Parameterized operator means and operator monotonicity of $\exp\{f(x)\}$

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Flow of Presentation

1. Introduction
   - Definition of operator monotone function and operator mean
   - Operator monotonicity of $S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}}$: Previous work
2. Extension of range of parameter (p, α) such that $S_{p,\alpha}(x)$ is operator monotone
   - Extended range of parameter (p, α)
3. Operator monotonicity of $\exp\{f(x)\}$
   - Identric mean
   - Characterization

Introduction

Introduction

Introduction

Positive Operator

$\mathcal{B}(\mathcal{H})$: The set of all bounded linear operators on a Hilbert space $\mathcal{H}$.

For $A \in \mathcal{B}(\mathcal{H})$,

$$A \geq 0 \overset{\text{def}}{\iff} \langle Ax, x\rangle \geq 0 \quad (\forall x \in \mathcal{H})$$

$$A > 0 \overset{\text{def}}{\iff} A \geq 0 \text{ and } A \text{ is invertible.}$$

For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$,

$$A \geq B \overset{\text{def}}{\iff} A - B \geq 0.$$

$\mathcal{B}(\mathcal{H})_+ = \{A \in \mathcal{B}(\mathcal{H}) : A \geq 0\}$.

Introduction

Operator Monotone Function

Let J be an interval of $\mathbb{R}$ and $f : J \to \mathbb{R}$ be a continuous function. A function $f(x)$ is called an operator monotone function on J, provided

$$A \leq B \Rightarrow f(A) \leq f(B)$$

for self-adjoint $A, B \in \mathcal{B}(\mathcal{H})$ whose spectra $\sigma(A)$ and $\sigma(B)$ lie in J.

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Example

(Löwner-Heinz inequality) $\quad f(x) = x^{\alpha} \quad (0 \leq \alpha \leq 1)$

$$f(x) = \log x$$

$$\left(\because \quad \frac{A^{\alpha} - I}{\alpha} \leq \frac{B^{\alpha} - I}{\alpha} \quad (0 < \alpha \leq 1) \Longrightarrow \log A \leq \log B \quad (\alpha \downarrow 0)\right)$$

Introduction

Operator Mean (Kubo-Ando 1980)

The map $\mathfrak{M} : (A, B) \in \mathcal{B}(\mathcal{H})_+^2 \mapsto \mathfrak{M}(A, B) \in \mathcal{B}(\mathcal{H})_+$ is called an operator mean if the operator $\mathfrak{M}(A, B)$ satisfies the following four conditions: for $A, B, C, D \in \mathcal{B}(\mathcal{H})_+$ and self-adjoint X

(1) $A \le C, B \le D \Longrightarrow \mathfrak{M}(A, B) \le \mathfrak{M}(C, D)$ (Joint monotonicity),

(2) $X(\mathfrak{M}(A, B))X \le \mathfrak{M}(XAX, XBX)$ (Transformer inequality),

(3) $A_n, B_n \in \mathcal{B}(\mathcal{H})_+, A_n \downarrow A, B_n \downarrow B \Longrightarrow \mathfrak{M}(A_n, B_n) \downarrow \mathfrak{M}(A, B)$ (Upper semi-continuity),

(4) $\mathfrak{M}(I, I) = I$.

Introduction

Theorem K-A (Kubo-Ando 1980)

(1) For any operator mean $\mathfrak{M}$, there uniquely exists an operator monotone function $f \geq 0$ on $[0, \infty)$ with $f(1) = 1$ such that

$$f(x)I = \mathfrak{M}(I, xI), \ x \geq 0.$$

(2) When $\mathfrak{M} \mapsto f$, $\mathfrak{N} \mapsto g$, then $\mathfrak{M}(A, B) \leq \mathfrak{N}(A, B) \Longleftrightarrow f(x) \leq g(x)$ for all $A, B \in \mathcal{B}(\mathcal{H})_+, \ x > 0.$

(3) When $A > 0$, $\mathfrak{M}(A, B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}.$

$f(x)$ is called the representing function of $\mathfrak{M}$.

Introduction

Power Mean

$$\mathfrak{P}_s(A, B) = A^{\frac{1}{2}}\left(\frac{1}{2}\left\{I + \left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^s\right\}\right)^{\frac{1}{s}} A^{\frac{1}{2}}$$

Representing function of $\mathfrak{P}_s$: $P_s(x) = \left(\frac{1 + x^s}{2}\right)^{\frac{1}{s}} \quad (-1 \le s \le 1)$

- $s = 1$ (Arithmetic Mean): $P_1(x) = \frac{1 + x}{2}$
- $s \to 0$ (Geometric Mean): $P_0(x) := \lim_{s \to 0} P_s(x) = x^{\frac{1}{2}}$
- $s = -1$ (Harmonic mean): $P_{-1}(x) = \left(\frac{1 + x^{-1}}{2}\right)^{\frac{1}{-1}} = \frac{2x}{1 + x}$

