A Universal Formulation of Uncertainty Relations in Quantum Theory

Jaeha Lee (IIS, The University of Tokyo)*

Abstract

The uncertainty principle [1], advocated by Heisenberg in 1927, is widely considered as one of the basic tenets of quantum theory, characterising the indeterministic nature of the microscopic world. The principle, originally derived from heuristic arguments with the aid of several Gedankenexperiments, has undergone much theoretical elaboration over the following century or so, thereby resulting in various relations [2–6], each of which accounting for certain realms of its diverse forms of manifestation. In this talk, a universal formulation [7, 8] of uncertainty relations is presented, which is established upon conceivably the simplest and most general framework of measurement of statistical nature. Quite assuringly, the new formulation entails several renowned previous results as corollaries to its special cases. Notably, it attains a seamless connection between two of the most prominent realms of quantum uncertainty, namely the uncertainty involving quantum measurements and that regarding the intrinsic indeterminacy inherent in quantum states.

1. A universal formulation [7, 8] of uncertainty relations

Let $S(\mathcal{H})$ denote the linear space of all the bounded self-adjoint operators on a Hilbert space $\mathcal{H}$, and $Z(\mathcal{H})$ denote the convex set of all the density operators on $\mathcal{H}$. In a parallel manner, let $R(\Omega)$ denote the linear space of all the real bounded measurable functions on a measurable space $(\Omega, \mathfrak{A})$, and $W(\Omega)$ denote the convex set of all the probability measures on $(\Omega, \mathfrak{A})$.

For each quantum state $\rho \in Z(\mathcal{H})$, define the Hilbert space of localised quantum observables $S_\rho(\mathcal{H}) := \overline{S(\mathcal{H})/\sim_\rho}$ by the completion of the quotient space under the equivalence relation $A \sim_\rho B \iff \|A - B\|_\rho = 0$, $A, B \in S(\mathcal{H})$, regarding the state-dependent seminorm $\|A\|_\rho := (\mathrm{Tr}[A^2\rho])^{1/2}$. Similarly, for each classical state $p \in W(\Omega)$, define the Hilbert space of localised classical observables $R_p(\Omega) := \overline{R(\Omega)/\sim_p}$ by the completion of the quotient space under the equivalence relation $f \sim_\rho g \iff \|f - g\|_p = 0$, $f, g \in R(\Omega)$, regarding the state-dependent seminorm $\|f\|_\rho := (\int_\Omega |f(\omega)|^2\, dp(\omega))^{1/2}$.

An affine map $M : Z(\mathcal{H}) \to W(\Omega)$, which shall be referred to as a quantum measurement, entails a natural dual map $M' : R(\Omega) \to S(\mathcal{H})$. The Kadison–Schwarz inequality reveals that the dual map passes to the quotient, whereby a quantum measurement is found to locally induce a pair of dual linear maps between the space of localised observables.

Proposition 1.1 (Pullback and Pushforward of a Measurement [7, 8]). *A quantum measurement $M : Z(\mathcal{H}) \to W(\Omega)$ induces a pair of non-expansive dual linear maps*

$$M_\rho^* : R_{M\rho}(\Omega) \to S_\rho(\mathcal{H}), \quad M_{\rho *} : S_\rho(\mathcal{H}) \to R_{M\rho}(\Omega), \tag{1}$$

respectively termed the pullback and the pushforward of M at $\rho \in Z(\mathcal{H})$, between the spaces of localised observables.

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Given the above result, define the error of the measurement M of A over ρ by the amount of contraction

$$\varepsilon_\rho(A;M) := \sqrt{\|A\|_\rho^2 - \|M_{\rho*}A\|_\rho^2} \tag{2}$$

induced by the pushforward. Note that the error is non-negative by construction.

Theorem 1.2 (Universal Uncertainty Relation [7,8])**.** *Let $M : Z(\mathcal{H}) \to W(\Omega)$ be a quantum measurement. For any $A, B \in S_\rho(\mathcal{H})$, $\rho \in Z(\mathcal{H})$, the inequality*

$$\varepsilon_\rho(A;M)\,\varepsilon_\rho(B;M) \geq \sqrt{\mathcal{R}^2 + \mathcal{I}^2} \tag{3}$$

holds with

$$\mathcal{R} := \left\langle \frac{\{A,B\}}{2} \right\rangle_\rho - \left\langle M_{\rho*}A,\ M_{\rho*}B \right\rangle_{M\rho}, \tag{4}$$

$$\mathcal{I} := \left\langle \frac{[A,B]}{2i} \right\rangle_\rho - \left\langle \frac{[M_\rho^* M_{\rho*}A, B]}{2i} \right\rangle_\rho - \left\langle \frac{[A, M_\rho^* M_{\rho*}B]}{2i} \right\rangle_\rho \tag{5}$$

being the two contributors to the lower bound.

2. The uncertainty principle as an impossibility theorem [7,8]

The above relation reveals a universal impossibility theorem (*alias* no-go theorem), which marks a fundamental incompatibility inherent in quantum theory.

