Compactness characterization of operators in the Toeplitz algebra

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Outline

1. Definition and notation
2. Lipschitz approximation and TO on the Fock space
3. TO on Bergman spaces over BSDs
4. Compactness in the Toeplitz algebra

Segal-Bargmann space

Consider $\mathbb{C}^n$ with Gaussian probability measure

$$d\mu(z) = \pi^{-n} e^{-|z|^2} dv(z),$$

- $|z|^2 = z \cdot \overline{z}$ and $z \cdot \overline{w} = z_1 \cdot \overline{w_1} + \cdots + z_n \cdot \overline{w_n}$.
- $dv =$ *Lebesgue volume form* on $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Segal-Bargmann space

The Segal-Bargmann space (or Fock space) [a] is the Hilbert space

$$H^2(\mathbb{C}^n, d\mu) := L^2(\mathbb{C}^n, d\mu) \cap \underbrace{\mathcal{H}(\mathbb{C}^n)}_{\text{entire functions}} .$$

of entire L^2-functions with reproducing kernel function:

$$K : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} : K(z, w) = \exp\left\{z \cdot \overline{w}\right\}.$$

[a] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. 14 (1961), 187-214.

Reproducing kernel property:

For all $z \in \mathbb{C}^n$ and all $f \in H^2(\mathbb{C}^n, d\mu)$ it holds

$$f(z) = \delta_z(f) = \left\langle f, K(\cdot, z)\right\rangle_{L^2(\mathbb{C}^n, d\mu)}.$$

In the following also the normalized reproducing kernels play a role:

$$k_w(z) := \frac{K(z, w)}{\|K(\cdot, w)\|} = \exp\left\{z \cdot \overline{w} - \frac{|w|^2}{2}\right\}.$$

Note: The distance induces by the Bergman metric

$$g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K(z, z), \qquad z \in \mathbb{C}^n$$

(up to a factor) coincides with the usual Euclidean distance:

$$d(z, w) := |z - w|.$$

The spaces $UC(\mathbb{C}^n)$ and $BMO^2(\mathbb{C}^n)$

On $\mathbb{C}^n$ we consider functions spaces:

- $\text{Lip}(\mathbb{C}^n)$="Lipschitz continuous functions w.r.t. d".
- $\text{UC}(\mathbb{C}^n)$="uniformly continuous functions".

Note

Both spaces contain unbounded functions and

$$\text{Lip}(\mathbb{C}^n) \subset \text{UC}(\mathbb{C}^n). \tag{$*$}$$

Define:

- $\text{BUC}(\mathbb{C}^n)$ ="bounded functions in $\text{UC}(\mathbb{C}^n)$".

We add a remark on the inclusion $(*)$ in a more general framework:

Definition (Metrically convex space)

A metric space (X, d) is called metrically convex if **(MC)** is true:

(MC):

Two closed balls $B(x, s)$ and $B(y, t)$ around $x \in X$ and $y \in X$ and with radii $s \geq 0$ and $t \geq 0$ intersect if and only if $d(x, y) \leq s + t$.

Example:

Any complete Riemannian manifold is a metrically convex space.

The following is known: [1]

Theorem

Let (X, d) be metrically convex. Then the space of all Lipschitz functions $\text{Lip}(X)$ is uniformly dense in $\text{UC}(X)$.

[1] e.g. see: Y. Benyamini, J. Lindenstrauss, *Geometric non-linear functional analysis*, AMS Colloquium Publication vol. 48, 2000.

Definition (heat transform)

Let $t > 0$, then the heat transform of (suitable) $f : \mathbb{C}^n \to \mathbb{C}$ is defined by:

$$\begin{aligned}\widetilde{f}^{(t)}(w) :&= \frac{1}{(4\pi t)^n} \int_{\mathbb{C}^n} f(w-z) e^{-\frac{|z|^2}{4t}} dv(z) \\ &= \text{"solution of the heat equation"}.\end{aligned}$$

Semi-group-property: $\widetilde{\{\widetilde{f}^{(s)}\}}^{(t)} = \widetilde{f}^{(t+s)}$, (if defined).