Introduction

Power Mean

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Representing function of $\mathfrak{P}_s$: $P_s(x) = \left(\frac{1+x^s}{2}\right)^{\frac{1}{s}}$ $(-1 \le s \le 1)$

Weighted Power Mean

$$\mathfrak{P}_{s,\alpha}(A,B) = A^{\frac{1}{2}}\left((1-\alpha)I + \alpha\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^s\right)^{\frac{1}{s}} A^{\frac{1}{2}}$$

Representing function of $\mathfrak{P}_{s,\alpha}$: $P_{s,\alpha}(x) = ((1-\alpha) + \alpha x^s)^{\frac{1}{s}}$

$(-1 \le s \le 1,\ 0 \le \alpha \le 1)$

$$\mathfrak{P}_{s,\frac{1}{2}} = \mathfrak{P}_s$$

Previous works

Theorem U-W-Y-Y (U.-Wada-Yamazaki-Yanagida 2015)

For each $r \in [-1,1]$ and $s \in [-1,1]$, let $F_{r,s}(x)$ be a non-negative function of $x \in [0,\infty)$ defined by

$$F_{r,s}(x) = \left(\int_0^1 \left((1-\alpha) + \alpha x^r\right)^{\frac{s}{r}} d\alpha\right)^{\frac{1}{s}} \quad \text{if } r \neq 0 \text{ and } s \neq 0$$

and its limit if $r = 0$ or $s = 0$. Then $F_{r,s}(x)$ is operator monotone.

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and its limit if $r=0$ or $s=0$. Then $F_{r,s}(x)$ is operator monotone.

Remark

- $$F_{r,s}(x)=\left(\frac{r(x^{r+s}-1)}{(r+s)(x^r-1)}\right)^{\frac{1}{s}}$$
- $-1\leq r_1\leq r_2\leq 1,\ -1\leq s_1\leq s_2\leq 1\Longrightarrow F_{r_1,s_1}(x)\leq F_{r_2,s_2}(x)$

Previous works

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- $-1 \le r_1 \le r_2 \le 1,\ -1 \le s_1 \le s_2 \le 1 \Longrightarrow F_{r_1,s_1}(x) \le F_{r_2,s_2}(x)$

Order among means from $F_{r,s}(x)$

$$\frac{2x}{x+1} \le x^{\frac{1}{2}} \le \frac{x-1}{\log x} \le \exp\left\{\frac{x\log x}{x-1}-1\right\} \le \frac{x+1}{2}$$

Arithmetic mean: $\frac{x+1}{2}$, Identric mean: $\exp\left\{\frac{x\log x}{x-1}-1\right\}$

Logarithmic mean: $\frac{x-1}{\log x}$, Geometric mean: $x^{\frac{1}{2}}$, Harmonic mean: $\frac{2x}{x+1}$

Power, Power Difference and Stolarsky Means

Power Difference Mean

$s = -1$ and $s = 1 \Longrightarrow$ Power Difference Mean

$$PD_r(x) = \frac{(r-1)(x^r-1)}{r(x^{r-1}-1)} \quad (-1 \le r \le 2)$$

Power Mean

$r = s \Longrightarrow$ Power Mean

$$F_{s,s}(x) = \left(\frac{x^s+1}{2}\right)^{\frac{1}{s}} = P_s(x) \quad (-1 \le s \le 1)$$

Stolarsky Mean

$r = 1$ and $s = p - 1 \Longrightarrow$ Stolarsky Mean

$$F_{1,p-1}(x) = \left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} = S_p(x) \quad (0 \le p \le 2)$$

Stolarsky Mean

Stolarsky Mean (Nakamura 1989)

The following function

$$S_p(x) = \left(\frac{p(x-1)}{x^p - 1}\right)^{\frac{1}{1-p}} \quad (x > 0)$$

is an operator monotone function if and only if $-2 \leq p \leq 2$.

The cases $p = 0, 1$ are defined as the limits:

$$S_0(x) := \lim_{p \to 0} \left(\frac{p(x-1)}{x^p - 1}\right)^{\frac{1}{1-p}} = \frac{x-1}{\log x},$$

$$S_1(x) := \lim_{p \to 1} \left(\frac{p(x-1)}{x^p - 1}\right)^{\frac{1}{1-p}} = \exp\left\{\frac{x \log x}{x-1} - 1\right\}.$$

Stolarsky Mean

$$S_p(x) = \left(\frac{p(x-1)}{x^p - 1}\right)^{\frac{1}{1-p}}$$

Example 2

$p = 2$ (Arithmetic Mean): $S_2(x) = \dfrac{x+1}{2}$

$p \to 1$ (Identric Mean): $S_1(x) := \lim_{p \to 1} S_p(x) = \exp\left\{\dfrac{x \log x}{x-1} - 1\right\}$

$p \to 0$ (Logarithmic Mean): $S_0(x) := \lim_{p \to 0} S_p(x) = \dfrac{x-1}{\log x}$

$p = -1$ (Geometric Mean): $S_{-1}(x) = x^{\frac{1}{2}}$

Problem

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- We showed that if $0 \leq p \leq 2$ then $F_{1,p-1}(x) = \left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}}$ is operator monotone.
- $\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}}$ is operator monotone function if and only if $-2 \leq p \leq 2$.
- A range of parameter of $F_{r,s}(x)$ is not optimal.
- We may extend a range of parameter of $F_{r,s}(x)$.

