Heat Kernel asymptotic expansions for the Heisenberg subLaplacian and the Grushin operator

Talk at Himeji Conference on PDEs

Der-Chen Chang
Georgetown University, Washington DC
(Joint work with Yutian Li, Baptist University HK)

March 5-7, 2015

Let $(\mathcal{M}, g)$ be a Riemannian manifold of n-dimension, $\rho(q_0, q)$ be the distance between two points q_0, $q \in \mathcal{M}$, and Δ be the Laplace-Beltrami operator, and $\mathcal{P}_t(q_0, q)$ be the heat kernel of Δ. The celebrated result of *Varadhan (1967)* reads

$$\lim_{t \to 0^+} t \log\left(\mathcal{P}_t(q_0, q)\right) = -\frac{1}{2}\rho^2(q_0, q). \tag{1.1}$$

This research topic were investigated by many authors. As $t \to 0^+$, $\mathcal{P}_t(q_0, q)$ has the following expansion

$$\mathcal{P}_t(q_0, q) \sim \frac{e^{-\frac{\rho^2(q_0,q)}{2t}}}{(2\pi t)^{\frac{n}{2}}} \left\{ a_0(q_0, q) + a_1(q_0, q)t^{1/2} + a_2(q_0, q)t + \cdots \right\}, \tag{1.2}$$

for q_0 and q are near points such that they are joined by a finite number of shortest geodesic along which they are not conjugate. The half-integer power terms vanish for manifold without boundary.

If q_0 and q are conjugate to each other along the shortest geodesic, the asymptotic behavior of $\mathcal{P}_t(q_0, q)$ will be different, namely, the leading power of t in the expansion changes from $t^{-\frac{n}{2}}$ to $t^{-\frac{n+k}{2}}$ for some positive number k. The number k appearing in the power is different for different situations.
When $\mathcal{M}$ is compact with $\partial\mathcal{M} \neq \varnothing$, consider the heat kernel trace,

$$\mathrm{trace}\,(\mathcal{P}_t) = \int_M \mathcal{P}_t(q,q)dV(q) = \sum_{j=1}^{\infty} e^{-t\lambda_j}, \tag{1.3}$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ are the eigenvalues of the Laplace-Beltrami operator with vanishing Dirichlet condition. *M.Kac's (1966)* famous question "*can one hear the shape of a drum*" concerns the extraction of geometric information of $\mathcal{M}$ from the asymptotic expansion of the heat kernel trace.

McKean and Singer (1967) showed that

$$\sum_{j=1}^{\infty} e^{-t\lambda_j} \sim C_0 t^{-\frac{n}{2}} + C_1 t^{-\frac{n-1}{2}} + C_2 t^{-\frac{n-2}{2}} + \cdots, \quad t \to 0^+, \tag{1.4}$$

where C_0, C_1 and C_2 are all global geometric quantities given by

$$\begin{aligned} C_0 &= \frac{V(\mathcal{M})}{(2\pi)^{\frac{n}{2}}}, \\ C_1 &= \frac{A(\partial\mathcal{M})}{4(2\pi)^{\frac{n-1}{2}}}, \\ C_2 &= \frac{\int_{\mathcal{M}} \mathcal{R}(x)dx}{6(2\pi)^{\frac{n-2}{2}}}, \end{aligned} \tag{1.5}$$

with $V(\mathcal{M})$, $A(\partial\mathcal{M})$ and $\mathcal{R}(x)$ being the volume of $\mathcal{M}$, the "area" of boundary of $\mathcal{M}$, and the scalar curvature, respectively.