Theorem 2.1 (Universal Uncertainty Principle [7, 8])**.** *Let M be a quantum measurement performed on a quantum system $\mathcal{H}$. Then, the implication*

$$\left\langle \frac{[A,B]}{2i} \right\rangle_\rho \neq 0 \implies \varepsilon_\rho(A;M) \neq 0 \quad \textit{or} \quad \varepsilon_\rho(B;M) \neq 0 \tag{6}$$

holds for any $A, B \in S_\rho(\mathcal{H})$, $\rho \in Z(\mathcal{H})$.

A stronger impossibility theorem, the form of which is shared in common in many of the traditional formulations of the principle, is then found to be valid under a certain constraint.

Theorem 2.2 (Traditional Uncertainty Principle [8])**.** *Let M be a quantum measurement performed on a quantum system $\mathcal{H}$. Then, the implication*

$$\left\langle \frac{[A,B]}{2i} \right\rangle_\rho \neq 0 \implies \varepsilon_\rho(A;M) \neq 0 \quad \textit{and} \quad \varepsilon_\rho(B;M) \neq 0 \tag{7}$$

holds for any $A, B \in \overline{\mathrm{ran}}\, M_\rho^* = (\ker M_{\rho*})^\perp$, $\rho \in Z(\mathcal{H})$.

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Title: Quaternion Matrix Computation

Speaker: Michael Ng (University of Hong Kong)

Abstract: In this talk, I share some recent results of quaternion matrix computation. Numerical examples and related applications are reported to illustrate our results.

Overviews of generalized probabilistic theories and specific results for regular polygon theories

Ryo Takakura (Kyoto University)

Quantum theory, whose foundation is formed by matrix theory (Hilbert space theory), is the most successful theory that describes nature. Among many of its remarkable predictions such as the existence of superposition or entanglement, probably the most drastic one is that nature is probabilistic: Even if we conduct a perfect preparation of a physical system and measurement, we do not always obtain one determined outcome. *Generalized probabilistic theories* (*GPTs*) are a mathematical framework that describes most intuitively these probabilistic behaviors of nature. While positive operators on a Hilbert space are needed in quantum theory, elements of a vector space and its dual (called states and effects corresponding to preparations of systems and measurements respectively) are tools for calculating probabilities in GPTs. The only requirement is the convexity for states and effects, and there is in general not assumed any Hilbert space structure or operator algebraic property. In this sense, GPTs are operationally the broadest framework to describe nature, and have been studied actively in recent years in the context of quantum foundations [1,2,3], followed by the intuition that seeing quantum theory from a broader perspective will contribute to elucidating its essence.

In this presentation, I explain the mathematical formulation of GPTs to show how they give the most intuitive description of nature. It will be revealed that its two basic assumptions: physical experiments consist of three procedures - preparing an object system, performing a measurement, and obtaining statistics – and their probabilistic mixtures are allowed, can be mathematically represented in terms of ordered vector spaces (more precisely, a base norm space and order unit space). In the embedding of these physical objects into ordered vector spaces, the requirement of the validity of probabilistic mixtures is reflected as the convexity of the sets of those objects (the state space and effect space) in the vector spaces. The mathematical formulation of GPTs through this embedding theorem is a generalization of that of quantum theory, where the preparation and measurement procedures are represented respectively as density operators and positive operators bounded by the identity operator (or POVMs) in the vector space of positive operators.

In addition to the fundamental description of GPTs, I also present results for one of the simplest classes of GPTs called regular polygon theories. Regular polygon theories are GPTs whose state spaces are in the shape of regular polygons, and are often regarded naturally as generalizations of the two-level quantum system (the qubit system). I explain whether or how several quantum behaviors such as uncertainty relations, the violation of Bell inequality, and the existence of thermodynamical entropy are observed also in regular polygon theories [3]. These observations may contribute to finding what is peculiar to quantum theory.

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Title: Inequalities of positive matrices and application to quantifying multipartite Gaussian correlations

Speaker: Jinchuan Hou (Taiyuan University of Technology)

Abstract: In this talk, we give several inequalities concerning positive-definite matrices and their determinants, and apply them to prove that the non-negative functional ${\mathcal M}^{(k)}$ defined by ${\mathcal M}^{(k)}(\rho_{A_1,A_2,\ldots,A_k})=1-\frac{\det(\Gamma_{\rho_{A_1,A_2\ldots,A_k}})}{\Pi_{j=1}^k\det(\Gamma_{\rho_{A_j}})}$ is a k-partite multi-mode Gaussian correlation measure for continuous-variable systems satisfying the unification condition, hierarchy condition, complete monogamy relation and tight monogamy relation, where $\rho_{A_1,A_2\ldots,A_k}$ is a k-partite $n_1+n_2+\cdots+n_k$-mode Gaussian state with $k\geq 2$ and $n_j\geq 1$ positive integers, ρ_{A_j} is the reduced state of $\rho_{A_1,A_2\ldots,A_k}$ in the jth subsystem A_j and Γ_ρ is the covariance matrix of Gaussian state ρ.