Definition:

The mean oscillation of the function f at time $t > 0$ is given by the non-negative function:

$$\begin{aligned}\mathrm{MO}_t(f,w) :&= \widetilde{|f|^2}^{(t)}(w) - |\widetilde{f}^{(t)}(w)|^2 \\ &= \widetilde{\left\{|f - \widetilde{f}^{(t)}(w)|^2\right\}}^{(t)}(w).\end{aligned}$$

Definition

The functions having bounded mean oscillation are given by:

$$\mathrm{BMO}_t^2(\mathbb{C}^n) := \left\{ f \; : \; \|f\|_{\mathrm{BMO}_t} := \sup_{z \in \mathbb{C}^n} \mathrm{MO}_t(f,z)^{\frac{1}{2}} < \infty \right\}. \qquad (*)$$

Remarks:

- The spaces $(*)$ are linear and independent of $t > 0$. Hence we denote them by $\mathrm{BMO}^2(\mathbb{C}^n)$.
- $\|\cdot\|_{\mathrm{BMO}_t}$ depend on $t > 0$ and only define semi-norms.
- The following inclusions hold:

$$\mathrm{BUC}(\mathbb{C}^n) \subset \mathrm{UC}(\mathbb{C}^n) \subset \mathrm{BMO}^2(\mathbb{C}^n) \subset L^2(\mathbb{C}^n, d\mu).$$

In particular: $\mathrm{BMO}^2(\mathbb{C}^n)$ contains unbounded functions.

Definition ($\mathrm{BO}(\mathbb{C}^n)$)

A continuous function $f \in C(\mathbb{C}^n)$ is of bounded oscillation if there is $C > 0$ such that for all $z, w \in \mathbb{C}^n$:

$$|f(z) - f(w)| \leq C + C|z - w|.$$

The relation between $\mathrm{BMO}^2(\mathbb{C}^n)$ and $\mathrm{BO}(\mathbb{C}^n)$ is as follows:

Lemma

The inclusion $\mathrm{BO}(\mathbb{C}^n) \subset \mathrm{BMO}^2(\mathbb{C}^n)$ holds. More precisely,

$$\mathrm{BMO}^2(\mathbb{C}^n) = \mathrm{BO}(\mathbb{C}^n) + F(\mathbb{C}^n) : f = \widetilde{f}^{(t)} + (f - \widetilde{f}^{(t)}),$$

where $F(\mathbb{C}^n) := \{f \in \mathrm{BMO}^2(\mathbb{C}^n) \ : \ \widetilde{|f|}^{(t)}$ is bounded$\}$.

We obtain the inclusions:

$$\mathrm{BUC}(\mathbb{C}^n) \subset \mathrm{UC}(\mathbb{C}^n) \subset \mathrm{BO}(\mathbb{C}^n) \subset \mathrm{BMO}^2(\mathbb{C}^n) \subset L^2(\mathbb{C}^n, d\mu).$$

Observation:

If $f \in \mathrm{BMO}^2(\mathbb{C}^n)$, then $\widetilde{f}^{(t)} \in \mathrm{Lip}(\mathbb{C}^n)$: it holds for all $z, w \in \mathbb{C}^n$:

$$\left|\widetilde{f}^{(t)}(z) - \widetilde{f}^{(t)}(w)\right| \leq 2\|f\|_{\mathrm{BMO}_t}|z - w|.$$

Roughly: $\mathrm{Lip}(\mathbb{C}^n)$ forms the "difference" of $\mathrm{UC}(\mathbb{C}^n)$ and $\mathrm{BUC}(\mathbb{C}^n)$:

Lemma

Let $t > 0$ and $f \in \mathrm{UC}(\mathbb{C}^n)$, then

- $\widetilde{f}^{(t)} \in \mathrm{Lip}(\mathbb{C}^n)$,
- $f - \widetilde{f}^{(t)} \in \mathrm{BUC}(\mathbb{C}^n)$.

Hence we have the decomposition:

$$\mathrm{UC}(\mathbb{C}^n) = \mathrm{Lip}(\mathbb{C}^n) + \mathrm{BUC}(\mathbb{C}^n).$$

In particular: *If a function $f \in \mathrm{UC}(\mathbb{C}^n)$ is unbounded, then the heat transform $\widetilde{f}^{(t)}$ is unbounded for all $t > 0$ as well.*

Theorem (W.B. and L.A. Coburn, 2012)

Let $f \in \mathrm{UC}(\mathbb{C}^n)$*, then the heat transform* $\{\tilde{f}^{(t)}\}_{t>0}$ *defines a flow of real analytic functions in* $\mathrm{Lip}(\mathbb{C}^n)$ *with*

$$\lim_{t\to 0} \tilde{f}^{(t)} = f$$

uniformly on $\mathbb{C}^n$*. The Lipschitz constant of* $\tilde{f}^{(t)}$ *is dominated by*

$$C_t := t^{-\frac{1}{2}} \| f(\cdot\, 2\sqrt{t}) \|_{\mathrm{BMO}_{1/4}}.$$

In particular, the inclusion $\mathrm{Lip}(\mathbb{C}^n) \cap C^\omega(\mathbb{C}^n) \subset \mathrm{UC}(\mathbb{C}^n)$ *is dense.*

Remark:

There is a completely analogous version of the theorem with $\mathbb{C}^n$ replaced by $\mathbb{R}^n$.