Consider a real-valued function $F : [0, \infty) \to \mathbf{R}$ of bounded variation. The Laplace-Stieltjes transform of F is defined by

$$\omega(s) = \int_0^\infty e^{-st} dF(t).$$

The asymptotics of ω relates to F in the following way. If $\rho \in \mathbf{R}_+$, then the *Hardy-Littlewood tauberian theorem* tells us that the following are equivalent

$$\begin{aligned} \omega(s) &\sim C s^{-\rho}, \qquad \text{as} \quad s \to 0 \\ F(t) &\sim \frac{C}{\Gamma(\rho+1)} t^\rho, \qquad \text{as} \quad t \to \infty. \end{aligned} \tag{1.6}$$

Consider the counting function of eigenvalues, *i.e.*, the number of eigenvalues not exceed λ

$$N(\lambda) = \#\{\lambda_j \leq \lambda\} = \sum_{\lambda_j \leq \lambda} 1.$$

Then $dN(\lambda)$ is the spectral measure:

$$\int_I dN(\lambda) = \#\,\text{numbers of eigenvalues containing in the interval } I.$$

Note that $N(\lambda)$ is a step function, and the measure $dN(\lambda)$ should be treated as

$$dN(\lambda) = \sum_{j=0}^{\infty} \delta(\lambda - \lambda_j),$$

and integral under this measure is understood as Lebesgue-Stieltjes integral. Now consider the Laplace-Stieltjes transform of the measure $dN(\lambda)$

$$\int_0^{\infty} e^{-st} dN(t) = \sum_{j=0}^{\infty} e^{\lambda_j t} \sim \frac{C}{t^{\frac{n}{2}}}$$

which is the heat kernel trace.

By the Hardy-Littlewood tauberian theorem, we have

$$N(\lambda) \sim \frac{C_0}{\Gamma(1+\frac{n}{2})}\lambda^{\frac{n}{2}} \qquad \text{as } \lambda \text{ sufficiently large}$$

which is the *Hermann Weyl's asymptotic formula*. Notice that $N(\lambda_n) = n$ by definition. Hence,

$$\lambda_n \sim \left[\frac{n}{C_0}\Gamma\big(1+\frac{n}{2}\big)\right]^{\frac{2}{n}} = \left[\frac{n}{V(\mathcal{M})}(2\pi)^{\frac{n}{2}}\Gamma\big(1+\frac{n}{2}\big)\right]^{\frac{2}{n}}$$
$$= 2\pi^2\left(\frac{n}{\omega_n V(\mathcal{M})}\right)^{\frac{2}{n}}.$$

Here $\omega_n = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}$ is the volume of a unit n-ball.

On an n-dimensional Riemannian manifold $\mathcal{M}$, one needs n independent smooth vector fields $\mathbf{X} = \{X_1, X_2, \cdots, X_n\}$ to introduce a metric g which is given by the $n \times n$ positive definite matrix $(g(X_i, X_j))_{n\times n} = (g_{ij})_{n\times n}$. The Laplace-Beltrami operator is given by

$$\Delta = \frac{1}{2}(\det g)^{1/2} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\frac{g^{ij}}{\sqrt{\det g}}\frac{\partial}{\partial x^j}\right), \qquad (2.1)$$

which is an elliptic operator. Here, $(g^{ij}) = (g_{ij})^{-1}$. We had already discussed some results on this situation.

When one or several vector fields are *missing*, say, given $\{X_j\}_{j=1}^m$ with $m < n$, the possible generalizations of the elliptic operators, Riemannian geometries and their relations are of particular interest.

Given $X_1, \dots, X_m$ in an n-dim manifold $\mathcal{M}$. Let $\gamma : I \to \mathcal{M}$ be a curve on $\mathcal{M}$. The curve γ is *horizontal* if $\dot{\gamma}(t) = \sum_{k=1}^{m} a_k(t) X_k$, or equivalently $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$, $\forall\, t \in I$.

Chow-Rashevskii's Theorem (1938, 1939)

If a manifold $\mathcal{M}$ is topologically connected and the distribution $\mathcal{D} = \mathrm{span}\{X_1, \dots, X_m\}$ is bracket generating, then any two points can be connected by a horizontal curve.

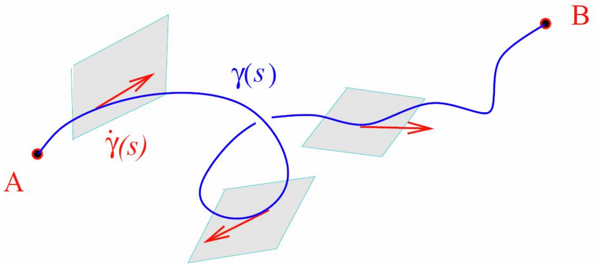

Figure 1. Chow's Theorem.