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- $\left(\frac{p(x-1)}{x^p - 1}\right)^{\frac{1}{1-p}}$ is operator monotone function if and only if $-2 \le p \le 2$.
- A range of parameter of $F_{r,s}(x)$ is not optimal.
- We may extend a range of parameter of $F_{r,s}(x)$.
- In the following, we treat $F_{r,s}(x)$ as $S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}}$.

$$F_{r,s}(x) = \left(\frac{r(x^{r+s} - 1)}{(r+s)(x^r - 1)}\right)^{\frac{1}{s}} \xrightarrow{r \to p,\ s \to \alpha - p} S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}}$$

Extension of range of parameter (p, α) such that $S_{p,\alpha}(x)$ is operator monotone

Extension of range of parameter (p, α) such that $S_{p,\alpha}(x)$ is operator monotone

$F_{r,s}(x) \longrightarrow S_{p,\alpha}(x)$

$$F_{r,s}(x) = \left(\frac{r(x^{r+s}-1)}{(r+s)(x^r-1)}\right)^{\frac{1}{s}} \xrightarrow{r\to p,\ s\to\alpha-p} S_{p,\alpha}(x) = \left(\frac{p(x^\alpha-1)}{\alpha(x^p-1)}\right)^{\frac{1}{\alpha-p}}$$

- $F_{r,s}(x)$ is operator monotone if $-1 \le r \le 1$ and $-1 \le s \le 1$
- $S_{p,\alpha}(x)$ is operator monotone if $-1 \le p \le 1$ and $-1 \le \alpha - p \le 1$

A range of parameter from $F_{r,s}(x)$

If $p \in [-1, 1]$ and $p - 1 \le \alpha \le p + 1$, then

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha-1)}{\alpha(x^p-1)}\right)^{\frac{1}{\alpha-p}} \quad (x > 0)$$

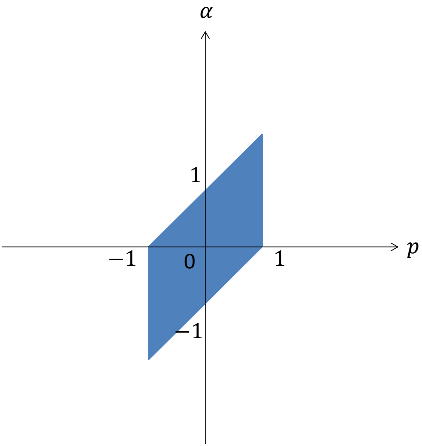
is an operator monotone function.

Remark

The range of parameter in which the above function is operator monotone is characterized in Nagisa-Wada (2015), but the range of parameter has not been determined explicitly yet.

$F_{r,s}(x) \longrightarrow S_{p,\alpha}(x)$

A range of parameter from $F_{r,s}(x)$



Nagisa-Wada (2015)

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)} \right)^{\frac{1}{\alpha - p}}$$

Nagisa-Wada (2015)

For real number a, b with $|a|, |b| \leq 2$ and $a \neq b$, we define the function $h : (0, \infty) \longrightarrow \mathbb{R}$ as follows:

$$h(x) = \frac{b(x^a - 1)}{a(x^b - 1)}.$$

Then h is operator monotone on $(0, \infty)$ if and only if

$$(a, b) \in \left\{ (a, b) \in \mathbb{R}^2 \;\middle|\; 0 < a - b \leq 1, a \geq -1, \text{and } b \leq 1 \right\}$$

$$\cup([0, 1] \times [-1, 0]) \setminus \{(0, 0)\}.$$

Nagisa-Wada (2015)

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}}$$

A range of parameter from Nagisa-Wada (2015)

$\dfrac{\alpha(x^p - 1)}{p(x^\alpha - 1)}$ is operator monotone if $(p, \alpha) \in [0, 1] \times [-1, 0]$.