Toeplitz operators on the Segal-Bargmann space

Consider the orthogonal projection

$$P : L^2(\mathbb{C}^n, d\mu) \to H^2(\mathbb{C}^n, d\mu).$$

Fix a function

$$f \in \mathcal{T}(\mathbb{C}^n) := \\ = \Big\{ f \in L^2(\mathbb{C}^n, d\mu) \; : \; f(w + \cdot) \in L^2(\mathbb{C}^n, d\mu) \;\; \text{for all } w \in \mathbb{C}^n \Big\}.$$

Definition: Toeplitz operator

The assignment

$$T_f : H^2(\mathbb{C}^n, d\mu) \supset \mathcal{D} \to H^2(\mathbb{C}^n, d\mu) : g \mapsto P(fg)$$

is called Toeplitz operator with symbol f and domain

$$\mathcal{D} := \mathrm{span}\Big\{ K(\cdot, w) \; : \; w \in \mathbb{C}^n \Big\} \overset{\text{dense}}{\subset} H^2(\mathbb{C}^n, d\mu).$$

Problems:

(A) Characterize boundedness of T_f in terms of the symbol f and provide norm estimates.

(B) Characterize compactness or Schatten-p-properties of T_f in terms of f.

(C) For which symbols are the following characterizations true:
- T_f bounded if and only if f bounded.
- T_f compact if and only if f vanishes at infinity?

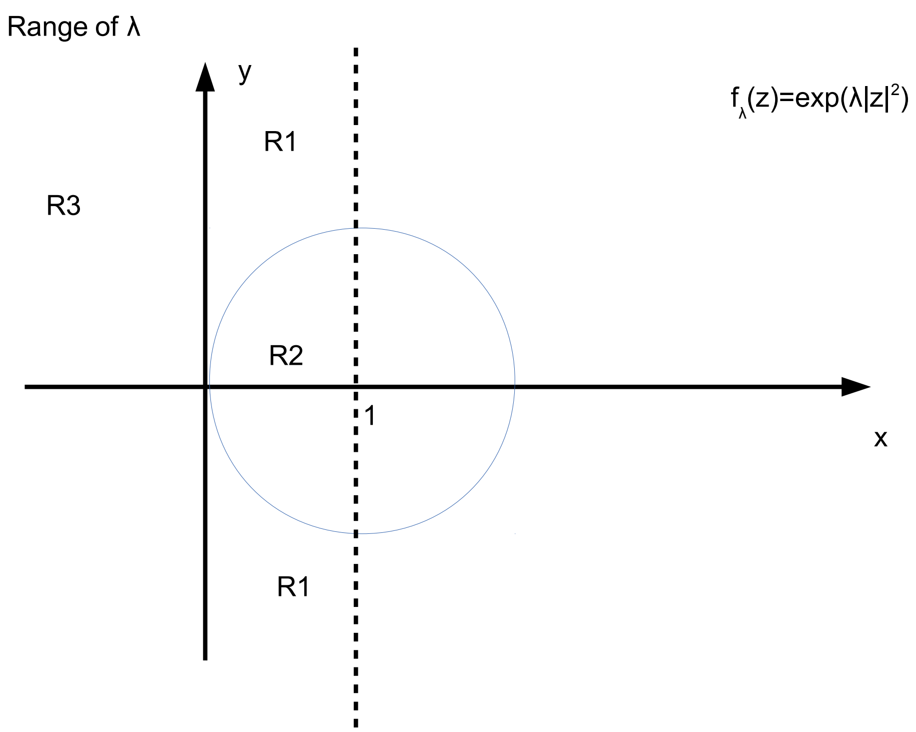
Example: Let $n = 1$ and with $\lambda \in \mathbb{C}$ consider the functions

$$f_\lambda(z) := e^{\lambda|z|^2}.$$

- $f_\lambda \in \mathcal{T}(\mathbb{C}^n)$ iff $\mathrm{Re}(\lambda) < \frac{1}{2}$.
- T_{f_λ} is diagonal with eigenvalue sequence $\{\gamma_j\}_{j=0,1,\cdots}$, where

$$\gamma_j := \frac{1}{(1-\lambda)^j}$$

Boundedness and compactness of T_{f_λ}



Theorem, (W.-B., L.A. Coburn, J. Isralowitz, 2010)

Let $f \in \mathrm{BMO}^2(\mathbb{C}^n)$:

(i) If T_f is bounded if and only if the heat transform $\widetilde{f}^{(t)}$ is bounded for all $t > 0$.

(ii) If T_f is compact if and only if $\widetilde{f}^{(t)} \in C_0(\mathbb{C}^n)$ [a] for all $t > 0$.

