Merging of positive maps: exposed and optimal maps, and their applications

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K, H	Hilbert spaces
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$B(K)^+$, $B(H)^+$	cones of positive operators on K, H
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- ϕ is *completely positive* (or CP) if it is k-positive for any $k \in \mathbb{N}$.
- ϕ is *decomposable* if $\phi(X) = \phi_1(X) + \phi_2(X)^t$, $X \in B(K)$, where ϕ_1, ϕ_2 are CP maps.

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Theorem (Størmer and Woronowicz)

Assume one of the following conditions holds:

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2. $\dim K = 2$ *and* $\dim H = 3$,
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Another examples of non-decomposable maps were given by Woronowicz, Tang, Ha, Osaka, Robertson, Kye and others.

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$$\langle Z, \phi \rangle_{\mathrm{d}} = \sum_i \mathrm{Tr}\left(\phi(X_i)\, Y_i^T\right)$$

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- $\mathscr{C}_\phi^t$ is a 'density matrix' of the functional

$$B(K) \otimes B(H) \ni Z \mapsto \langle Z, \phi \rangle_{\mathrm{d}}$$

i.e.

$$\langle Z, \phi \rangle_{\mathrm{d}} = \mathrm{Tr}(\mathscr{C}_\phi^t Z).$$

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- For $S \subset B(B(K), B(H))$, consider its dual cone $S^\circ \subset B(K) \hat{\otimes} T^1(H)$

$$S^\circ = \{Z \in B(K) \hat{\otimes} T^1(H) : \langle Z, \phi \rangle_{\mathrm{d}} \geq 0 \text{ for all } \phi \in \mathfrak{P}\}.$$

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- $CP \subset \mathfrak{P}$ completely positive maps, $CP^\circ = B(K \otimes H)^+$ (Choi theorem)
- $\mathrm{Dec} \subset \mathfrak{P}$ decomposable maps, Dec° is composed of PPT positive matrices

$$Z \in \mathrm{Dec}^\circ \quad \Leftrightarrow \quad Z \geq 0 \text{ and } Z^\Gamma \geq 0,$$

where

$$(X \otimes Y)^\Gamma = X \otimes Y^t.$$

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- Z is a PPT matrix if and only if $\langle Z, \phi \rangle_{\mathrm{d}} > 0$ for every decomposable ϕ.
- Z is a PPT entangled matrix if and only if there is a nondecomposable map ϕ, such that $\langle Z, \phi \rangle_{\mathrm{d}} < 0$.
  This provides also a nice criterion for nondecomposability.

Duality and exposed positive maps

For each $S \subset \mathfrak{P}$ one can define a dual face $S' \subset \mathfrak{P}^\circ$ by

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It follows from the above theorem that the problem of the description of positive maps can be reduced to the problem of characterization of exposed positive maps.

Examples

- (MM'2011) For finite dimensional dimensional K and H and any $A : K \to H$, the maps

$$\mathrm{Ad}_A : X \mapsto AXA^*, \qquad \mathrm{Ad}_A \circ \mathrm{t} : X \mapsto AX^{\mathrm{t}}A^*$$

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- Choi map is an extremal **non**exposed positive map.
- Other examples are due to Cruściński and Sarbicki, Ha and Kye, and others..

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Theorem (Miller, Olkiewicz)

S is a bistochastic, exposed and nondecomposable (even atomic) map.

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Theorem (Rutkowski et al.)

Λ_d is a bistochastic positive, nondecomposable and optimal map.

Miller-Olkiewicz map as a merging

For $V = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, consider 'denormalized' version of S

$$\phi(X) = VS(X)V^*$$

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$$\phi_{\mathrm{ess}} : \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & 0 & x_{13} \\ 0 & x_{22} & x_{32} \\ x_{31} & x_{23} & x_{33} \end{pmatrix}$$

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- merging of identity and transposition

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Merging of positive maps

Let K_1, K_2, H_1, H_2 be Hilbert spaces and

$$\phi_1 : B(K_1) \to B(H_1), \qquad \phi_2 : B(K_2) \to B(H_2)$$

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Let $K_3 = H_3 = \mathbb{C}$, and consider spaces

$$K = K_1 \oplus K_2 \oplus K_3, \qquad H = H_1 \oplus H_2 \oplus H_3$$

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$$K = K_1 \oplus K_2 \oplus K_3, \qquad H = H_1 \oplus H_2 \oplus H_3$$