Given $X_1, \ldots, X_m$ in an n-dimensional manifold $\mathcal{M}$. Assume that $\mathbf{X} = \{X_1, \ldots, X_m\}$ satisfies bracket generating condition: "*the horizontal vector fields* $\mathbf{X}$ *and their brackets span* $T\mathcal{M}$". In 1967, Lars Hörmander proved that the differential operators

$$\Delta_{\mathbf{X}} = \sum_{j=1}^{m} X_j^2$$

is hypoelliptic: if $f \in C^\infty(\mathcal{M})$ then the solutions to

$$\Delta_{\mathbf{X}} u = f$$

is also in $C^\infty(\mathcal{M})$.

A subRiemannian structure over a manifold $\mathcal{M}$ is a pair $(\mathcal{D}, \langle\cdot,\cdot\rangle)$, where $\mathcal{D}$ is a *bracket generating distribution* and $\langle\cdot,\cdot\rangle$ a fibre inner product defined on $\mathcal{D}$. The length of the horizontal curve γ is

$$\ell(\gamma) := \int_0^\tau \sqrt{\langle\dot\gamma(s),\dot\gamma(s)\rangle}ds = \int_0^\tau \sqrt{a_1^2(s)+\cdots+a_m^2(s)}ds.$$

The shortest length $d_{cc}(A,B)$ is called the Carnot-Carathéodory distance between A, $B \in \mathcal{M}$ which is given by

$$d_{cc}(A,B) := \inf \ell(\gamma)$$

where the infimum is taken over all absolutely continuous horizontal curves joining A and B. Hence, we may define a geometry on $\mathcal{M}$ which is so-called *subRiemannian geometry*.

Set

$$X_j = \sum_{k=1}^{n} a_{jk}(x) \frac{\partial}{\partial x_k}, \quad j = 1, \dots, m.$$

Then

$$H = \frac{1}{2} \sum_{j=1}^{m} \Big(\sum_{k=1}^{n} a_{jk}(x) \xi_k \Big)^2$$

is the Hamiltonian function on the cotangent bundle $T^*\mathcal{M}$. A *bicharacteristic curve* $(\mathbf{x}(s), \xi(s)) \in T^*\mathcal{M}$ is a solution of the Hamilton's system:

$$\dot{x}_k(s) = H_{\xi_k}, \qquad \dot{\xi}_k(s) = -H_{x_k},$$

with boundary conditions,

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(\tau) = \mathbf{x},$$

for given points $\mathbf{x}_0, \mathbf{x} \in \mathcal{M}$. The projection $\mathbf{x}(s)$ of a bicharacteristic curve on $\mathcal{M}$ is a *geodesic*.

Remark 2.1

Let $\mathcal{M} = \mathbf{R}^2 \times \frac{1}{2}\mathbb{S}^1$, $(x, y) \in \mathbf{R}^2$, $\theta \in \mathbb{S}^1$. *The distribution*

$$\mathcal{D}^2 = span\{X = \frac{\partial}{\partial p},\ Y = \frac{\partial}{\partial y} + p\frac{\partial}{\partial x}\}, \quad p = \tan\theta,\ [X, Y] = \frac{\partial}{\partial x}$$

satisfies Chow's condition which can be applied to our daily life.

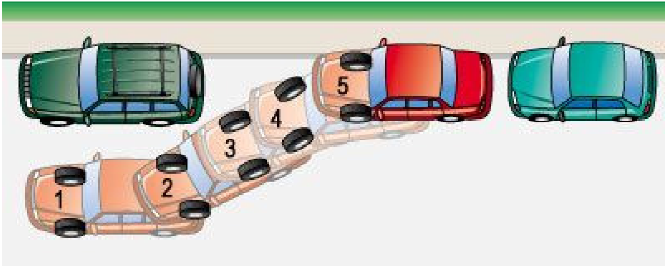

Figure 2. Parallel Parking.