[a] with the notation $C_0(\mathbb{C}^n) :=$ *continuous functions vanishing at infinity.*

Example: Again consider the functions $f_\lambda(z) = \exp(\lambda|z|^2)$. We calculate the heat transform:

$$\widetilde{f_\lambda}^{(t)}(z) = \frac{1}{1-4t\lambda} \exp\left\{ \frac{\lambda - 4t|\lambda|^2}{|1-4t\lambda|^2} |z|^2 \right\}.$$

Observation: If $\mathrm{Re}(\lambda) > 0$ and $|\mathrm{Im}(\lambda)| >> 0$, then the Real part of the exponent change sign as $t \downarrow 0$.

Moreover: T_{f_λ} is compact for $\mathrm{Re}(\lambda) < \frac{1}{2}$ and $\mathrm{Im}(\lambda) >> 0$.

We address question (C):

Theorem, (W.-B., L.A. Coburn, 2014)

Let $f \in \mathrm{UC}(\mathbb{C}^n)$, then

(a) T_f is bounded if and only if f is bounded on $\mathbb{C}^n$.

(b) T_f is compact if and only if $f \in C_0(\mathbb{C}^n)$.

Proof of (b): The implication "$\Longleftarrow$" is standard. We omit it.

"$\Longrightarrow$": Let T_f be compact. Since $\mathrm{UC}(\mathbb{C}^n) \subset \mathrm{BMO}^2(\mathbb{C}^n)$ we conclude from the last Theorem:

$$\widetilde{f}^{(t)} \in C_0(\mathbb{C}^n), \qquad \text{for all} \quad t > 0.$$

Since $f \in \mathrm{UC}(\mathbb{C}^n)$ we have the uniform convergence

$$\lim_{t \to 0} \widetilde{f}^{(t)} = f$$

and therefore $f \in C_0(\mathbb{C}^n)$. □

Remark: The theorem fails if one replaces $\mathrm{UC}(\mathbb{C}^n)$ by $\mathrm{BMO}^2(\mathbb{C}^n)$.

Toeplitz operators on Bergman spaces over BSDs

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain.

Definition (BSD)

Ω is called a bounded symmetric domain (BSD) if each $w \in \Omega$ is an isolated fixpoint of an involutive holomorphic diffeomorphism of Ω onto itself. [a]

[a] A BSD is a Hermitian space of non-compact type

Harish-Chandra realization of Ω:

- Ω contains 0 and is invariant under the dilation $z \mapsto \lambda z$ where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.
- There is a polydisc D^r such that

$$\Omega = KD^r, \qquad r = rank\ of\ \Omega,$$

where $K \subset \mathrm{Aut}(\Omega)$ is the isotropy subgroup of 0.

Example:

Let $\Omega := \{z \in \mathbb{C}^n \ : \ |z| < 1\}$ be the open unit ball. Then $r = 1$.

Remark:

The action of the Lie group $\mathrm{Aut}(\Omega)$ on Ω is transitive. For each $w \in \Omega$ there is $\varphi_w \in \mathrm{Aut}(\Omega)$ such that

$$\varphi_w \circ \varphi_w = \mathrm{id} \qquad \text{and} \qquad \varphi_w(0) = w.$$

Bergman metric on Ω:

Let dv be the usual Lebesgue measure on Ω normalized to one, i.e. $v(\Omega) = 1$. Consider the Bergman space:

$$H^2(\Omega, dv) := \Big\{ f \in L^2(\Omega, dv) \ : \ f \ \text{ is holomorphic on } \Omega \Big\}.$$

Note: $H^2(\Omega, dv)$ forms a Hilbert sub-space of $L^2(\Omega, dv)$ with reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}$: for $w \in \Omega$ and $f \in H^2(\Omega, dv)$:

$$f(w) = \int_\Omega f(z)K(w,z)dv(z)$$

Properties of K

- $K(\cdot, w) \in H^2(\Omega, dv)$ for all $w \in \Omega$ and $K(z,w) = \overline{K(w,z)}$,
- $K(z,0) = K(0,z) \equiv 1$,
- $K(z,z) > 0$ and $\lim_{z\to\partial\Omega} K(z,z) = \infty$.

Example: Let $\Omega := \{z \in \mathbb{C}^n \ : \ |z| < 1\}$ be the open unit ball.

The Bergman kernel of Ω is given by:

$$K(z,w) = \frac{1}{(1 - z \cdot \bar{w})^{n+1}}.$$

In particular,

$$\lim_{z\to\partial\mathbb{B}^n} K(z,z) = \lim_{z\to\partial\mathbb{B}^n} \frac{1}{(1 - |z|^2)^{n+1}} = \infty.$$

Definition (Bergman metric)

The function $z \mapsto K(z,z)$ induces a complete Riemannian metric (Bergman metric) on Ω, via

$$g_{ij} = \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K(z,z),$$

where $i,j = 1, \cdots, n$ and $z \in \Omega$.