Each element $X \in B(K)$ can be represented in the matrix form

$$X = \left(\begin{array}{c|c|c} X_{11} & X_{12} & X_{13} \\ \hline X_{21} & X_{22} & X_{23} \\ \hline X_{31} & X_{32} & X_{33} \end{array}\right)$$

where $X_{ij} \in B(K_j, K_i)$. In particular

$$X_{i3} \in B(\mathbb{C}, K_i) = K_i, \qquad X_{3j} \in B(K_j, \mathbb{C}) = K_j^*, \qquad X_{33} \in \mathbb{C}.$$

Merging of positive maps

Consider a $\phi : B(K) \to B(H)$ given by

$$\phi(X) = \left(\begin{array}{c|c|c} \phi_1(X_{11}) + \omega_2(X_{22})P_1 & 0 & B_1X_{13} + C_1X_{31}^t \\ \hline 0 & \phi_2(X_{22}) + \omega_1(X_{11})P_2 & B_2X_{23} + C_2X_{32}^t \\ \hline X_{31}B_1^* + X_{13}^tC_1^* & X_{32}B_2^* + X_{23}^tC_2^* & X_{33} \end{array}\right)$$

where

- $B_i, C_i : K_i \to H_i, \quad i = 1,2,$ linear operators
- $\omega_i : B(K_i) \to \mathbb{C}, \quad i = 1,2,$ positive functionals
- $P_i \in B(H_i), \quad i = 1,2,$ projection onto the range of $\phi_i(\mathbb{I}_{B(K_i)})$

Definition

We say that the map ϕ is a merging of ϕ_1, ϕ_2 by means of ingredients B_i, C_i, ω_i.

Positivity of merging

Question: Is a merging of positive maps ϕ_1 and ϕ_2 positive?

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$$\mu_i(\eta_i, y_i) = \sqrt{\langle y_i, \phi_i(\eta_i \eta_i^*) y_i \rangle} \qquad \varepsilon_i(\eta_i, y_i) = |\langle y_i, B_i \eta_i \rangle| + |\langle y_i, C_i \overline{\eta_i} \rangle|$$

$$\delta_i(\eta_i, y_i) = \sqrt{\mu_i(\eta_i, y_i)^2 - \varepsilon_i(\eta_i, y_i)^2} \qquad \sigma_i(\eta_i, y_{i'}) = \sqrt{\omega_i(\eta_i \eta_i^*)}\, \| P_{i'} y_{i'} \|$$

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Question: Is a merging of positive maps ϕ_1 and ϕ_2 positive?
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Theorem

The merging ϕ of positive maps ϕ_1, ϕ_2 by means of B_i, C_i, ω_i is a positive map if and only if the following conditions are satisfied

(i) $\varepsilon_i(\eta_i, y_i) \le \mu_i(\eta_i, y_i)$ *for* $i = 1, 2$, $\eta_i \in K_i$, $y_i \in H_i$,

(ii) *for every* $\eta_1 \in K_1$, $\eta_2 \in K_2$, $y_1 \in H_1$, $y_2 \in H_2$,

$$\delta_1(\eta_1, y_1)\delta_2(\eta_2, y_2) + \sigma_1(\eta_1, y_2)\sigma_2(\eta_2, y_1) \ge \varepsilon_1(\eta_1, y_1)\varepsilon_2(\eta_2, y_2)$$

Examples: ϕ_{A_1,A_2}

$$\phi_1(X) = A_1 X A_1^*, \qquad \phi_2(X) = A_2 X^t A_2^*$$

$$B_1 = A_1, \quad B_2 = 0, \quad C_1 = 0, \quad C_2 = A_2$$

$$\omega_1(X) = \mathrm{Tr}(A_1 X A_2^*), \quad \omega_2(X) = \mathrm{Tr}(A_2 X^t A_2^*)$$

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$$\phi(X) = \left(\begin{array}{c:c:c} A_1 X_{11} A_1^* + \mathrm{Tr}(A_2 X_{22}^{\mathrm{t}} A_2^*) E_1 & 0 & A_1 X_{13} \\ \hdashline 0 & A_2 X_{22}^{\mathrm{t}} A_2^* + \mathrm{Tr}(A_1 X_{11} A_1^*) E_2 & A_2 X_{32}^{\mathrm{t}} \\ \hdashline X_{31} A_1^* & X_{23}^{\mathrm{t}} A_2^* & X_{33} \end{array}\right)$$

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where

- $A_i : K_i \to H_i$ are Hilbert-Schmidt operators, $i = 1, 2$.
- E_i is the projection in $B(H_i)$ onto the range of A_i for $i = 1, 2$.