Operator monotone functions and operator inequalities

Yoichi Udagawa

Let $\mathcal{B}(\mathcal{H})_+$ be a set of all positive elements of $\mathcal{B}(\mathcal{H})$. A continuous function $f(x)$ defined on an interval I in $\mathbb{R}$ is called an operator monotone function if $A \leq B \Rightarrow f(A) \leq f(B)$ for every pair $A, B \in \mathcal{B}(\mathcal{H})$ with spectra in I. In [1], Kubo and Ando showed that for any operator mean $\mathfrak{M}(\cdot, \cdot)$, there uniquely exists an operator monotone function $f \geq 0$ on $[0, \infty)$ with $f(1) = 1$ such that $f(x)I = \mathfrak{M}(I, xI)$. Moreover, they found that $\mathfrak{M}(A, B)$ has explicit form $\mathfrak{M}(A, B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$ when $A > 0$. The function f is called the representing function of $\mathfrak{M}$, and in the following, $\mathfrak{M}_f$ denotes the operator mean whose representing function is f. It is well-known that an operator monotone function $f \geq 0$ on $[0, \infty)$ with $f(1) = 1$ has an integral representation

$$f(x) = \int_{[0,1]} \left[(1-t) + tx^{-1}\right]^{-1} d\mu(t), \qquad (\star)$$

where μ is a probability measure on $[0, 1]$. Note that f has an analytic continuation to the cut plane $\mathbb{C} \setminus (-\infty, 0]$. By this representation, we obtain

$$\mathfrak{M}_f(A, B) = \int_{[0,1]} \left[(1-t)A^{-1} + tB^{-1}\right]^{-1} d\mu(t).$$

We introduce the accretive operator version of an operator mean, and give some properties of it. We also establish the reverse Arithmetic-Geometric-Harmonic inequality for $A = X + iY,\ B = X - iY\ (X > 0,\ Y = Y^*)$[2].

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Title：QUANTUM TOMOGRAPHY WITH GAUSSIAN NOISE

Speaker: Kan He (Taiyuan University of Technology)

Abstract：We introduce a framework for quantum tomography with Gaussian noise. The measurement scheme is based on the symmetric informationally complete positive operator-valued measure. To make the framework realistic, we impose the Gaussian noise on the measured states numbers. The precision of framework is presented on graphs through numerical simulations.

Loewner's theorem for maps on operator domains

Michiya Mori (University of Tokyo)

This talk is based on a joint work [2] with Peter Šemrl (Ljubljana). In this talk, we consider the Loewner order of bounded self-adjoint operators on a complex Hilbert space (or hermitian matrices): For a pair of bounded self-adjoint operators X, Y, the symbol $X \leq Y$ means that $Y - X$ is positive semidefinite.

A real function f defined on an open interval (a, b) is said to be matrix monotone of order n if for every pair of $n \times n$ hermitian matrices X, Y whose eigenvalues belong to (a, b) we have $X \leq Y \Rightarrow f(X) \leq f(Y)$. If f is a matrix monotone function of order n for all positive integers n, we say that f is operator monotone. The study of operator monotone functions was initiated by Loewner [1]. His famous theorem states that a function $f : (a, b) \to \mathbb{R}$ is operator monotone if and only if f has an analytic continuation to the upper half-plane Π which maps Π into itself, and this is true if and only if f has an analytic extension to $(\mathbb{C} \setminus \mathbb{R}) \cup (a, b)$ which maps the upper half-plane Π into itself, and the extension to the lower half-plane is obtained by the reflection across the real line. A number of alternative proofs can be found in Simon's book [3].

We are going to give a variant of Loewner's theorem. Let H be a complex Hilbert space. To avoid trivialities, we assume that H has dimension at least 2. We denote by $B(H)$ the algebra of all bounded linear operators on H, and by $S(H)$ the subset of all self-adjoint operators. Let U, V be subsets of $S(H)$. A map $\phi : U \to V$ preserves order (in one direction) if for every pair $X, Y \in U$ we have $X \leq Y \Rightarrow \phi(X) \leq \phi(Y)$, and it is an order embedding (or preserves order in both directions) if for every pair $X, Y \in U$ we have $X \leq Y \iff \phi(X) \leq \phi(Y)$. It is easy to see that an order embedding is injective. If ϕ is a bijective order embedding, it is called an order isomorphism.

A nonempty subset $U \subset S(H)$ will be called an operator domain if it is open and connected. Here, the topology on $B(H)$ is induced by the operator norm. Let U be an operator domain. A map $\phi : U \to S(H)$ is defined to be a local order isomorphism if for every $X \in U$ there are operator domains V, W with $X \in V \subset U, W \subset S(H)$, such that $\phi(V) = W$ and $\phi : V \to W$ is an order isomorphism. The generalized upper half-plane $\Pi(H)$ is the collection of all operators of the form $X + iY$, where $X \in S(H)$ and Y is a positive invertible operator in $S(H)$. The generalized lower half-plane $\Pi(H)^* = \{X^* : X \in \Pi(H)\}$ is the set of all bounded operators on H whose imaginary part is negative and invertible.

We are now ready to formulate our main theorem.

Theorem 1. Let $U \subset S(H)$ be an operator domain. The following conditions are equivalent for a map $\phi : U \to S(H)$.