Leandre (1987) proved a result for the subelliptic heat kernels

$$\lim_{t\to 0^+} t\,\log\big(\mathcal{P}_t(q_0,q)\big) = -\frac{1}{2}d_{cc}^2(q_0,q). \tag{2.2}$$

Here, $d_{cc}(q_0,q)$ denotes the subRiemannian distance between q_0 and q. A refined asymptotic formula was then given by *Ben Arous (1989)*, who showed that

$$\mathcal{P}_t(q_0,q) \sim \frac{1}{t^{\frac{n}{2}}} e^{-\frac{d_{cc}^2(q_0,q)}{2t}} \left[a_0(q_0,q) + \mathcal{O}(t^{1/2})\right], \quad t\to 0^+.$$

for $q_0 \neq q$ and q is not on the cut-locus of q_0. These results could be regarded as generalizations of (1.1) and (1.2) to subelliptic operators.

The above results suggest that subelliptic heat kernels have the similar small-time behavior as elliptic ones. However, the subelliptic operators also show some new phenomena.

On the diagonal, *i.e.*, when $q_0 = q$, *Ben Arous and Leandre (1991)* proved

$$\mathcal{P}_t(q,q) \sim \frac{C}{t^{\frac{Q}{2}}} + O(t^{\frac{1-Q}{2}}), \quad t \to 0^+.$$

The asymptotic behavior of $\mathcal{P}_t(q_0,q)$ is not known for $q \neq q_0$ and q is a cut point of q_0. Recently, *Barilari, Boscain, Neel (2012)*, who showed that

$$\mathcal{P}_t(q_0,q) \sim \frac{1}{(2\pi t)^{\frac{n+k}{2}}} e^{-\frac{d_{cc}^2(q_0,q)}{2t}} \left\{a_0(q_0,q) + o(1)\right\}, \quad t \to 0^+,$$

if $q \neq q_0$ and q is a cut point as well as a conjugate point of q_0 along some shortest geodesic. Here, k is a positive number that reflects "how conjugate" the two points are, in particular, when there is a k-parameter family of shortest geodesics.

The n-dimensional **Heisenberg group** is a nilpotent Lie group of step two on the manifold

$$\mathbf{H}^n \cong \mathbb{C}^n \times \mathbb{R} = \{(z, y) = (z_1, z_2, \cdots, z_n, y) : \ z \in \mathbb{C}^n, \ y \in \mathbb{R}\},$$

with the group law

$$(z, y) \circ (w, s) = \Big(z + w, y + s + 2\,\mathrm{Im} \sum_{j=1}^{n} a_j z_j \overline{w_j}\Big),$$

where a_j's are positive parameters. Without loss of generality, we restrict ourselves on $\mathbb{H}^1 = \{(x_1 + ix_2, y)\}$, and assume $a_1 = 1/2$. The vector fields

$$X_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial y},$$

are left-invariant under the group law and bracket-generating, *i.e.*, $[X_1, X_2] = 2\frac{\partial}{\partial y}$ recovers the missing direction.

Therefore, the *Heisenberg subLaplacian*

$$\Delta_H = \frac{1}{2}(X_1^2 + X_2^2), \tag{3.1}$$

is subelliptic by *Hörmander's theorem*. The heat kernel of Δ_H is well studied in literatures, see *e.g.*, *Calin-Chang-Furutani-Iwasaki (2011)*. We can fix $q_0 = (0,0,0)$ and let the other point $q(x_1, x_2, y)$ vary. The heat kernel $\mathcal{P}_t(x_1, x_2, y)$ is given as a Laplace integral

$$\mathcal{P}_t(x_1, x_2, y) = \frac{1}{(2\pi t)^2} \int_{-\infty}^{\infty} e^{-\frac{f(\tau)}{t}} V(\tau)\, d\tau, \tag{3.2}$$

where the phase function (also known as the "*modified complex action function*") is

$$f(\tau) = -i\tau y + \frac{1}{2}\|x\|^2 \tau \coth\tau, \quad \text{with} \quad \|x\|^2 = x_1^2 + x_2^2. \tag{3.3}$$

The amplitude function (also known as the "*volume element*") is $V(\tau) = \frac{\tau}{\sinh\tau}$.