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$$\begin{aligned} \mu_1(\eta_1, y_1) &= |\langle y_1, A_1\eta_1\rangle|, & \mu_2(\eta_2, y_2) &= |\langle y_2, A_2\overline{\eta_2}\rangle|, \\ \varepsilon_1(\eta_1, y_1) &= |\langle y_1, A_1\eta_1\rangle|, & \varepsilon_2(\eta_2, y_2) &= |\langle y_2, A_2\overline{\eta_2}\rangle|, \\ \delta_1(\eta_1, y_1) &= 0, & \delta_2(\eta_2, y_2) &= 0, \\ \sigma_1(\eta_1, y_2) &= \|A_1\eta_1\|\,\|y_2\|, & \sigma_2(\eta_2, y_1) &= \|A_2\overline{\eta_2}\|\,\|y_1\|. \end{aligned}$$

Examples: ϕ_{A_1,A_2}

$$\phi(X) = \left(\begin{array}{c|c|c} A_1X_{11}A_1^* + \mathrm{Tr}(A_2X_{22}^{t}A_2^*)E_1 & 0 & A_1X_{13} \\ \hline 0 & A_2X_{22}^{t}A_2^* + \mathrm{Tr}(A_1X_{11}A_1^*)E_2 & A_2X_{32}^{t} \\ \hline X_{31}A_1^* & X_{23}^{t}A_2^* & X_{33} \end{array}\right)$$

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$$\delta_1\delta_2 + \sigma_1\sigma_2 \geq \varepsilon_1\varepsilon_2$$

Hence ϕ is positive.

Examples: Ω_{K_1,K_2}

$$\phi_1(X) = \mathrm{Tr}(X)\mathbb{I}_{B(K_1)}, \qquad \phi_2(X) = X^t$$

$$B_1 = \mathrm{id}_{K_1}, \quad B_2 = 0, \quad C_1 = 0, \quad C_2 = \mathrm{id}_{K_2}$$

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$$\mu_1(\eta_1, y_1) = \|y_1\|\|\eta_1\|, \qquad \mu_2(\eta_2, y_2) = |\langle y_2, \overline{\eta_2}\rangle|,$$

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$$\delta_1(\eta_1, y_1) = \sqrt{\|y_1\|^2\|\eta_1\|^2}, \qquad \delta_2(\eta_2, y_2) = 0,$$

$$\sigma_1(\eta_1, y_2) = \|\eta_1\|\|y_2\|, \qquad \sigma_2(\eta_2, y_1) = \|\overline{\eta_2}\|\|y_1\|.$$

Properties of merging

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Define $\psi_i : B(K_i) \to B(H_i)$ and $\chi_i : B(K_i) \to B(H_i)$ by

$$\psi_i(X) = B_i X B_i^*, \qquad \chi_i(X) = C_i X^t C_i^*, \qquad X \in B(K_i).$$

Corollary

If the merging of positive maps ϕ_1, ϕ_2 by means of B_i, C_i, ω_i is positive, then $\psi_i + \chi_i \le \phi_i$ for $i = 1, 2$.

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If the merging of positive maps ϕ_1, ϕ_2 by means of B_i, C_i, ω_i is positive, then $\psi_i + \chi_i \leq \phi_i$ for $i = 1,2$.

No notrivial merging of two extremal nondecomposable maps produces a positive map. Therefore, in order to get some nontrivial positive map by the merging procedure one should consider maps ϕ_1 and ϕ_2 with some 'regularity' properties. However, for properly chosen 'regular' maps there is a possibility for nontrivial merging. Surprisingly, merging of 'regular' maps can produce highly 'nonregular' positive maps.