The Bergman metric induces a distance function

$$\beta(\cdot,\cdot) : \Omega \times \Omega \to \mathbb{R}_+.$$

Remark:

The β-metric topology on Ω is equivalent to the usual Euclidean topology inherited from $\mathbb{C}^n$.

Example

Let $\Omega = \mathbb{D} \subset \mathbb{C}$ be the unit disc. Then $\mathrm{Aut}(\mathbb{D})$ =Möbius tranforms and it is known that:

$$\beta(0, z) = \frac{1}{\sqrt{2}} \log\left(\frac{1+|z|}{1-|z|}\right) = \textbf{hyperbolic metric}.$$

For BSDs $\Omega \subset \mathbb{C}^n$ more is known about the Bergman kernel K:

There is a function (Jordan triple determinant)

$$h = h(z, w) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$$

such that $h(\cdot, w)$ is a polynomial and:

- $h(z, 0) = 1$ and $h(z, w) = \overline{h(w, z)}$ for all $z, w \in \mathbb{C}^n$.
- $h(z, z) > 0$ for all $z \in \Omega$ and $h(z, z) = 0$ for all $z \in \partial\Omega$.

Lemma

Let $p > 0$ be the genus of the BSD $\Omega \subset \mathbb{C}^n$, then the reproducing kernel K of Ω has the form

$$K(z, w) = h(z, w)^{-p}, \qquad z, w \in \Omega.$$

With $\lambda > p - 1$ we define the norm:

$$\|f\|_\lambda^2 := c_\lambda \int_\Omega |f(z)|^2 h(z, z)^{\lambda - p} dv(z).$$

Here the constant $c_\lambda > 0$ is chosen with $\|e\|_\lambda = 1$ where $e(z) \equiv 1$.

Note: The norm $\|\cdot\|_p$ coincides with the $L^2(\Omega, dv)$-norm.

Lemma

The normalizing constant $c_\lambda > 0$ in the definition of $\|\cdot\|_\lambda$ has the explicit form

$$c_\lambda = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\lambda - \frac{n}{r})},$$

where $\Gamma_\Omega(\lambda)$ is the Gindikin Gamma function.

Next goal:

In the above framework we aim to define suitable replacements of the function spaces

- $\mathrm{UC}(\mathbb{C}^n) \Longrightarrow \mathrm{UC}(\Omega)$,
- $\mathrm{Lip}(\mathbb{C}^n) \Longrightarrow \mathrm{Lip}(\Omega)$,
- $\mathrm{BO}(\mathbb{C}^n) \Longrightarrow \mathrm{BO}(\Omega)$,
- $\mathrm{BMO}^2(\mathbb{C}^n) \Longrightarrow \mathrm{BMO}^2(\Omega)$,
- $\cdots$

Lipschitz approximation and Berezin-Harish Chandra flow

Definition

Let $\mathrm{UC}(\Omega)$ and $\mathrm{Lip}(\Omega)$ be the spaces of uniformly continuous and Lipschitz functions on Ω with respect to the Bergman metric β.

To define the space

$$\mathrm{BMO}^2(\Omega)$$

we need a "good replacement" for the heat transform on $\mathbb{C}^n$.

Definition (weighted Bergman space)

The weighted Bergman space with weight $\lambda > p - 1$ is defined by:

$$H^2_\lambda(\Omega, dv) = \Big\{ f \in \mathcal{H}(\Omega) \, : \, \|f\|_\lambda < \infty \Big\}.$$

In particular, these include the unweighted Bergman space:

$$H^2_p(\Omega, dv) = H^2(\Omega, dv)$$

Lemma

The weighted Bergman space $H^2_\lambda(\Omega, dv)$ with $\lambda > p-1$ has a reproducing kernel of the form

$$K_\lambda(z,w) = h(z,w)^{-\lambda}.$$

Let $g \in L^1(\Omega, dv)$ and $\varphi_w \in \mathrm{Aut}(\Omega)$ with $w \in \Omega$ be an involutive automorphism with

$$\varphi_w(0) = w.$$

Definition (Berezin-Harish-Chandra flow)

Assume that $\lambda \geq p$ and $w \in \Omega$. We define a family of integral transforms of g by

$$B_\lambda(g)(w) := c_\lambda \int_\Omega g \circ \varphi_w(z) h(z,z)^{\lambda-p} dv(z).$$

Lemma

Let $g \in L^1(\Omega, dv)$ and $\lambda \geq p$. The Berezin-Harish Chandra flow can also be expressed in the form

$$B_\lambda(g)(w) = c_\lambda \int_\Omega g(z) |k_w^\lambda(z)|^2 h(z,z)^{\lambda-p} dv(z),$$

where $k_w^\lambda \in H^2_\lambda(\Omega, dv)$ with $w \in \Omega$ is the normalized reproducing kernel:

$$k_w^\lambda(z) = \frac{K_\lambda(z,w)}{\|K_\lambda(\cdot,w)\|_\lambda} = \frac{h(z,w)^{-\lambda}}{h(w,w)^{-\frac{\lambda}{2}}}.$$

Moreover, $B_\lambda(g)(w)$ is a real analytic function on Ω.