- $(p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 | 0 \leq p \leq 1, -1 \leq \alpha \leq 0$ and $\alpha \leq p - 1\}$

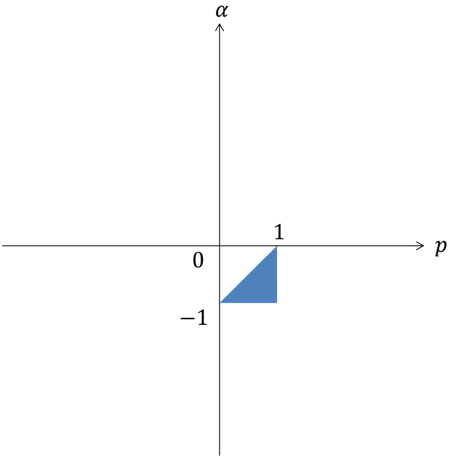
$$\Longrightarrow \frac{-1}{\alpha - p} \in \left[\frac{1}{2}, 1\right]$$

- $\left(\dfrac{\alpha(x^p - 1)}{p(x^\alpha - 1)}\right)^{\frac{-1}{\alpha - p}} = \left(\dfrac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}}$ is operator monotone if

$$(p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 | 0 \leq p \leq 1, -1 \leq \alpha \leq 0 \text{ and } \alpha \leq p - 1\}$$

Nagisa-Wada (2015)

A range of parameter from Nagisa-Wada (2015)



A range of parameter from $F_{r,s}(x)$ and Nagisa-Wada (2015)

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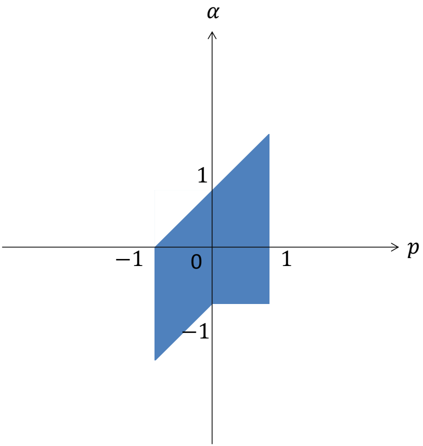


Figure. 1

Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

Result

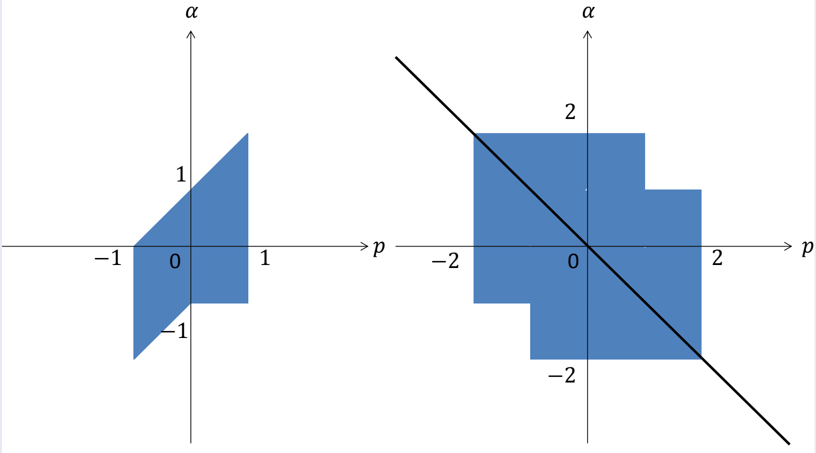


Figure. 1

Figure. 2

Trivial part

$$S_{p,\alpha}(x) = \left(\frac{p(x^{\alpha} - 1)}{\alpha(x^p - 1)} \right)^{\frac{1}{\alpha - p}}$$

$\alpha = -p$

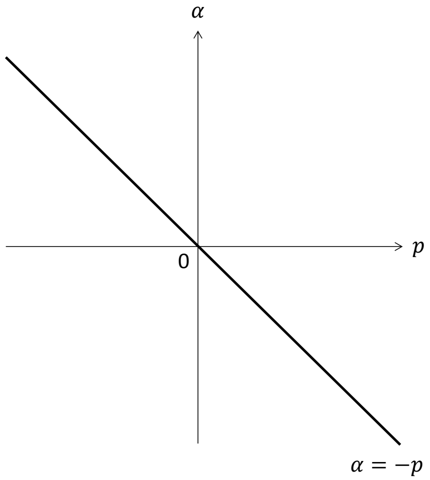
$$S_{p,-p}(x) = \left(\frac{p(x^{-p} - 1)}{(-p)(x^p - 1)} \right)^{\frac{1}{-2p}}$$
$$= \left(\frac{p(1 - x^p)}{(-p)x^p(x^p - 1)} \right)^{\frac{1}{-2p}} = \left(\frac{1}{x^p} \right)^{\frac{1}{-2p}} = x^{\frac{1}{2}}.$$

$\Longrightarrow$ $S_{p,\alpha}(x)$ is operator monotone if

$$\alpha = -p.$$

Trivial part

$\alpha = -p$



Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

Löwner's theorem

Let f be a real-valued function. Then the following are equivalent :

(1) f is operator monotone,

(2) f has an analytic continuation to upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \Im z > 0\}$, and $z \in \mathbb{C}^+$ implies $f(z) \in \mathbb{C}^+$. ($\Im z$ means the imaginary part of z.)

