{d}{d^2 - 1} \left(\begin{array}{c} \cup \\ \cup \end{array} + \begin{array}{c} \cup \\ \cup \end{array} \right) \right. \right. + \frac{1}{d^2 - 1} \left(\begin{array}{c} \cup \\ \cup \end{array} + \begin{array}{c} \cup \\ \cup \end{array} \right) \right|$$

properties of f_n

$$\boxed{f_n} \circlearrowleft = \Delta_n = (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}, \quad \mu_n = \frac{\Delta_{n-1}}{\Delta_n} = \frac{1}{d - \mu_{n-1}}$$

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$$\begin{aligned} \boxed{f_n} \bigcirc_{n-1} &= \boxed{f_{n-1}} \bigcirc_{n-1} - \mu_{n-1} \begin{array}{c} \boxed{f_{n-1}} \\ | \\ \boxed{f_{n-1}} \\ | \\ \bigcirc_{n-1} \end{array} = (d - \mu_{n-1})f_{n-1} \quad (*) \\ &= \frac{1}{\mu_n} f_{n-1} \end{aligned}$$

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$$f_n U_i = U_i f_n = 0 \quad (i < n)$$

$$\mathbf{f}_n \mathbf{e}_i = \mathbf{e}_i - \mathbf{e}_i (\mathbf{e}_1 \vee \dots \vee \mathbf{e}_{n-1})$$

$$\boxed{f_{n+1}} \text{ (cup and cap) } = \boxed{f_n} \text{ (cup and cap) } - \mu_n \left(\boxed{f_n} \text{ (cup and cap) } \right) = 0$$

admissible 3-valent graph

$\{a, b, c\}$: integers

$$a + b + c : \text{even} \iff a + b - c, b + c - a, c + a - b : \text{even}$$

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Vertices are expressed via f_n

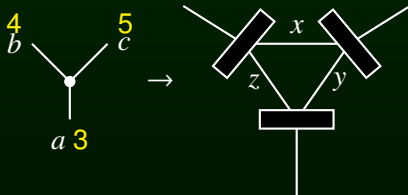
f_n is often omitted

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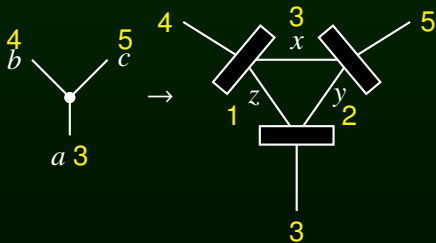
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the vertex for $\{3, 4, 5\}$ is admissible

spin network

Spin network is a graph whose vertices \bullet are all admissible.

Every edge is labelled by a number that is just a number of lines in it ,

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Fibonacci model Put $\mathbf{0} = 0, \tau = 2, A = \exp(\frac{3i\pi}{5})$.

$$\Delta_1 = d = -(A^2 + A^{-2}) = \frac{1 + \sqrt{5}}{2} \equiv g = g^2 - 1 = A^4 + 1 + A^{-4} = \Delta_2,$$

$$\Delta_3 = -A^6 - A^{-6} + g = 1 = \Delta_0, \quad \Delta_4 = (-1)^4 \frac{A^{10} - A^{-10}}{A^2 - A^{-2}} = 0.$$

The last fact implies $f_4 = 0$ and hence the admissible $\{2, 2, 4\}$ is vanished.


Thus $\mathcal{F} = \{0, 2\}$ and possible numbers are $\{2, 2, 0\}, \{2, 2, 2\}$,

$$\tau \otimes \tau \rightarrow \mathbf{0}, \quad \tau \otimes \mathbf{0} \rightarrow \tau, \quad \tau \otimes \tau \rightarrow \tau$$

which represents the Fibonacci particle set.

cross and topological spin

$$\sigma_i \equiv \left| \begin{array}{c} \cdot \\ \cdot \\ \vdots \end{array} \right|_1 \left| \begin{array}{c} \cdot \\ \cdot \\ \vdots \end{array} \right|_{j-1} \times \left| \begin{array}{c} \cdot \\ \cdot \\ \vdots \end{array} \right|_{j+2} \left| \begin{array}{c} \cdot \\ \cdot \\ \vdots \end{array} \right|_n = A \mathbf{1}_n + A^{-1} U_i \quad \text{and} \quad \sigma_i f_n = A f_n + A^{-1} 0 = A f_n = f_n \sigma_i,$$