- The map ϕ is a local order isomorphism.
- The map ϕ has a unique continuous extension to $U \cup \Pi(H)$ that maps $\Pi(H)$ biholomorphically onto itself.
- There exist open connected sets $\mathcal{U}, \mathcal{V} \subset B(H)$ such that $U \cup \Pi(H) \cup \Pi(H)^* \subset \mathcal{U}$ and ϕ has an extension to a biholomorphic map from $\mathcal{U}$ onto $\mathcal{V}$ that maps $\Pi(H)$ onto itself and $\Pi(H)^*$ onto itself.

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Biholomorphy in our statement replaces holomorphy in the classical Loewner theory, and local order isomorphisms appear in the place of operator monotone functions. This is inevitable. It is not difficult to construct order preserving maps on operator domains that do not have holomorphic extensions to the generalized upper half-plane even under the additional assumptions of bijectivity and continuity.

We have a much better result in the matrix case. We denote by M_n the set of all $n \times n$ complex matrices and by S_n the set of all $n \times n$ hermitian matrices. The corresponding generalized upper half-plane Π_n is the collection of all $n \times n$ complex matrices whose imaginary part is a positive invertible matrix. To avoid unnecessary repetition we present a version with only the generalized upper half-plane involved.

Theorem 2. Let $n \geq 2$ and $U \subset S_n$ be a matrix domain. A map $\phi : U \to S_n$ is an order embedding if and only if ϕ has a unique continuous extension to $U \cup \Pi_n$ that maps Π_n biholomorphically onto itself.

In this talk, I will also give explicit formulae for biholomorphic automorphisms of the generalized upper half-plane and local order isomorphisms, and explain basic properties of such maps.

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Title：Systems of Sylvester-like matrix equations

Speaker：Professor Qing-Wen Wang (Shanghai University)

Abstract： In this talk, we mainly introduce some new results on systems of Sylvester-type matrix equation over the quaternion algebra.

Circuit equation for quantum walks

Etsuo Segawa
(Yokohama National University)

1 Introduction

Irreducible and non-periodic random walks converge to a stationary state [1]. This is very essential to give fruitful aspect of random walks, for example the cut off phenomena and connection to the electric circuit. Quantum walks are introduced as quantum analogue of random walks (see [2] and its references therein.) Then it is natural to consider the corresponding property of quantum walks. However since the eigenvalues of the time evolution operator of quantum walks live on the unit circle in the complex plane, it is hard to obtain the stationary state directly. A relation between the quantum walk and the traditional stationary Schrödinger equation on $\mathbb{L}^\infty(\mathbb{Z})$ is discussed [3]. Then we found that the extension of the squared summable Hilbert space to the uniformly bounded functional state as the total space of the time evolution is one of the useful idea to obtain the stationary state of quantum walks [4]. To this end, we attach the semi-infinite lines, on which the walk is free, to the finite graph and set the initial state on the tails, which is uniformly bounded but no longer square summable, so that a quantum walker penetrates into the internal graph at every time step. Note that since the dynamics on the tails are free, once a quantum walker goes out to the tails, then it never come back to the internal graph, which can be interpreted as the outflow from the internal graph. It is mathematically shown that this dynamical system converges to a fixed point as the stationary state due to the balance between the inflow and outflow [4].

In this talk, we introduce a generalized Laplacian matrix L_z which has the information on the boundary with parameter $z \in \mathbb{C}$. We show that the study on the unitary operator of this model can be switched to that of the finite generalized Laplacian matrix expresses the stationary state.

2 Setting

Let $G = (V, A)$ be a connected and finite graph with the vertex set V and the symmetric arc set A. Set the boundary $\delta V \subset V$ and connect the semi-infinite path to the each boundary vertex. The resulting graph is denoted by $\tilde{G} = (\tilde{V}, \tilde{A})$.

Definition 1 (Grover walk)**.**

(1) Total state space: $\mathbb{C}^{\tilde{A}}$

(2) Time evolution: Let $\Psi_n \in \mathbb{C}^{\tilde{A}}$ *be the n-th iteration such that* $\Psi_{n+1} = U\Psi_n$*. Let the set of the arcs whose terminal vertices are* $u \in \tilde{V} = \{a_1, \dots, a_{\tilde{d}(u)}\}$*. Then the time evolution U is denoted by*

$$\begin{bmatrix} \Psi_{n+1}(\bar{a}_1) \\ \vdots \\ \Psi_{n+1}(\bar{a}_{\tilde{d}(u)}) \end{bmatrix} = \mathrm{Gr}(\tilde{d}(u)) \begin{bmatrix} \Psi_n(a_1) \\ \vdots \\ \Psi_n(a_{\tilde{d}(u)}) \end{bmatrix}$$

Here $\bar{a} \in \tilde{A}$ *is the inverse arc of* $a \in \tilde{A}$.

(3) The initial state: let $|z| = 1$.

$$\Psi_0(a) = \begin{cases} z^{-\mathrm{dist}(t(a),G)} & : a \in \text{tail}, \ \mathrm{dist}(o(a), G) > \mathrm{dist}(t(a), G) \\ 0 & : \textit{otherwise} \end{cases}$$

where $t(a)$ *and* $o(a)$ *are the terminal and origin vertices of* $a \in \tilde{A}$.