Case I: diagonal, *i.e.*, $x_1 = x_2 = y = 0$.

In this case, $f(\tau) = 0$ and the integral in (3.2) reduces to

$$\mathcal{P}_t(0,0,0) = \frac{1}{(2\pi t)^2} \int_{-\infty}^{\infty} \frac{\tau}{\sinh \tau} d\tau = \frac{1}{8t^2}.$$

The last integral can be evaluated explicitly by residue calculus.

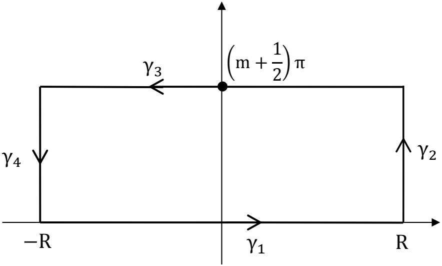

Figure 3. The path γ_j, $j = 1, 2, 3, 4$.

Case II: $x = (x_1, x_2) = (0, 0)$ **with** $y \neq 0$**.**

We may assume $y > 0$ by symmetry. Now the heat kernel is simply given as

$$\mathcal{P}_t(0,0,y) = \frac{1}{(2\pi t)^2} \int_{-\infty}^{\infty} e^{i\frac{\tau y}{t}} \frac{\tau}{\sinh \tau} d\tau,$$

which can be evaluated via the *residue calculation*. Note that each $ik\pi$, $k = 1, 2, \ldots$, is a simple pole of $V(\tau)$ with the residue being $e^{-\frac{k\pi y}{t}}(-1)^k ik\pi$. Then, we have

$$\mathcal{P}_t(0,0,y) = \frac{2\pi i}{4\pi^2 t^2} \sum_{k=1}^{\infty} \mathrm{Res}\Big(\frac{ze^{i\frac{yz}{t}}}{\sinh(z)}, k\pi i\Big) = \frac{1}{2t^2} \sum_{k=1}^{\infty} k(-1)^{k+1} e^{-\frac{k\pi y}{t}}.$$

Note that $k\pi y = \frac{\ell_k^2}{2}$ with $\ell_k =$ length of the kth geodesic joining $(0,0,y)$ and the origin. In particular, $\ell_1 = d_{cc}((0,0,0);(0,0,y))$, see *e.g.*, *Calin-Chang-Greiner (2007)*. Hence,

$$\mathcal{P}_t(0,0,y) \sim \frac{1}{2t^2} e^{-\frac{\ell_1^2}{2t}} + \text{exponentially small terms}, \quad t \to 0^+.$$

Case III-1: $y = 0$.

In this case, the heat kernel (3.2) takes the form

$$\mathcal{P}_t(x_1, x_2, y) = \frac{1}{(2\pi t)^2} \int_{-\infty}^{\infty} e^{-\frac{\tau \coth(\tau) \|x\|^2}{2t}} \frac{\tau}{\sinh \tau} d\tau.$$

Since the exponent $f(\tau)$ is real, the method for asymptotic expansion in previous two cases does not work. We may handle the last integral as $t \to 0^+$ by *Laplace method*.

We recall some properties of the modified complex function

$$f(\tau) = \frac{1}{2} \|x\|^2 \tau \coth(\tau).$$

The function $f(\tau)$ is real positive and has only one minimum point at $\tau = 0$, *i.e.*, $f'(\tau) = 0 \Leftrightarrow \tau = 0$. This critical point corresponds to the unique geodesic connecting the point $(x_1, x_2, 0)$ and $(0, 0, 0)$, which is a line segment joining these two points.