The diagram shows two lines, labeled 'a' and 'b', crossing each other. The line labeled 'a' starts from the top-left and goes to the bottom-right. The line labeled 'b' starts from the top-right and goes to the bottom-left. Below the crossing, there is a rectangular box containing the symbol f_n . A vertical line extends downwards from the bottom center of this box.

$$= A^{ab} f_n$$

cross and topological spin

$$\sigma_i \equiv \left| \begin{array}{c} \cdot \\ \cdot \\ \vdots \end{array} \right|_{1 \dots j-1} \times \left| \begin{array}{c} \cdot \\ \cdot \\ \vdots \end{array} \right|_{j+2 \dots n} = A \mathbf{1}_n + A^{-1} U_i \quad \text{and} \quad \sigma_i f_n = A f_n + A^{-1} 0 = A f_n = f_n \sigma_i,$$

$$\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \boxed{f_n} \\ | \end{array} = A^{ab} f_n$$

$$\text{bracket poly. comp.:} \quad \left| \rho = (A^{-1} + Ad) \right| = (A^{-1} - A^3 - A^{-1}) \left| \right| = -A^3 \left| \right|$$

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Then putting $R[n]:$  $S[n]:$ 

cross and topological spin

$$\sigma_i \equiv \left| \begin{array}{c} \cdot \\ \cdot \end{array} \right|_1 \left| \begin{array}{c} \cdot \\ \cdot \end{array} \right|_{j-1} \times \left| \begin{array}{c} \cdot \\ \cdot \end{array} \right|_{j+2} \left| \begin{array}{c} \cdot \\ \cdot \end{array} \right|_n = A \mathbf{1}_n + A^{-1} U_i \quad \text{and} \quad \sigma_i f_n = A f_n + A^{-1} 0 = A f_n = f_n \sigma_i,$$

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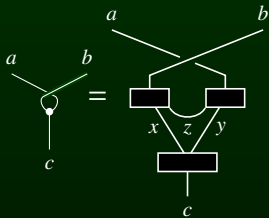
$$\begin{array}{c} \rho \\ \square \\ f_n \end{array} = \begin{array}{c} \rho \\ \square \\ f_n \end{array} = S[n](R[n])^n f_n = (-A^3)^n A^{n(n-1)} f_n = (-1)^n A^{n(n+2)} f_n.$$

R matrix

$$\widetilde{R}_{ab}^c = (-1)^z A^{-z(z+2)+xy-yz-xz} = (-1)^{\frac{a+b-c}{2}} A^{\frac{c(c+2)-a(a+2)-b(b+2)}{2}} \quad \text{by}$$

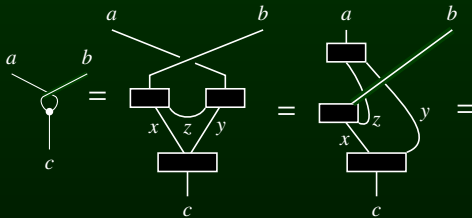
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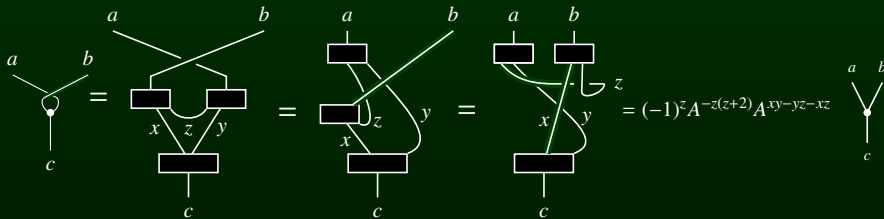
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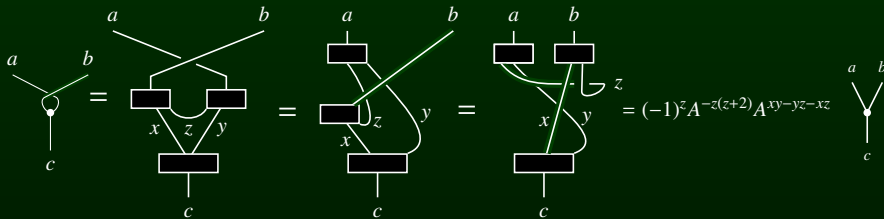
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Fibonacci model