Example (x^α $(0 < \alpha \leq 1)$).

Let $f(x) := x^\alpha$ $(0 < \alpha \leq 1)$. If $z \in \mathbb{C}^+$, namely $0 < \arg z < \pi$, then

$$0 < \arg z^\alpha = \alpha \arg z < \alpha\pi \leq \pi.$$

Therefore, $f(x) := x^\alpha$ $(0 < \alpha \leq 1)$ is operator monotone.

Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}} \quad (-2 \leq p < 1,\ 1 < \alpha \leq 2)$$

$-2 \leq p < 1,\ 1 < \alpha \leq 2$

$$z \in \mathbb{C}^+ \Longrightarrow 0 < \arg\left(\frac{p(z-1)}{z^p - 1}\right)^{\frac{1}{1-p}} < \pi$$

($\because$ $S_p(x)$ is operator monotone)

Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

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$$z \in \mathbb{C}^+ \Longrightarrow 0 < \arg\left(\frac{p(z-1)}{z^p - 1}\right)^{\frac{1}{1-p}} < \pi$$

($\because$ $S_p(x)$ is operator monotone)

$$\therefore\ 0 < \arg\left(\frac{p(z-1)}{z^p-1}\right) < (1-p)\pi\ (-2 \le p < 1)$$

$$0 < \arg\left(\frac{z^p-1}{p(z-1)}\right) < (p-1)\pi\ (1 < p \le 2)$$

Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}} \quad (-2 \le p < 1,\ 1 < \alpha \le 2)$$

$-2 \le p < 1,\ 1 < \alpha \le 2$

$$z \in \mathbb{C}^+ \Longrightarrow 0 < \arg\left(\frac{p(z-1)}{z^p - 1}\right)^{\frac{1}{1-p}} < \pi$$

$(\because S_p(x)$ is operator monotone$)$

$$\therefore\ 0 < \arg\left(\frac{p(z-1)}{z^p - 1}\right) < (1-p)\pi\ (-2 \le p < 1)$$

$$0 < \arg\left(\frac{z^\alpha - 1}{\alpha(z-1)}\right) < (\alpha - 1)\pi\ (1 < \alpha \le 2)$$

Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}} \quad (-2 \le p < 1,\ 1 < \alpha \le 2)$$

$-2 \le p < 1,\ 1 < \alpha \le 2$

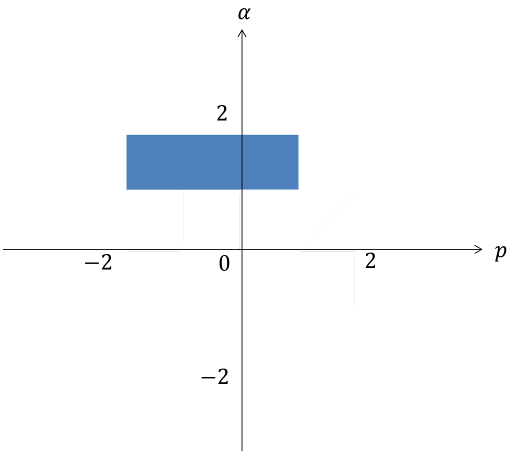
$$\begin{aligned} 0 < \arg &\left(\frac{p(z^\alpha - 1)}{\alpha(z^p - 1)}\right)^{\frac{1}{\alpha - p}} \\ &= \frac{1}{\alpha - p}\left\{\arg\left(\frac{p(z-1)}{z^p - 1}\right) + \arg\left(\frac{z^\alpha - 1}{\alpha(z-1)}\right)\right\} \\ &< \frac{1}{\alpha - p}\{(\alpha - 1)\pi + (1-p)\pi\} = \pi \end{aligned}$$

$\Longrightarrow$ $S_{p,\alpha}(x)$ is operator monotone if

$$-2 \le p < 1,\ 1 < \alpha \le 2.$$

Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

$-2\leq p<1,\ 1<\alpha\leq 2$



Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}} \quad (-1 < p \leq 2, -2 \leq \alpha < -1)$$