Let $\Psi_n(z) := z^n \Psi_n$. Then it is shown that $\Phi_z(a) := \exists \lim_{n\to\infty} \Phi_n(a)$ for any $a \in \tilde{A}$ by [4]. Let $\chi : \mathbb{C}^{\tilde{A}} \to \mathbb{C}^A$ by $(\chi\psi)(a) = \psi(a)$ for any $a \in A$. Let us focus on our interest to the internal $\phi_z := \chi\Phi_z$. To characterize ϕ_z, we introduce the following generalized Laplacian matrix.

Definition 2 (Generalized Laplacian matrix)**.** *Set* $j_\pm(z) = (z \pm z^{-1})/2$. *The generalized Laplacian is defined by*

$$L_z = M - j_+(z)D + j_-(z)\Pi_{\delta V}.$$

Here M and D are the adjacency and the degree matrices of G and $\Pi_{\delta V}$ is the projection matrix of δV.

3 Results

Theorem 3.1 (Circuit equation)**.** *The stationary state restricted to G, ϕ_z, is expressed as follows. There exists a potential function $\nu_z \in \mathbb{C}^V$ such that*

$$j_-(z)\phi_z(a) = z\, \nu_z(t(a)) - \nu_z(o(a)). \tag{1}$$

The potential function ν_z satisfies the following Poisson equation.

$$L_z \nu_z = j_-(z)\, \boldsymbol{\alpha}_{in}. \tag{2}$$

Let $B_* := \{z \neq 0 \ : \ \det(L_z) = 0\} \cup \{\pm 1\}$.

Theorem 3.2. *Let E_{PON} be the principal submatirx of U with respect to A_0; that is, $E_{PON} = \chi U \chi^*$. Let us denote the set* $\mathrm{spec}^\star(E_{PON})$ *as*

$$\mathrm{spec}^\star(E_{PON}) := \{z^{-1} \mid z \in \mathrm{spec}(E_{PON}) \setminus \{0\}\}.$$

Then we have

$$\mathrm{spec}^\star(E_{PON}) \cup \{\pm 1\} = \mathbb{B}_*. \tag{3}$$

In particular,

$$\mathbb{B}_* \cap \delta\mathbb{D} = j_+^{-1}(\sigma_{per}) \cup \{\pm 1\}. \tag{4}$$

Here

$$\sigma_{per} := \{\lambda \in \mathrm{spec}(P_0) \mid \{f \ : \ \mathrm{supp}(f) \subset V_0 \setminus \delta V\} \cap \ker(\lambda - P_0) \neq \emptyset\}.$$

Since ϕ_n can be expressed by the polynomial of E_{PON}, we can show the following.

Theorem 3.3 (Stationary state)**.** *Let λ_{max} with $|\lambda_{max}| < 1$ be the eigenvalue of E_{PON} whose distance to the unit circle is closest. Let $\partial_z^* : \mathbb{C}^V \to \mathbb{C}^A$ such that $(\partial_z^* f)(a) = z f(t(a)) - f(o(a))$. The function*

$$\phi(z) := \partial_z^* L_z^{-1} \boldsymbol{\alpha}_{in}$$

can be extended to $\{z \in \mathbb{C} \ : \ |z| < 1/|\lambda_{max}|\}$ in the entrywise. In particular, the stationary state with the inflow $e^{i\xi}$ is expressed by $\phi(e^{i\theta})$.

In particular, taking $z \to 1$, we show that the stationary state describes the electric circuit [5] using Kato's perturbation theory [6].

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Title: A completely self-scaling G-transformation for weighted least square problems

Speaker: Bing Zheng（Lanzhou University)

Abstract:

The G-transformation is an efficient method for solving the weighted least squares problems. However, the underflows and overflows were not considered in the original G-transformation. In order to keep its stability, some specified scaling strategies has been proposed for guarding against the underflows. Note that these specific strategies are not easy to be implemented in actual operations, in this talk, we present a completely self-scaling G-transformation (CSSGT) which not only avoids these specified scaling strategies, but maintain the stability of operations. Complexity analysis of our self-scaling G-transformation shows that its cost of computation is less than that of the G-transformation, which implies the high efficiency of our proposed SSGT. The stability of the SSGT was theoretically confirmed by a detailed error analysis. Some numerical experiments are performed to illustrate the effects of the self-scaling strategy.

Title: Asymptotic Freeness of Layerwise Jacobians Caused by Invariance of Multilayer Perceptron

Speaker: Tomohiro Hayase (Cluster Inc.)

Abstract: Free Probability Theory provides rich knowledge for handling mathematical difficulties caused by random matrices that appear in research related to deep neural networks, such as the dynamical isometry, Fisher information matrix, and training dynamics. The asymptotic freeness assumption plays a fundamental role when propagating spectral distributions through the layers. In this talk, we introduce the asymptotic freeness of layerwise Jacobians of multilayer perceptron initialized with Haar orthogonal matrices.