$$A = \exp\left(\frac{3i\pi}{5}\right)$$

$$\widetilde{R}_{22}^0 = (-1)^2 A^{-8} = \exp\left(\frac{-24i\pi}{5}\right) = \exp\left(\frac{-4i\pi}{5}\right)$$

$$\widetilde{R}_{22}^1 = (-1)^1 A^{-4} = -\exp\left(\frac{-12i\pi}{5}\right) = -\exp\left(\frac{-2i\pi}{5}\right) = -\exp\left(\frac{3i\pi}{5}\right)$$

θ net and tetra net

Though the computations are complicated, the following results are known:

θ net and tetra net

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$$\tilde{\Theta}(a, b, c) = \begin{array}{c} b \\ \circlearrowleft \\ a \quad c \end{array} = \begin{array}{c} y \\ \circlearrowleft \\ z \quad x \end{array} = \begin{array}{c} \text{triangle net} \\ \begin{array}{c} y \\ \circlearrowleft \\ z \quad x \end{array} \end{array} = \frac{\Delta_{x+y+z}! \Delta_{x-1}! \Delta_{y-1}! \Delta_{z-1}!}{\Delta_{y+z-1}! \Delta_{z+x-1}! \Delta_{x+y-1}!}$$

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Though the computations are complicated, the following results are known:

triangle net

$$\tilde{\Theta}(a, b, c) = \text{diagram} = \text{diagram} = \frac{\Delta_{x+y+z}! \Delta_{x-1}! \Delta_{y-1}! \Delta_{z-1}!}{\Delta_{y+z-1}! \Delta_{z+x-1}! \Delta_{x+y-1}!}$$

tetra net

$$\text{Tet}_{efg}^{abc} = \text{diagram} = \frac{\prod_{i,j} [b_j - a_i]!}{[a]![b]![c]![d]![e]![f]!} \sum_{m \leq k \leq M} \frac{(-1)^k [k+1]!}{\prod_i [k - a_i]! \prod_j [b_j - k]!}$$

where a, b, c, d, e, f are positive and $[n] = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}$ (the quantum integer),

$$a_1 = \frac{a+d+e}{2}, a_2 = \frac{b+c+e}{2}, a_3 = \frac{a+b+f}{2}, a_4 = \frac{c+d+f}{2}, m = \max\{a_i\},$$

$$b_1 = \frac{b+d+e+f}{2}, b_2 = \frac{a+c+e+f}{2}, b_3 = \frac{a+b+c+d}{2}, M = \min\{b_j\}$$

quantum 6- j symbol

$$\begin{array}{c} a \\ \circlearrowleft \\ b \quad c \end{array} = \frac{\tilde{\theta}(a, b, c)}{\Delta_a} \begin{array}{c} \square \\ f_a \\ \square \end{array}$$

quantum 6- j symbol

$$\begin{array}{c} a \\ | \\ \circ \\ | \\ b \quad c \\ | \end{array} = \frac{\tilde{\theta}(a, b, c)}{\Delta_a} \begin{array}{c} | \\ \square \\ | \\ f_a \end{array}$$