$-1 < p \leq 2, \ -2 \leq \alpha < -1$

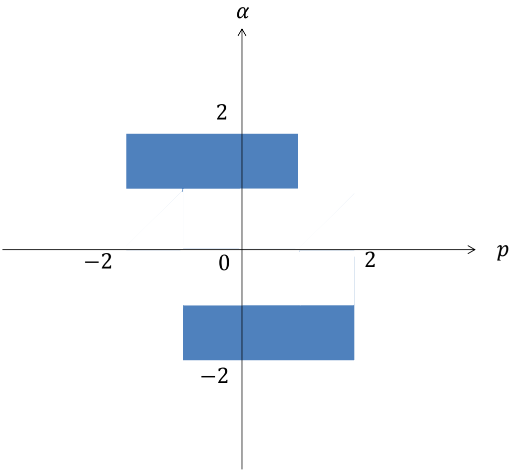
- $S_{-p}(x^{-1})^{-1} = \left(\frac{x(x^p - 1)}{p(x - 1)}\right)^{\frac{1}{1+p}} \quad (-2 \leq p \leq 2)$ is operator monotone.
- $0 < \arg\left(\frac{z(z^p - 1)}{p(z - 1)}\right) < (1 + p)\pi \ (-1 < p \leq 2)$
- $\arg\left(\frac{p(z^\alpha - 1)}{\alpha(z^p - 1)}\right)^{\frac{1}{\alpha - p}} = \frac{1}{p - \alpha}\left\{\arg\left(\frac{z(z^p - 1)}{p(z - 1)}\right) + \arg\left(\frac{\alpha(z - 1)}{z(z^\alpha - 1)}\right)\right\}$

$\Longrightarrow$ $S_{p,\alpha}(x)$ is operator monotone if

$$-1 < p \leq 2, \ -2 \leq \alpha < -1.$$

Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

$-2 \le p < 1,\ 1 < \alpha \le 2$ and $-1 < p \le 2,\ -2 \le \alpha < -1$



Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}}$$

Löwner's theorem

$S_{p,\alpha}(x)$ is symmetric for p, α:

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}} \Longleftrightarrow S_{\alpha,p}(x) = \left(\frac{\alpha(x^p - 1)}{p(x^\alpha - 1)}\right)^{\frac{1}{p - \alpha}}$$

Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

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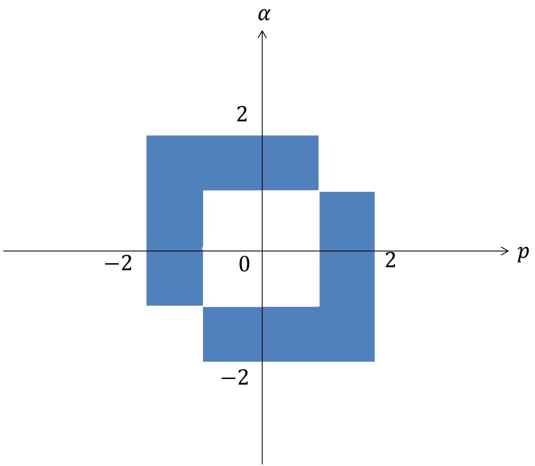
$\therefore$ We can extend a range of parameter simmetrically ;

$$(-2 \le \boldsymbol{p} < 1,\ 1 < \boldsymbol{\alpha} \le 2) \longrightarrow (-2 \le \boldsymbol{\alpha} < 1,\ 1 < \boldsymbol{p} \le 2),$$

$$(-1 < \boldsymbol{p} \le 2,\ -2 \le \boldsymbol{\alpha} < -1) \longrightarrow (-1 < \boldsymbol{\alpha} \le 2,\ -2 \le \boldsymbol{p} < -1)$$

Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

Extended range from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$



Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}}$$

Theorem 1 (2-parameter Stolarsky mean)

Let

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}} \quad (x > 0).$$

Then $S_{p,\alpha}(x)$ is operator monotone if $(p, \alpha) \in \mathcal{A} \subset \mathbb{R}^2$, where

$$\mathcal{A} = \big([-2, 1] \times [-1, 2]\big) \cup \big([-1, 2] \times [-2, 1]\big) \cup \left\{(p, \alpha) \in \mathbb{R}^2 \;\middle|\; \alpha = -p\right\}$$

Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$

Figure of $\mathcal{A}$

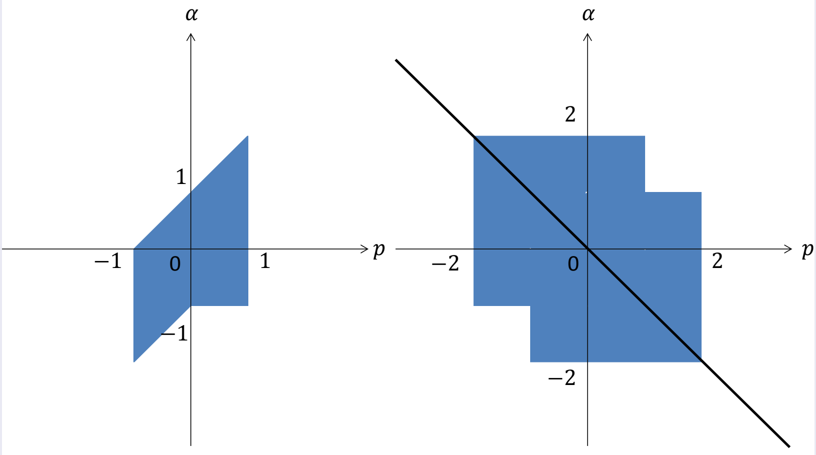


Figure. 1

Figure. 2

Operator monotonicity of $\exp\{f(x)\}$

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Identric mean

$$\mathcal{I}(x) = \exp\left\{\frac{x\log x}{x-1} - 1\right\}\left(= \frac{1}{e}x^{\frac{x}{x-1}}\right)$$

is an operator monotone function on $(0, \infty)$.