Nondecomposable merging

Theorem

If ϕ_1 is 2-positive and ϕ_2 is 2-copositive, then there are operators B_i, C_i and functionals ω_i such that merging of ϕ_1 and ϕ_2 by means of C_i, D_i, ω_i is a positive nondecomposable map.

Corollary

Consequently, for each pair of positive maps satisfying assumptions of the above theorem, there is a merging which is an entanglement witness for some PPT state

3×3 *example of PPT entagled state*

By considering EW from the previous slide we obtain the following example of (unnormalized) PPT entangled matrix

$$
Z = \left(\begin{array}{ccc|ccc|ccc}
\gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\overline{b_1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot & \cdot & -\overline{b_2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -c_2 & \cdot \\
\hline
\cdot & \cdot & -\overline{c_1} & \cdot & \cdot & \cdot & s_1^2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\overline{c_2} & \cdot & s_2^2 & \cdot \\
-b_1 & \cdot & \cdot & \cdot & -b_2 & \cdot & \cdot & \cdot & \gamma^{-1}s^2
\end{array}\right)
$$

where $\gamma > 0$, $b_1, c_1 \in \mathbb{C}$,

$$
s_i = \max\{|b_i|, |c_i|\}, \qquad s = \max\left\{\sqrt{|b_1|^2 + |b_2|^2}, \sqrt{|c_1|^2 + |c_2|^2}\right\}
$$

Exposed positive maps

Theorem (M,Rutkowski)

For $A_i : K_i \to H_i$, $i = 1,2$, the map $\phi_{A_1,A_2} : B(K_1 \oplus K_2 \oplus \mathbb{C}) \to B(H_1 \oplus H_2 \oplus \mathbb{C})$ given by

$$X \mapsto \left(\begin{array}{c|c|c} A_1X_{11}A_1^* + \mathrm{Tr}(A_2X_{22}^tA_2^*)E_1 & 0 & A_1X_{13} \\ \hline 0 & A_2X_{22}^tA_2^* + \mathrm{Tr}(A_1X_{11}A_1^*)E_2 & A_2X_{32}^t \\ \hline X_{31}A_1^* & X_{23}^tA_2^* & X_{33} \end{array}\right)$$

is exposed in the cone of positive maps.

Remark

Strong spanning property was shown by Chruscinski and Sarbicki to be a useful sufficient condition for exposedness. Note, that ϕ_{A_1,A_2} does not satisfy this property for general choice of A_1, A_2.

Optimal positive maps maps

A positive map $\phi : B(K) \to B(H)$ is called optimal if there is no CP map ψ such that $\psi \leq \phi$.

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Spanning property: There are vectors $\eta_k \in K$ and $y_k \in H$, $k = 1, \ldots, N$, such that

- $\langle y_k, \phi(\eta_k \eta_k^*) y_k \rangle = 0$ for $k = 1, \ldots, N$,
- $\mathrm{span}\{\eta_k \otimes y_k : k = 1, \ldots, N\} = K \otimes H$.

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Kye: Spanning property is equivalent to $\{\phi\}'' \cap CP = \emptyset$.

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Theorem (M,Rutkowski)

The map $\Omega_{K_1,K_2} : B(K_1 \oplus K_2 \oplus \mathbb{C}) \to B(H_1 \oplus H_2 \oplus \mathbb{C})$ *given by*

$$X \mapsto \left(\begin{array}{c|c|c} \big(\mathrm{Tr}(X_{11}) + \mathrm{Tr}(X_{22})\big) \mathbb{I}_{B(K_1)} & 0 & X_{13} \\ \hline 0 & X_{22}^{t} + \mathrm{Tr}(X_{11}) \mathbb{I}_{B(K_2)} & X_{32}^{t} \\ \hline X_{31} & X_{23}^{t} & X_{33} \end{array} \right)$$

satisfies spanning property.