Title: Iterative Algorithms for Reducing Inversion of Discrete Algebraic Riccati Matrix Equation

Speaker: Jianzhou Liu (Xiangtan University)

Joining work with Zheng Wang, Zhiming Xie and Li Wang

Abstract: In practical engineering, many control problems usually can be transformed into solutions of the discrete algebraic Riccati equation (DARE) which has two matrix inverse operations formally. In this talk, first, by the relationship between properties of the matrix Schur complement and partitioned representation of inverse matrix, we change the DARE with twice inversions into an equivalent form with once inversion, and propose a corresponding iterative algorithm. Next, for a special case of DARE, we deformed this DARE into a new equivalent one. For the equivalent form, we propose a new iterative algorithm in an inversion free way. Furthermore, for these algorithms, we prove their monotone convergence, and give the analysis of their errors. Last, comparing with some existing work on this topic, corresponding numerical examples are given to illustrate the superiority and effectiveness of our results.

Title: Fast Algorithm for Maxwell’s Equations for 3D Photonic Crystals

Speaker: Tiexiang Li (Southeast University)

Abstract: In this talk, we propose a Fast Algorithm for Maxwell's Equations (FAME) package for solving Maxwell's equations for modeling three-dimensional (3D) photonic crystals, especially those with nonorthogonal Bravais lattices. Yee's scheme assisted by linear interpolation can handle anisotropic media, by which the frequency domain Maxwell's equations are discretized into a standard eigenvalue problem (SEP). FAME combines the null-space free method with fast Fourier transform (FFT)-based matrix-vector multiplications to solve the SEP. We successfully use FAME on a single V100 GPU to solve a set of GEPs with more than 19 million dimensions in 127 to 191 seconds per problem. These results demonstrate the potential of our proposed package to enable large-scale numerical simulations for novel physical discoveries and engineering applications of photonic crystals.

Title: Non-Local Robust Quaternion Matrix Completion for Large-Scale Color Image and Video Inpainting

Speaker: Zhigang Jia (Jiangsu Normal University)

Abstract: The image nonlocal self-similarity (NSS) prior refers to the fact that a local patch often has many nonlocal similar patches to it across the image and has been widely applied in many recently proposed machining learning algorithms for image processing. However, there is no theoretical analysis on its working principle in the literature. In this talk, we discover a potential causality between NSS and low-rank property of color images, which is also available to grey images. A new patch group based NSS prior learning scheme is proposed to learn explicit NSS models of natural color images. The numerical low-rank property of patched matrices is also rigorously proved. The NSS-based QMC algorithm computes an optimal low-rank approximation to the high-rank color image, resulting in high PSNR and SSIM measures and particularly the better visual quality. A new tensor NSS-based QMC method is also presented to solve the color video inpainting problem based on quaternion tensor representation. The numerical experiments on large-scale color images and videos indicate the advantages of NSS-based QMC over the state-of-the-art methods.

Title：Nonnegative Low Rank Tensor Approximation and its Application to Multidimensional Images

Speaker: Guangjing Song (Weifang University)

Abstract：The main aim of this talk is to develop a new algorithm for computing Non-negative Low Rank Tensor (NLRT) approximation for nonnegative tensors that arise in many multi-dimensional imaging applications. Nonnegativity is one of the important property as each pixel value refer to nonzero light intensity in image data acquisition. Our approach is different from classical nonnegative tensor fac-torization (NTF) which has been studied for many years. For a given nonnegative tensor, the classical NTF approach is to determine nonnegative low rank tensor by computing factor matrices or tensors (for example, CPD finds factor matrices while Tucker decomposition finds core tensor and factor matrices), such that the distance between this nonnegative low rank tensor and given tensor is as small as possible. The proposed NLRT approach is different from the classical NTF. It determines a nonnegative low rank tensor without using decompositions or factorization methods. The minimized distance by the proposed NLRT method can be smaller than that by the NTF method, and it implies that the proposed NLRT method can obtain a better low rank tensor approximation. The proposed NLRT approximation algorithm is derived by using the alternating averaged projection on the product of low rank matrix manifolds and non-negativity property. We show the convergence of the alternating projection algorithm. Experimental results for synthetic data and multi-dimensional images are presented to demonstrate the performance of the proposed NLRT method is better than that of existing NTF methods.

Algebraic midpoints and Means

Toshikazu Abe (Ibaraki Univesity)

1 Introduction

We say that a $n \times n$ matrix A is positive if $\langle \boldsymbol{x}, A\boldsymbol{x} \rangle \geq 0$ for all $\boldsymbol{x} \in \mathbb{C}^n$. We denote by $\mathbb{P}_n$ the set of $n \times n$ positive invertible matrices. For $A, B \in \mathbb{P}_n$, we use the notation $A \geq B$ to mean that the matrix $A - B$ is positive. In particular, $\mathbb{P}_1 = \mathbb{R}_+$ is the set of all positive real numbers.

The map $M : \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n$ is called a mean, if the following conditions (M1) to (M5) are fulfilled.

(M1) If $a \leq b$, then $a \leq M(a, b) \leq b$.

(M2) $M(a, b) = M(b, a)$.

(M3) $M(a, b)$ is monotone increasing in a, b.

(M4) $M(x^*ax, x^*bx) = x^*M(a, b)x$ for all $a, b \in \mathbb{P}_n$ and nonsingular $x \in \mathbb{M}_n(\mathbb{C})$.

(M5) $M(a, b)$ is continuous in a, b.