Operator monotonicity of $\exp\{f(x)\}$

Identric mean

$$\mathcal{I}(x) = \exp\left\{\frac{x\log x}{x-1} - 1\right\} \left(= \frac{1}{e} x^{\frac{x}{x-1}} \right)$$

is an operator monotone function on $(0, \infty)$.

Problem

- $\exp(x)$ is not an operator monotone function.
- $\mathcal{I}(x) = \exp\left\{\frac{x\log x}{x-1} - 1\right\}$ is a composite function with $\exp(x)$, but it is an operator monotone function.
- We consider a condition of $f(x)$ such that $\exp\{f(x)\}$ is operator monotone.

Operator monotonicity of $\exp\{f(x)\}$

Theorem 2

Let $f(x)$ *be a continuous function on* $(0, \infty)$*. If* $f(x)$ *is not a constant or* $\log(\alpha x)$ $(\alpha > 0)$*, then the following are equivalent:*

(1) $\exp\{f(x)\}$ *is an operator monotone function,*

(2) $f(x)$ *is an operator monotone function, and there exists an analytic continuation satisfying*

$$0 < v(r, \theta) < \theta,$$

where

$$f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \; (0 < r, \; 0 < \theta < \pi).$$

Operator monotonicity of $\exp\{f(x)\}$

Corollary 3

Let $f(x)$ be a continuous function on $(0, \infty)$, and assume $f(x)$ is not a constant or $\log(\alpha x)$ $(\alpha > 0)$. If $f(x)$ is not an operator monotone function or is an operator monotone function which does not satisfy

$$v(r, \theta) < \pi,$$

then $\exp\{f(x)\}$ is not operator monotone, where

$$f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \ (0 < r, \ 0 < \theta < \pi).$$

Operator monotonicity of $\exp\{f(x)\}$

Example (Harmonic mean)

$$H(x) = \frac{2x}{x+1}$$

is an operator monotone function on $[0, \infty)$, but $\exp\{H(x)\}$ is not operator monotone.

$\because$

$$\Im H(x) = v(r, \theta) = \frac{2r\sin\theta}{r^2 + 1 + 2r\cos\theta}.$$

$$r = 1,\ \theta = \frac{5\pi}{6}$$

$$\Longrightarrow\ v\left(1, \frac{5\pi}{6}\right) = 2 + \sqrt{3} > \frac{5\pi}{6}.$$

Operator monotonicity of $\exp\{f(x)\}$

Example (Logarithmic mean)

$$L(x)=\frac{x-1}{\log x}$$

is an operator monotone function on $[0,\infty)$, but $\exp\{L(x)\}$ is not operator monotone.

$\because$

$$\Im L(x)=v(r,\theta)=\frac{(r\log r)\sin\theta-\theta(r\cos\theta-1)}{(\log r)^2+\theta^2}$$

$r=\exp\left\{\frac{\pi}{2}\right\},\ \theta=\frac{\pi}{2}$

$$\Longrightarrow\ v\left(\exp\left\{\frac{\pi}{2}\right\},\frac{\pi}{2}\right)>\frac{\pi}{2}.$$

Operator monotonicity of $\exp\{f(x)\}$

Example (dual of Logarithmic mean)

$$DL(x) = \frac{x\log x}{x-1}$$

is an operator monotone function on $[0,\infty)$, and $\exp\{DL(x)\}$ is operator monotone too.

$$\Im DL(z) := v(r,\theta) = \frac{r}{r^2+1-2r\cos\theta}\{\theta(r-\cos\theta)-(\log r)\sin\theta\}.$$

[1] $v(r,\theta) < \theta \iff r\{\theta\cos\theta - (\log r)\sin\theta\} < \theta.$

$$\begin{aligned} r\{\theta\cos\theta-(\log r)\sin\theta\} &\le r\{\sin\theta-(\log r)\sin\theta\} \\ &= r(1-\log r)\sin\theta \\ &\le \sin\theta < \theta. \end{aligned}$$

$$\left(\because\ \theta\cos\theta \le \sin\theta < \theta\ (0<\theta<\pi),\ r(1-\log r)\le 1\ (0<r)\right)$$

Operator monotonicity of $\exp\{f(x)\}$

Example (dual of Logarithmic mean)

$$DL(x) = \frac{x \log x}{x - 1}$$

is an operator monotone function on $[0, \infty)$, and $\exp\{DL(x)\}$ is operator monotone too.

$$\Im DL(z) := v(r, \theta) = \frac{r}{r^2 + 1 - 2r\cos\theta}\{\theta(r - \cos\theta) - (\log r)\sin\theta\}.$$

[2] $0 < v(r, \theta) \iff (\log r)\sin\theta < \theta(r - \cos\theta)$.