Case 3×3 – *positivity*

The general form of $\phi : M_3(\mathbb{C}) \to M_3(\mathbb{C})$:

$$\phi\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} f_1 x_{11} + w_2 x_{22} & 0 & b_1 x_{13} + c_1 x_{31} \\ 0 & f_2 x_{22} + w_1 x_{11} & b_2 x_{23} + c_2 x_{32} \\ \overline{b_1} x_{31} + \overline{c_1} x_{13} & \overline{b_2} x_{32} + \overline{c_2} x_{23} & x_{33} \end{pmatrix}.$$

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The general form of $\phi : M_3(\mathbb{C}) \to M_3(\mathbb{C})$:

$$\phi\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} f_1 x_{11} + w_2 x_{22} & 0 & b_1 x_{13} + c_1 x_{31} \\ 0 & f_2 x_{22} + w_1 x_{11} & b_2 x_{23} + c_2 x_{32} \\ \overline{b_1} x_{31} + \overline{c_1} x_{13} & \overline{b_2} x_{32} + \overline{c_2} x_{23} & x_{33} \end{pmatrix}.$$

$$\mu_i = f_i^{1/2}, \quad \sigma_i = w_i^{1/2}, \quad \varepsilon_i = |b_i| + |c_i|, \quad \delta_i = (\mu_i^2 - \varepsilon_i^2)^{1/2}.$$

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$$\phi(X) = \begin{pmatrix} (\varepsilon_1^2 + \delta_1^2) x_{11} + \sigma_2^2 x_{22} & 0 & b_1 x_{13} + c_1 x_{31} \\ 0 & (\varepsilon_2^2 + \delta_2^2) x_{22} + \sigma_1^2 x_{11} & b_2 x_{23} + c_2 x_{32} \\ \overline{b_1} x_{31} + \overline{c_1} x_{13} & \overline{b_2} x_{32} + \overline{c_2} x_{23} & x_{33} \end{pmatrix}.$$

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Proposition

The above map is positive if and only if $\sigma_1\sigma_2 + \delta_1\delta_2 \geq \varepsilon_1\varepsilon_2$.

Case 3×3 – *complete (co)positivity*

$$\phi(X) = \begin{pmatrix} (\varepsilon_1^2 + \delta_1^2) x_{11} + \sigma_2^2 x_{22} & 0 & b_1 x_{13} + c_1 x_{31} \\ 0 & (\varepsilon_2^2 + \delta_2^2) x_{22} + \sigma_1^2 x_{11} & b_2 x_{23} + c_2 x_{32} \\ \overline{b_1} x_{31} + \overline{c_1} x_{13} & \overline{b_2} x_{32} + \overline{c_2} x_{23} & x_{33} \end{pmatrix}.$$

$$\phi(X) = \begin{pmatrix} (\varepsilon_1^2+\delta_1^2)x_{11}+\sigma_2^2 x_{22} & 0 & b_1 x_{13}+c_1 x_{31} \\ 0 & (\varepsilon_2^2+\delta_2^2)x_{22}+\sigma_1^2 x_{11} & b_2 x_{23}+c_2 x_{32} \\ \overline{b_1} x_{31}+\overline{c_1} x_{13} & \overline{b_2} x_{32}+\overline{c_2} x_{23} & x_{33} \end{pmatrix}.$$

Proposition

The following conditions are equivalent:

(i) ϕ *is completely positive (respectively completely copositive);*

(ii) ϕ *is* 2*-positive (respectively* 2*-copositive);*

(iii) $c_1 = c_2 = 0$ *(respectively* $b_1 = b_2 = 0$*) and* $\delta_1\delta_2 \geq \varepsilon_1\varepsilon_2$.

Case 3×3 *– decomposability vs. nondecomposability*

$$\phi(X) = \begin{pmatrix} (\varepsilon_1^2 + \delta_1^2)x_{11} + \sigma_2^2 x_{22} & 0 & b_1 x_{13} + c_1 x_{31} \\ 0 & (\varepsilon_2^2 + \delta_2^2)x_{22} + \sigma_1^2 x_{11} & b_2 x_{23} + c_2 x_{32} \\ \overline{b_1} x_{31} + \overline{c_1} x_{13} & \overline{b_2} x_{32} + \overline{c_2} x_{23} & x_{33} \end{pmatrix}.$$

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$$\vec{b} = (|b_1|, |b_2|)^t, \quad \vec{c} = (|c_1|, |c_2|)^t, \quad s_i = \max\{|b_i|, |c_i|\}, i = 1, 2,$$

$$s = \max\{\|\vec{b}\|, \|\vec{c}\|\}, \quad \delta = (\delta_1^2 + \delta_2^2)^{1/2}, \quad \varepsilon = (\varepsilon_1^2 + \varepsilon_2^2)^{1/2}$$