In this talk, we study the relationship between algebraic midpoints and means.

2 Binary operations and algebraic midpoints

A magma $(S, \oplus)$ is a set S with a binary operation $\oplus$. We say that $(S, \oplus)$ is uniquely 2-divisible if, for any $a \in S$ there exists a unique element $b \in S$ such that $a = b \oplus b$. The element b is called the half of a. In this talka, for uniquely 2-divisible magma $(S, \oplus)$, we denote by $\frac{1}{2} \otimes a$ the half of $a \in S$.

Definition 1. A magma $(G, \oplus)$ is called a gyrogroup if it satisfies the following (G1) to (G5).

(G1) $(G, \oplus)$ has the identity e.

(G2) For any $a \in (G, \oplus)$, a has the inverse $\ominus a$.

(G3) For any $a, b, c \in G$, there exists a unique element $\mathrm{gyr}[a, b]c$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c.$$

(G4) For any $a, b \in G$, the map $\mathrm{gyr}[a, b] : G \to G$ defined by $c \mapsto \mathrm{gyr}[a, b]c$ for any c is an automorphism of the magma $(G, \oplus)$, that is $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$. The map $\mathrm{gyr}[a, b]$ is called a gyroautomorphism of $(G, \oplus)$ generated by a and b.

(G5) For any $a, b \in G$, $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$.

A gyrogroup $(G, \oplus)$ is gyrocommutative if the following (G6) is satisfied.

(G6) For any $a, b \in G$, $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$.

Definition 2. Let $(G, \oplus)$ be a uniquely 2-divisible gyrocommutative gyrogroup. For $a, b \in G$, $\frac{1}{2} \otimes (a \oplus \mathrm{gyr}[a, \ominus b]b)$ is called the gyromidpoint of a and b.

Definition 3. Let $(S, \oplus)$ be a uniquely 2-divisible commutative semi-group. For $a, b \in S$, $\frac{1}{2} \otimes (a \oplus b)$ is called the semi-group midpoint of a and b.

3 Means and algebraic midpoints

Theorem 4. *Let $(\mathbb{P}_n, \oplus)$ be a uniquely 2-divisible commutative semi-group. If $M(a, b) = \frac{1}{2} \otimes (a \oplus b)$ is a mean, then the following conditions* (i) *and* (ii) *are equivalent to each other.*

(i) *$(\mathbb{P}_n, \oplus)$ is (left and right) cancellative.*

(ii) *$b \neq c$ implies $M(a, b) \neq M(a, c)$.*

Theorem 5. *Let $(\mathbb{R}_+, \oplus)$ be a uniquely 2-divisible gyrocommutative gyrogroup. If $M(a, b) = \frac{1}{2} \otimes (a \boxplus b)$ is a mean on $\mathbb{R}_+$, then $(\mathbb{R}_+, \oplus)$ is a group. In particular, $(\mathbb{R}_+, \oplus)$ is isomorphic to $(\mathbb{R}, +)$.*

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Title: Monolithic projection-based method with staggered time discretization for solving non-Oberbeck-Boussinesq natural convection flows

Speaker: Xiaomin Pan (Shanghai University)

Joining work with Ki-Ha Kim (Yonsei University) and Jung-Il Choi (Yonsei University)

Abstract： In this talk we present an efficient monolithic projection-based method with staggered time discretization (MPM-STD) to examine the non-Oberbeck-Boussinesq (NOB) effects in several natural convection problems involving dramatic temperature-dependent changes in fluid properties. The proposed approach employs the Crank-Nicolson scheme along with staggered time discretization to discretize the momentum and energy equations. The momentum and energy equations are decoupled by evaluating the velocity vector at integral time levels (n+1) whereas the scalar variables (pressure and temperature) at half-integral time levels (n+1/2). The observed density variations in all terms result in a variable-coefficient Poisson equation, which is difficult to solve efficiently. The convergence is accelerated via adoption of an appropriate pressure-correction scheme that transforms the aforementioned Poisson equation to a constant-coefficient form. The numerical simulations concerning two-dimensional (2D) periodic NOB Rayleigh-Bénard convection (RBC) in glycerol confirmed the second-order temporal and spatial accuracies of the proposed method. By simulating the 2D differentially heated cavity problem in air and the RBC problem in liquid (water or glycerol) considering NOB effects, it is concluded that the proposed MPM-STD significantly mitigates the time-step restriction, thereby increasing the computational efficiency, which exceeds that of existing semi-implicit and explicit schemes. Moreover, the potential of the proposed approach with regard to solving challenging three-dimensional turbulent problems is demonstrated by performing direct simulations of turbulent RBCs under NOB effects involving temperature differences up to 60 K with corresponding Rayleigh number.