$(1 \leq r)$ $\qquad (\log r)\sin\theta < (r-1)\theta < \theta(r - \cos\theta)$.

$(0 < r < 1)$

$$\begin{aligned}(\log r)\sin\theta &< (r-1)\sin\theta \\ &\leq (r-1)\theta\cos\theta \\ &= \theta(r\cos\theta - \cos\theta) < \theta(r - \cos\theta).\end{aligned}$$

Operator monotonicity of $\exp\{f(x)\}$

Example

$$-L(x)^{-1} := IL(x) = -\frac{\log x}{x-1}$$

is an operator monotone function on $(0, \infty)$, and $\exp\{IL(x)\}$ is operator monotone too.

$$\Im IL(z) := v(r,\theta) = \frac{(r\log r)\sin\theta - \theta(r\cos\theta - 1)}{r^2 + 1 - 2r\cos\theta}.$$

[1] $v(r,\theta) < \theta \iff (\log r)\sin\theta + \theta\cos\theta < r\theta.$

$$\begin{aligned}(\log r)\sin\theta + \theta\cos\theta &\leq (\log r)\sin\theta + \sin\theta \\ &= \sin\theta(\log r + 1) \\ &\leq r\sin\theta < r\theta.\end{aligned}$$

$$\left(\because\ \theta\cos\theta \leq \sin\theta < \theta\ (0 < \theta < \pi),\ \log r \leq r - 1\ (0 < r)\right)$$

Operator monotonicity of $\exp\{f(x)\}$

Example

$$-L(x)^{-1} := IL(x) = -\frac{\log x}{x-1}$$

is an operator monotone function on $(0, \infty)$, and $\exp\{IL(x)\}$ is operator monotone too.

$$\Im IL(z) := v(r,\theta) = \frac{(r\log r)\sin\theta - \theta(r\cos\theta - 1)}{r^2 + 1 - 2r\cos\theta}.$$

[2] $0 < v(r,\theta) \iff r\{\theta\cos\theta - (\log r)\sin\theta\} < \theta.$

$$\begin{aligned} r\{\theta\cos\theta - (\log r)\sin\theta\} &\leq r\{\sin\theta - (\log r)\sin\theta\} \\ &= \sin\theta\{r(1-\log r)\} \\ &\leq \sin\theta < \theta. \end{aligned}$$

$$\left(\because\ \theta\cos\theta \leq \sin\theta < \theta\ (0 < \theta < \pi),\ r(1-\log r) \leq 1\ (0 < r)\right)$$

Thank you for your attention!

A part to which range of parameter (p, α) cannot be extended

$$S_{p,\alpha}(x) = \left(\frac{p(x^{\alpha}-1)}{\alpha(x^p-1)}\right)^{\frac{1}{\alpha-p}}$$

Power mean

$$S_{p,2p}(x) = \left(\frac{p(x^{2p}-1)}{2p(x^p-1)}\right)^{\frac{1}{2p-p}}$$
$$= \left(\frac{(x^p+1)(x^p-1)}{2(x^p-1)}\right)^{\frac{1}{p}} = \left(\frac{x^p+1}{2}\right)^{\frac{1}{p}}$$

- $\left(\frac{x^p+1}{2}\right)^{\frac{1}{p}}$ is operator monotone if and only if $-1 \leq p \leq 1$
- We cannot extend a range of parameter such that $S_{p,\alpha}(x)$ is operator monotone when $\alpha = 2p$

A part to which range of parameter (p, α) cannot be extended

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)} \right)^{\frac{1}{\alpha - p}}$$

Parameterized Identric mean

$$S_{p,p}(x) := \lim_{\alpha \to p} S_{p,\alpha}(x) = \exp\left\{ \frac{1}{p} \left(\frac{x^p \log x^p}{x^p - 1} - 1 \right) \right\}$$

- When $p = \frac{5}{4}$, $S_{\frac{5}{4},\frac{5}{4}}(x)$ is not operator monotone.
- When $\alpha \to p$, we cannot extend a range of parameter more than $|p| \geq \frac{5}{4}$ such that $S_{p,\alpha}(x)$ is operator monotone.

$\left(\because f(x) \text{is operator monotone} \Rightarrow f(x^p)^{\frac{1}{p}} (p \in [-1, 1]) \text{is operator monotone} \right)$