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$$\begin{array}{c} a \\ | \\ \bigcirc \\ | \\ b \quad c \end{array} = \frac{\tilde{\theta}(a, b, c)}{\Delta_a} \begin{array}{c} | \\ \boxed{f_a} \\ | \end{array} \quad \begin{array}{c} \bigcirc \\ | \\ \boxed{f_i} \\ | \end{array} = \begin{array}{c} \bigcirc \\ | \\ \boxed{f_i} \\ | \\ \boxed{f_i} \\ | \end{array} = \frac{\Delta_i^2}{\tilde{\theta}(d, a, i)\tilde{\theta}(b, c, i)} \begin{array}{c} \bigcirc \\ | \\ \bigcirc \\ | \\ d \quad c \\ | \\ i \end{array}$$

quantum 6- j symbol

$$\begin{array}{c} a \\ | \\ \circ \\ | \\ b \quad c \\ | \end{array} = \frac{\tilde{\theta}(a, b, c)}{\Delta_a} \begin{array}{c} | \\ \square \\ | \\ f_a \end{array}$$

$$I = \frac{\Delta_i}{\tilde{\theta}(a, b, i)\tilde{\theta}(c, d, i)} \begin{array}{c} \circ \\ | \\ \circ \\ | \\ i \end{array} \begin{array}{c} a \quad b \\ | \\ i \\ | \\ d \quad c \\ | \\ i \end{array}$$

quantum 6- j symbol

$$\begin{array}{c} a \\ | \\ \bigcirc \\ | \\ b \quad c \\ | \\ f_a \end{array} = \frac{\tilde{\theta}(a, b, c)}{\Delta_a} \begin{array}{c} | \\ \square \\ | \end{array}$$

$$\begin{array}{c} \bigcirc \\ | \\ a \quad b \\ | \\ d \quad c \\ | \\ i \end{array} = \frac{\text{Tet}_{bcj}^{dai} \Delta_i}{\tilde{\theta}(a, b, i) \tilde{\theta}(c, d, i)} \begin{array}{c} \bigcirc \\ | \\ a \quad b \\ | \\ i \\ | \\ \bigcirc \\ | \\ d \quad c \\ | \\ i \end{array}$$

quantum 6- j symbol

$$\begin{array}{c} a \\ | \\ \bigcirc \\ | \\ c \end{array} = \frac{\tilde{\theta}(a, b, c)}{\Delta_a} \begin{array}{c} \boxed{f_a} \\ | \end{array}$$

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$$\begin{array}{c} a \quad b \\ \diagdown \quad / \\ \bullet \\ | \\ j \\ \bullet \\ \diagup \quad \diagdown \\ d \quad c \end{array} = \sum_i \left\{ \begin{array}{c} d \ a \ i \\ b \ c \ j \end{array} \right\}_A \begin{array}{c} a \quad b \\ \diagdown \quad / \\ \bullet \\ | \\ i \\ \bullet \\ \diagup \quad \diagdown \\ d \quad c \end{array} = \sum_i \frac{\text{Tet}_{bcj}^{dai} \Delta_i}{\tilde{\theta}(a, b, i) \tilde{\theta}(c, d, i)} \begin{array}{c} a \quad b \\ \diagdown \quad / \\ \bullet \\ | \\ i \\ \bullet \\ \diagup \quad \diagdown \\ d \quad c \end{array}$$

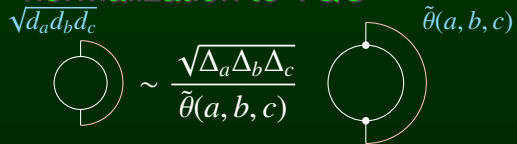
$$\begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagdown \quad / \\ \bullet \\ | \\ d \end{array} = \sum_i (\widetilde{F_{abc}^d})_i^j \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagdown \quad / \\ \bullet \\ | \\ i \\ \bullet \\ \diagup \quad \diagup \\ d \end{array} = \sum_i \frac{\text{Tet}_{bcj}^{dai} \Delta_i}{\tilde{\theta}(a, b, i) \tilde{\theta}(c, d, i)} \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagdown \quad / \\ \bullet \\ | \\ i \\ \bullet \\ \diagup \quad \diagup \\ d \end{array}$$

normalization to TQC

$$\sqrt{d_a d_b d_c}$$



normalization to TQC

$$\sqrt{d_a d_b d_c} \quad \sim \quad \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\tilde{\theta}(a, b, c)} \quad \tilde{\theta}(a, b, c)$$


normalization to TQC


$$\text{circle with one external line} \sim \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\tilde{\theta}(a, b, c)} \text{circle with two external lines}$$