Proposition

1. *If* $\vec{b}$ *and* $\vec{c}$ *are linearly dependent, then* ϕ *is decomposable.*
2. *If* $s(\varepsilon^2 + \delta^2)^{1/2} < \|\vec{b}\|^2 + \|\vec{c}\|^2$*, then* ϕ *is nondecomposable.*

Case 3×3 *– decomposability vs. nondecomposability*

$$\phi(X) = \begin{pmatrix} (\varepsilon_1^2+\delta_1^2)x_{11}+\sigma_2^2 x_{22} & 0 & b_1x_{13}+c_1x_{31} \\ 0 & (\varepsilon_2^2+\delta_2^2)x_{22}+\sigma_1^2 x_{11} & b_2x_{23}+c_2x_{32} \\ \overline{b_1}x_{31}+\overline{c_1}x_{13} & \overline{b_2}x_{32}+\overline{c_2}x_{23} & x_{33} \end{pmatrix}.$$

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Proposition

1. *If $\vec{b}$ and $\vec{c}$ are linearly dependent, then ϕ is decomposable.*
2. *If $s(\varepsilon^2+\delta^2)^{1/2} < \|\vec{b}\|^2+\|\vec{c}\|^2$, then ϕ is nondecomposable.*

If $\|\vec{b}\| = \|\vec{c}\|$, then the inequality in 2. is equivalent to linear independence of $\vec{b}$ and $\vec{c}$.

Case 3×3 - *extremality*

Theorem

The following are equivalent:

1. ϕ *is exposed,*
2. ϕ *is extremal,*
3. *each of the following conditions is satisfied*
   - 3.1 $\vec{b} \neq 0$ *and* $\vec{c} \neq 0$,
   - 3.2 $\delta_1 = \delta_2 = 0$,
   - 3.3 $\sigma_1\sigma_2 = \varepsilon_1\varepsilon_2$,
   - 3.4 $\langle \vec{b}, \vec{c} \rangle = 0$.

Case 3×3 *- optimality*

Theorem

The following are equivalent:

1. ϕ *is optimal,*
2. ϕ *satisfies spanning property,*
3. *each of the following conditions is satisfied*
   - 3.1 $\vec{b} \neq 0$ *and* $\vec{c} \neq 0$,
   - 3.2 $\sigma_1\sigma_2 + \delta_1\delta_2 = \varepsilon_1\varepsilon_2$,
   - 3.3 $\langle \vec{b}, \vec{c} \rangle = 0$.

Applications

1. Concrete: construction of a new family of PPT entangled states.

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2. Possible: construction of NPT bound entangled states (?) - work in progress

Main idea of the proof

- $K = K_1 \oplus K_2 \oplus \mathbb{C}$, $H = H_1 \oplus H_2 \oplus \mathbb{C}$
- $\mathcal{Z} = \{(\xi, \eta) \in K \times H : \langle \eta, \phi(\xi\xi^*)\eta \rangle = 0$

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- By Kye's characterization of exposed faces, $\phi : B(K) \to B(H)$ is exposed iff
  $\forall\, \psi \in \mathfrak{P} : \ (\forall\, (\xi, \eta) \in \mathcal{Z} : \langle \eta, \psi(\xi\xi^*)\eta \rangle = 0) \quad \Rightarrow \quad \psi \in \mathbb{R}^+ \phi.$

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- $\langle \eta, \phi(\xi\xi^*)\eta \rangle$ is equal to

$$\|A_1\xi_1\|^2\|E_2\eta_2\|^2 + \|A_2\overline{\xi_2}\|^2\|E_1\eta_1\|^2 + |\langle \eta_1, A_1\xi_1\rangle|^2 + |\langle \eta_2, A_2\overline{\xi_2}\rangle|^2$$

if $\alpha = 0$, and

$$|\alpha|^{-2}\left(\left||\alpha|^2\overline{\beta} + \overline{\alpha}\langle \eta_1, A_1\xi_1\rangle + \alpha\langle \eta_2, A_2\overline{\xi_2}\rangle\right|^2 + \left\|\alpha E_1\eta_1 \otimes A_2\overline{\xi_2} - \overline{\alpha}A_1\xi_1 \otimes E_2\eta_2\right\|^2\right),$$

if $\alpha \neq 0$.