Remark: The same construction for $\Omega = \mathbb{C}^n$ leads to the heat transform, i.e. in this case

$$B_\lambda(g) = \widetilde{g}^{(\lambda)}.$$

Example

Let $\Omega = \mathbb{B}^n$:=Euclidean unit ball in $\mathbb{C}^n$. Then

- The rank of $\mathbb{B}^n$ is $r = 1$ and the genus $p = n + 1$.
- The Gindikin Gamma function $\Gamma_\Omega(\lambda)$ coincides with the usual Gamma function $\Gamma(\lambda)$.

If $\lambda = n + 1 + \alpha$ where $\alpha \geq 0$, then

$$B_{n+1+\alpha}(g)(w) = \\ = \frac{1}{\pi^n} \frac{\Gamma(n+1+\alpha)}{\Gamma(\alpha+1)} \int_{\mathbb{B}^n} g(z) \frac{(1-|w|^2)^{n+1+\alpha}(1-|z|^2)^\alpha}{|1 - z \cdot \overline{w}|^{2(n+1+\alpha)}} dv(z).$$

Remark

In this setting $B_{n+1+\alpha}(g)$ also is called α-Berezin transform of g.

Definition (Mean oscillation)

Let $\lambda \geq p$ and $f \in L^2(\Omega, dv)$ where Ω is a BSD. The λ-mean oscillation of f is defined by

$$\mathrm{MO}_\lambda(f, z) = B_\lambda(|f|^2)(z) - |B_\lambda(f)(z)|^2 \\ = B_\lambda\Big(|f - B_\lambda(f)(z)|^2\Big)(z) \geq 0.$$

Moreover, we form the semi-norms

$$\|f\|_{\mathrm{BMO}_\lambda} := \sup_{z \in \Omega} \sqrt{\mathrm{MO}_\lambda(f, z)}.$$

The functions of bounded λ-mean oscillation are given by:

$$\mathrm{BMO}^2_\lambda(\Omega) := \Big\{ f \in L^2(\Omega, dv) \; : \; \|f\|_{\mathrm{BMO}_\lambda} < \infty \Big\}$$

Definition (bounded oscillation on Ω)

A function $f \in C(\Omega)$ is said to be of bounded oscillation with respect to the Bergman metric β and we write

$$f \in \mathrm{BO}(\Omega),$$

if and only if there is $C > 0$ such that

$$\big|f(z) - f(w)\big| \leq C\Big(1 + \beta(z,w)\Big)$$

for all $z, w \in \Omega$.

Let $\lambda \geq p$ then one can prove the following inclusions:

$$\mathrm{Lip}(\Omega) \subset \mathrm{UC}(\Omega) \subset \mathrm{BO}(\Omega) \subset \mathrm{BMO}^2_\lambda(\Omega).$$

Remark:

The space $\mathrm{UC}(\Omega)$ contains unbounded functions. However, we always have the inclusions

$$\mathrm{UC}(\Omega) \subset \bigcap_{r>0} L^r(\Omega, dv)$$

and therefore $B_\lambda(f)$ is defined for all $f \in \mathrm{UC}(\Omega)$ and $\lambda \geq p$.

Completely analogous to the Euclidean case we have:

Lemma (W.-B. and L.A. Coburn, 2012)

Let $f \in \mathrm{UC}(\Omega)$, then $\lim_{\lambda \to \infty} B_\lambda(f) = f$ uniformly on Ω.

The following questions remain:

- Is it true that $B_\lambda(f) \in \mathrm{Lip}(\Omega)$ for all $\lambda \geq p$?
- How do the Lipschitz constants behave as $\lambda \to \infty$?

Theorem (W.-B. and L.A. Coburn, 2012)

Let $\Omega \subset \mathbb{C}^n$ be a BSD of genus p equipped with the Bergman metric and let $f \in \mathrm{UC}(\Omega)$.