normalization to TQC

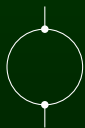
$$\begin{array}{c} \text{---} \\ | \\ \bigcirc \\ | \\ \text{---} \end{array} \sim \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\tilde{\theta}(a, b, c)} \begin{array}{c} \text{---} \\ | \\ \bigcirc \\ | \\ \text{---} \end{array}$$

$$\begin{array}{c} a \quad b \\ \diagdown \quad / \\ \text{---} \\ | \\ c \end{array} \sim \frac{\sqrt[4]{\Delta_a \Delta_b \Delta_c}}{\sqrt{\tilde{\theta}(a, b, c)}} \begin{array}{c} a \quad b \\ \diagdown \quad / \\ \bullet \\ | \\ c \end{array}$$

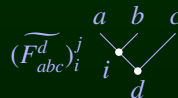
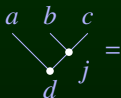
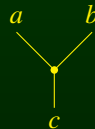
normalization to TQC



$$\sim \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\tilde{\theta}(a, b, c)}$$



$$\sim \frac{\sqrt[4]{\Delta_a \Delta_b \Delta_c}}{\sqrt{\tilde{\theta}(a, b, c)}}$$



normalization to TQC

$$\text{circle with top and bottom lines} \sim \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\tilde{\theta}(a, b, c)} \text{circle with top and bottom dots}$$

$$\text{Y-junction with lines } a, b, c \text{ and a dot} \sim \frac{\sqrt[4]{\Delta_a \Delta_b \Delta_c}}{\sqrt{\tilde{\theta}(a, b, c)}} \text{Y-junction with lines } a, b, c \text{ and a dot}$$

$$\frac{\sqrt{\tilde{\theta}(a, d, j) \tilde{\theta}(b, c, j)}}{\sqrt[4]{\Delta_a \Delta_j \Delta_d \Delta_b \Delta_c \Delta_j}} \text{Y-junction with lines } a, b, c \text{ and } d, j \text{ and a dot} = \frac{\sqrt{\tilde{\theta}(a, b, i) \tilde{\theta}(c, d, i)}}{\sqrt[4]{\Delta_a \Delta_b \Delta_i \Delta_c \Delta_d \Delta_i}} (\widetilde{F_{abc}^d})_i \text{Y-junction with lines } a, b, c \text{ and } i, d \text{ and a dot}$$

normalization to TQC

$$\begin{array}{c}
 \text{circle with top line} \sim \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\tilde{\theta}(a, b, c)} \text{circle with top and bottom lines} \\
 \\
 \begin{array}{c} a \quad b \\ \diagdown \quad / \\ c \end{array} \sim \frac{\sqrt[4]{\Delta_a \Delta_b \Delta_c}}{\sqrt{\tilde{\theta}(a, b, c)}} \begin{array}{c} a \quad b \\ / \quad \diagdown \\ c \end{array} \\
 \\
 \frac{\sqrt{\tilde{\theta}(a, d, j) \tilde{\theta}(b, c, j)}}{\sqrt[4]{\Delta_a \Delta_j \Delta_d \Delta_b \Delta_c \Delta_j}} \begin{array}{c} a \quad b \quad c \\ \diagdown \quad / \quad / \\ d \quad j \end{array} = \frac{\sqrt{\tilde{\theta}(a, b, i) \tilde{\theta}(c, d, i)}}{\sqrt[4]{\Delta_a \Delta_b \Delta_i \Delta_c \Delta_d \Delta_i}} (\widetilde{F_{abc}^d})_i^j \begin{array}{c} a \quad b \quad c \\ / \quad \diagdown \quad / \\ i \quad d \end{array}
 \end{array}$$

Thus, we can obtain F also from the spin network:

$$(\widetilde{F_{abc}^d})_i^j = \sqrt{\frac{\Delta_i \Delta_j}{\tilde{\theta}(a, b, i) \tilde{\theta}(c, d, i) \tilde{\theta}(a, d, j) \tilde{\theta}(b, c, i)}} \text{Tet}_{bcj}^{dai}$$

In case of Fibonacci model, two types of F coincide.