Surjective isometries and Hermitian operators on vector-valued Lipschitz algebras

Shiho Oi (Niigata University)*

1. Introduction

Let (X, d) be a compact metric space and $(E, \|\cdot\|_E)$ be a complex Banach space. A map $F : X \to E$ is said to be Lipschitz if

$$L(F) := \sup_{x \neq y \in X} \left\{ \frac{\|F(x) - F(y)\|_E}{d(x, y)} \right\} < \infty.$$

We denote a space of all Lipschitz maps from X into E by $\mathrm{Lip}(X, E)$. When $E = \mathbb{C}$, we simply write $\mathrm{Lip}(X)$. The Lipschitz space $\mathrm{Lip}(X, E)$ is a Banach space with

$$\|F\|_L = \sup_{x \in X} \|F(x)\|_E + L(F), \quad F \in \mathrm{Lip}(X, E).$$

In particular, $\mathrm{Lip}(X, E)$ endowed with $\|\cdot\|_L$ is a Banach algebra if E is a Banach algebra. When a Banach algebra E has the unit 1_E, then $\mathrm{Lip}(X, E)$ has the unit, which is a constant map $1(x) = 1_E$ for any $x \in X$. When no confusion is caused, we denote unity of a unital Banach algebra by 1.

In this talk, we consider surjective complex linear isometries on $\mathrm{Lip}(X, \mathcal{A})$, where $\mathcal{A}$ is a unital C^*-algebra. It would be interesting to see whether every surjective linear isometry on algebras is closely related to a Jordan $*$-isomorphism.

2. Hermitian operators on $\mathrm{Lip}(X, E)$

In order to study a representation of surjective complex linear isometries on the vector-valued Lipschitz algebras, we study hermitian operators on $\mathrm{Lip}(X, E)$. Lumer [3] defined Hermitian operators on Banach space with respect to semi-inner product. For a complex Banach space $(E, \|\cdot\|_E)$, if a map $[\cdot, \cdot] : E \times E \to \mathbb{C}$ satisfies

1. $[\lambda x + y, z] = \lambda[x, z] + [y, z]$,
2. $[x, x] = \|x\|^2$,
3. $|[x, y]|^2 \leq [x, x][y, y]$,

for any $x, y, z \in E$ and $\lambda \in \mathbb{C}$, a map $[\cdot, \cdot]$ is said to be semi-inner product on E compatible with the norm of E. A bounded linear operator T on E is called a Hermitian operator if there is a semi-inner product on E compatible with the norm of E such that $[T(a), a] \in \mathbb{R}$ for any $a \in E$. For more details on Hermitian operators, we refer the reader to [2] and [3].

We give a characterization of Hermitian operators on $\mathrm{Lip}(X, E)$ as the following. This is the generalization of [1].

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Theorem 2.1. *Let X be a compact metric space and E a complex Banach space. Then $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(X, E)$ is a Hermitian operator if and only if there exists a Hermitian operator $\phi : E \to E$ such that*

$$TF(x) = \phi(F(x)), \quad F \in \mathrm{Lip}(X, E), \quad x \in X.$$

For each $a \in \mathcal{A}$, a left multiplication operator $M_a : \mathcal{A} \to \mathcal{A}$ is defined by $M_a b = ab$ for every $b \in \mathcal{A}$. We denote the set of all hermitian elements of $\mathcal{A}$ by $H(\mathcal{A})$. For any $h \in H(\mathcal{A})$, we define a multiplication operator $M_{1\otimes h} : \mathrm{Lip}(X, \mathcal{A}) \to \mathrm{Lip}(X, \mathcal{A})$ by

$$M_{1\otimes h}(F) = (1 \otimes h)F, \quad F \in \mathrm{Lip}(X, \mathcal{A}).$$

For any $*$-derivation $D : \mathcal{A} \to \mathcal{A}$, we define a map $\widehat{D} : \mathrm{Lip}(X, \mathcal{A}) \to \mathrm{Lip}(X, \mathcal{A})$ by

$$\widehat{D}(F)(x) = D(F(x)), \quad F \in \mathrm{Lip}(X, \mathcal{A}), \quad x \in X.$$

As a corollary of Theorem 2.1 and [6], we obtain the following.

Corollary 2.2. *Suppose that $T : \mathrm{Lip}(X, \mathcal{A}) \to \mathrm{Lip}(X, \mathcal{A})$ is a map. Then T is a Hermitian operator if and only if there exists $h \in H(\mathcal{A})$ and a $*$-derivation D on $\mathcal{A}$ such that*

$$T = M_{1\otimes h} + i\widehat{D}.$$

3. Surjective linear isometries on $\mathrm{Lip}(X, \mathcal{A})$

Let B_j be Banach algebras for $j = 1, 2$. Suppose that U is a surjective complex linear isometry from B_1 onto B_2 and T is a Hermitian operator on B_1. It is well-known fact that the map UTU^{-1} is a Hermitian operator on B_2. By applying this fact, we get the main theorem. This theorem is a generalization of [4].

Theorem 3.1. *Let X_i be compact metric spaces and $\mathcal{A}_i$ unital factor C^*-algebras for $i = 1, 2$. The map $U : \mathrm{Lip}(X_1, \mathcal{A}_1) \to \mathrm{Lip}(X_2, \mathcal{A}_2)$ is a surjective complex linear isometry such that $U(1) = 1$ if and only if there exist a unital surjective complex linear isometry $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ and a surjective isometry $\varphi : X_2 \to X_1$ such that*

$$UF(y) = \psi(F(\varphi(y))), \quad F \in \mathrm{Lip}(X_1, \mathcal{A}_1), y \in X_2.$$

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