Sketch of the proof

- Thus $(\xi, \eta) \in \mathcal{Z}$ iff one of the following conditions holds

$$
\begin{array}{lll}
\alpha = 0,\ A_1\xi_1 = 0,\ A_2\overline{\xi_2} = 0 & & \\
\alpha = 0,\ A_1\xi_1 \neq 0,\ A_2\overline{\xi_2} = 0 & \text{and} & \eta_1 \perp A_1\xi_1,\ E_2\eta_2 = 0 \\
\alpha = 0,\ A_1\xi_1 = 0,\ A_2\overline{\xi_2} \neq 0 & \text{and} & E_1\eta_1 = 0,\ \eta_2 \perp A_2\overline{\xi_2} \\
\alpha = 0,\ A_1\xi_1 \neq 0,\ A_2\overline{\xi_2} \neq 0 & \text{and} & E_1\eta_1 = 0,\ E_2\eta_2 = 0 \\
\alpha \neq 0,\ A_1\xi_1 = 0,\ A_2\overline{\xi_2} = 0 & \text{and} & \beta = 0 \\
\alpha \neq 0,\ A_1\xi_1 \neq 0,\ A_2\overline{\xi_2} = 0 & \text{and} & \langle A_1\xi_1, \eta_1\rangle = -\overline{\alpha}\beta,\ E_2\eta_2 = 0 \\
\alpha \neq 0,\ A_1\xi_1 = 0,\ A_2\overline{\xi_2} \neq 0 & \text{and} & E_1\eta_1 = 0,\ \langle A_2\overline{\xi_2}, \eta_2\rangle = -\alpha\beta \\
\alpha \neq 0,\ A_1\xi_1 \neq 0,\ A_2\overline{\xi_2} \neq 0 & \text{and} & \begin{cases} E\eta_1 = -\dfrac{\overline{\alpha}\beta}{\|A_1\xi_1\|^2 + \|A_2\overline{\xi_2}\|^2} A_1\xi_1, \\ E\eta_2 = -\dfrac{\alpha\beta}{\|A_1\xi_1\|^2 + \|A_2\overline{\xi_2}\|^2} A_2\overline{\xi_2} \end{cases}
\end{array}
$$

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- Now, assume $\langle \eta, \psi(\xi\xi^*)\eta \rangle = 0$ for all $(\xi, \eta) \in \mathcal{Z}$.

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- One shows that there are sesquilinear vector valued forms

$$\Psi_{kl} : (K_1 \oplus K_2) \times (K_1 \oplus K_2) \to B(H_l, H_k), \quad k, l = 1, 2$$

and linear maps $R_k, Q_k : K_1 \oplus K_2 \to H_k$ for $k = 1, 2$ such that $\psi(\xi\xi^*)$ is equal to

$$\begin{pmatrix} \Psi_{11}(\xi_0, \xi_0) & \Psi_{12}(\xi_0, \xi_0) & \overline{\alpha} R_1\xi_0 + \alpha Q_1\overline{\xi_0} \\ \Psi_{21}(\xi_0, \xi_0) & \Psi_{22}(\xi_0, \xi_0) & \overline{\alpha} R_2\xi_0 + \alpha Q_2\overline{\xi_0} \\ \alpha(R_1\xi_0)^* + \overline{\alpha}(Q_1\overline{\xi_0})^* & \alpha(R_2\xi_0)^* + \overline{\alpha}(Q_2\overline{\xi_0})^* & \lambda|\alpha|^2 \end{pmatrix}$$

for any $\xi \in K$ where $\xi = \xi_0 + \alpha e$ for a unique $\xi_0 = \xi_1 + \xi_2 \in K_1 \oplus K_2$ and $\alpha \in \mathbb{C}$.

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for any $\xi \in K$ where $\xi = \xi_0 + \alpha e$ for a unique $\xi_0 = \xi_1 + \xi_2 \in K_1 \oplus K_2$ and $\alpha \in \mathbb{C}$.
- Finally, by a sequence of reasonings using linearity-antilinearity interplay, one that all ingredients are multiples by λ of respective terms of ϕ.

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