Then the integral transforms $\{B_\lambda(f)\}_{\lambda \geq p}$ define a flow of real analytic functions in $\mathrm{Lip}(\Omega)$ with

$$\lim_{\lambda \to \infty} B_\lambda(f) = f$$

uniformly on Ω. The Lipschitz constant of $B_\lambda(f)$ is dominated by

$$C_\lambda := 2\sqrt{\frac{\lambda}{p}}\|f\|_{\mathrm{BMO}_\lambda}.$$

In particular, the inclusion $\mathrm{Lip}(\Omega) \cap C^\omega(\Omega) \subset \mathrm{UC}(\Omega)$ is dense.

Idea: Study the family of Bergman metrics coming from the reproducing kernels of the weighted Bergman spaces.

Application to Toeplitz operators

Consider the orthogonal projection:

$$P : L^2(\Omega, dv) \longrightarrow H^2(\Omega, dv)$$

Definition: Toeplitz operator (TO)

Let $f \in L^2(\Omega, dv)$, then the TO with symbol f is defined by:

$$T_f : H^2(\Omega, dv) \supset \mathcal{D} \longrightarrow H^2(\Omega, dv) : g \mapsto P(gf) = T_f(g)$$

with dense domain

$$\mathcal{D} := \mathrm{span}\Big\{K(\cdot, w) \ : \ w \in \Omega\Big\}.$$

In other words: T_f is the integral operator:

$$\big[T_f g\big](z) = \int_\Omega f(w)g(w)K(z,w)dv(w).$$

Theorem (H. Issa, 2011)

Let $\lambda > p-1$. Then there is $C > 0$ [a] such that for all $\lambda > C$ there is $D_\lambda > 0$ with

$$\|B_\lambda(g)\|_\infty \leq D_\lambda \|T_g\| \qquad (*)$$

for all $g \in L^2(\Omega, dv)$.

In particular:

If T_g is a bounded operator, then $B_\lambda(g)$ is a bounded function for sufficiently large weight $\lambda > 0$.

[a] C depends on the type of the domain Ω

Idea: Express the left hand side of $(*)$ as an operator trace

$$B_\lambda(g)(z) = \mathbf{trace}\big(T_g S_{\lambda,z}\big)$$

and use the trace estimate

$$|\mathbf{trace}(AB)| \leq \|A\| \|B\|_1.$$

Theorem (W.-B., L. A. Coburn (2014))

Let $f \in \mathrm{UC}(\Omega)$, then we have

(i) T_f is compact if and only if $f \in C_0(\Omega)$.

Proof: (i): The implication "$\Longleftarrow$" is standard.

"$\Longrightarrow$": Assume that T_f is compact and let $\varepsilon > 0$. From the above Theorems we can choose $C_\varepsilon > 0$ such that for $\lambda > C_\varepsilon$:

$$\begin{cases} (a): \ \|f - B_\lambda(f)\|_\infty < \varepsilon \\ (b): \ \|B_\lambda(f)\|_\infty < D_\lambda \|T_f\|. \end{cases}$$

Moreover, choose $f_\varepsilon \in C_c(\Omega)$ with

$$\|T_f - T_{f_\varepsilon}\| \leq \frac{\varepsilon}{D_\lambda} \quad \Longrightarrow \quad \|B_\lambda(f - f_\varepsilon)\|_\infty \overset{(b)}{\leq} D_\lambda \cdot \frac{\varepsilon}{D_\lambda} = \varepsilon.$$

Finally, use $B_\lambda(f_\varepsilon) \in C_0(\Omega)$ and

$$\|f - B_\lambda(f_\varepsilon)\|_\infty \leq \|f - B_\lambda(f)\|_\infty + \|B_\lambda(f - f_\varepsilon)\|_\infty < 2\varepsilon.$$

Remarks:

(a) If $f \in \mathrm{UC}(\Omega)$, then we also have the equivalence:

$$T_f \text{ bounded} \quad \Longleftrightarrow \quad f \text{ bounded}.$$

(b) Replacing the Bergman metric distance on Ω by the Euclidean distance d we define

$$\mathrm{UC}_d(\Omega) := \textit{uniformly continuous functions w.r.t. } d.$$

One can check that:

$$C(\overline{\Omega})_{|\Omega} = \mathrm{UC}_d(\Omega) \subsetneq \mathrm{UC}(\Omega)$$

Compactness and the Toeplitz algebra

Theorem (W.-B., L.A. Coburn, J. Isralowitz, 2010)

Let $g \in \mathcal{T}(\mathbb{C}^n)$, then

A: *If T_g is compact, then it holds for all $\frac{1}{2} < s < 2$:*

$$\widetilde{g}^{(s)} \in C_0(\mathbb{C}^n).$$

B: *The Toeplitz operator T_g is compact if for some $0 < s < \frac{1}{2}$*

$$\widetilde{g}^{(s)} \in C_0(\mathbb{C}^n).$$

Question: Is the following true:

$$T_g \textit{ compact if and only if } \widetilde{g}^{(\frac{1}{2})} \in C_0(\mathbb{C}^n)?$$

Problem:

How to obtain a compactness characterization for a larger class of bounded operators on the Fock space?

Observation: We can express the heat transform of a function g at time $t = 1$ as follows:

$$\widetilde{g}^{(1)}(z) = \left\langle T_g k_z, k_z \right\rangle,$$

where k_z= *"normalized reproducing kernel."* for $z \in \mathbb{C}$.

Definition

Let A be a bounded operator on $H^2(\mathbb{C}^n, d\mu)$, then we define the Berezin transform of A by

$$\widetilde{A}(z) := \left\langle A k_z, k_z \right\rangle.$$

Lemma

The following assignment is one-to-one:

$$\mathcal{L}(H^2(\mathbb{C}^n, d\mu)) \ni A \mapsto \widetilde{A} \in C_b^{\omega}(\mathbb{C}^n).$$

Recall: With some restriction of the symbol class we obtain:

Theorem (L. Coburn, J. Isralowitz, B. Li)

Assume that $g \in \mathrm{BMO}^2(\mathbb{C}^n)$, then T_g is compact if and only if $\widetilde{T_g} = \widetilde{g}^{(1)} \in C_0(\mathbb{C}^n)$.

Question: $A \in \mathcal{L}(H^2(\mathbb{C}^n, d\mu))$ compact iff $\widetilde{A} \in C_0(\mathbb{C}^n)$?

Example:

Consider the reflection $[Rf](z) := f(-z)$, then R is unitary and

$$\widetilde{R}(z) = e^{-2|z|^2} \in C_0(\mathbb{C}^n).$$

Consider the Toeplitz algebra

$$\mathcal{A} := \textit{norm closure of the algebra generated by } \{T_f \,:\, f \in L^\infty\}.$$

Theorem (W. B., J. Isralowitz (2012))

Let $A \in \mathcal{L}(H^2(\mathbb{C}^n, d\mu))$ then (i) and (ii) are equivalent:

(i) A is compact.

(ii) $A \in \mathcal{A}$ and $\widetilde{A} \in C^\omega(\mathbb{C}^n)$ vanishes at infinity.

Example: It follows that the reflection operator R with

$$[Rf](z) := f(-z), \qquad \text{and} \qquad \widetilde{R}(z) = e^{-2|z|^2} \in C_0(\mathbb{C}^n)$$

is not in $\mathcal{A}$. Moreover, it is known that

$$\inf \left\{ \| T_f - R \| \,:\, f \in L^\infty \right\} \geq 1.$$

Some ideas of the proof:

Put $H := H^2(\mathbb{C}^n, d\mu)$ and for $z \in \mathbb{C}^n$ consider the weighted shift:

$$W_z f := k_z \cdot f(\cdot + z), \qquad f \in H.$$

We obtain a map:

$$\mathbb{C}^n \ni z \mapsto W_z \in \mathcal{U}(H) = \textit{unitary operators on } H.$$

Given $S \in \mathcal{L}(H)$ write:

$$\Psi_S : \mathbb{C}^n \longrightarrow \mathcal{L}(H) : \Psi_S(z) := S_z = W_z S W_z^{-1}.$$

Aim:

Extend Ψ_S to a suitable compactification of $\mathbb{C}^n$ and relate its "boundary values" to the essential spectrum of S.

Consider the Banach algebra $\mathcal{B} \subset L^\infty(\mathbb{C}^n)$ defined by

$$\mathcal{B} := \textit{bounded uniformly continuous functions on } \mathbb{C}^n.$$

We obtain the inclusion

$$\mathbb{C}^n \subset M_{\mathcal{B}} = \textit{character space of } \mathcal{B}.$$

Proposition

Assume that $S \in \mathcal{A}$ (=Toeplitz algebra) with $\|A\| < 1$, then

$$\Psi_S : \mathbb{C}^n \to \Big(\text{unit ball of } \mathcal{L}(H), \text{SOT}\Big)$$

has a continuous extension from $\mathbb{C}^n$ to $M_{\mathcal{B}}$.

End of the proof

Use the following results:

Lemma

Let $S \in \mathcal{A}$, then (a) and (b) are equivalent:

(a) $\widetilde{S}(z) \to 0$ as $|z| \to \infty$.

(b) $\Psi_S(z) = 0$ for all $z \in M_{\mathcal{B}} \setminus \mathbb{C}^n$.

and

Lemma

Let $S \in \mathcal{A}$ and let $\|S\|_e$ denote the essential norm of S. Then

$$\|S\|_e \cong \sup_{z \in M_{\mathcal{B}} \setminus \mathbb{C}^n} \|\Psi_S(z)\|.$$

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Thank you for your attention!