Moreover,

$$f(0) = \frac{1}{2}\|x\|^2 = \frac{1}{2}d_{cc}^2(x_1, x_2, 0),$$

Now, the functions f and V have the following Taylor's expansions at the point $\tau = 0$:

$$f(\tau) = \frac{1}{2}\|x\|^2 \tau \coth \tau = \sum_{k=0}^{\infty} \alpha_k \tau^k$$

$$V(\tau) = \frac{\tau}{\sinh \tau} = \sum_{k=0}^{\infty} \beta_k \tau^k,$$

where

$$\alpha_0 = \frac{\|x\|^2}{2},\ \ \alpha_2 = \frac{\|x\|^2}{6},\ \ \alpha_4 = -\frac{\|x\|^2}{90},\ \ \alpha_6 = \frac{\|x\|^2}{945},\ \ \cdots,\ \ \alpha_{2j+1} = 0,$$

$$\beta_0 = 1,\ \ \beta_2 = -\frac{1}{6},\ \ \beta_4 = \frac{7}{360},\ \ \beta_6 = -\frac{31}{15120},\ \ \cdots,\ \ \beta_{2j+1} = 0.$$

By Laplace method, we have the small t asymptotic expansion

$$\begin{aligned} \mathcal{P}_t(x_1, x_2, y) &\sim \frac{1}{(2\pi t)^2} e^{-\frac{f(0)}{t}} \sum_{n=0}^{\infty} 2\Gamma\left(\frac{n+1}{2}\right) C_n t^{\frac{n+1}{2}} \\ &\sim \frac{1}{2\pi^2 t^{3/2}} e^{-\frac{\|x\|^2}{2t}} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) C_n t^{\frac{n}{2}} \end{aligned}$$

where the coefficients C_n can be expressed in terms of α_k and β_k with $0 \le k \le n$. The first few coefficients are given by

$$C_0 = \frac{\beta_0}{\alpha_0} = \frac{2}{\|x\|^2}, \qquad C_2 = \frac{1}{\alpha_0^3}\left(\beta_2 - \frac{3\alpha_2\beta_0}{\alpha_0}\right) = -\frac{28}{3\|x\|^6}, \ldots$$

and $C_1 = 0, \quad C_3 = 0 \ldots$

Case III-2: $x \neq 0$ and $y \neq 0$.

The action function $f(\tau)$ is analytic in τ. The Laplace method can not be applied. Hence, we may apply *Debye's method of steepest descent* to derive the asymptotic expansion. Consider the complex integration

$$\widehat{I}(t) = \int_{\mathcal{C}} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau.$$

Assume that $f(\tau)$ and $V(\tau)$ are analytic functions on **C**. The original idea of this method is to deform $\mathcal{C}$ to the *steepest descent curve* Γ. so that the following conditions hold:
(a) Γ passes the critical points of $f'(\tau)$, *i.e.* points such that $f'(\tau) = 0$;
(b) the imaginary part of $f(\tau)$ is a constant along Γ, (here $\mathrm{Im} f(z) = 0$ on Γ).

Suppose that such a path Γ can be obtained from the original path $\mathcal{C}$ through a deformation so that

$$\widehat{I}(t) = \int_{\mathcal{C}} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau = \int_{\Gamma} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau. \tag{3.4}$$

Let $\eta = f(\tau) - f(\tau_1)$ for a critical point τ_1. Then

$$\widehat{I}(t) = e^{-\frac{f(\tau_1)}{t}} \int_0^T V(\tau) \frac{d\tau}{d\eta} e^{-\frac{\eta}{t}} d\eta \tag{3.5}$$

where $T > 0$. In most cases, $T > 0$ is at $+\infty$ unless the integration path strikes another saddle point. Then the asymptotic expansion maybe obtained if (3.5) satisfies the hypotheses in *Watson's lemma*.

Watson's Lemma

Assume that
(1). $f(\tau)$ is analytic when $|\tau| \leq a+\delta$, where $a>0,\ \delta>0$, except at a branch-point at the origin, and

$$f(\tau) = \sum_{m=1}^{\infty} a_m \tau^{\frac{m}{r}-1}$$

when $|\tau| \leq a,\ r$ being positive;
(2). $|f(\tau)| < Ke^{b\tau}$, where $K,\ b$ are independent of τ, when $\tau \geq a$;
(3). $|z|$ is sufficiently small and $|\arg(z)| \leq \frac{\pi}{2} - \delta$ where $\delta > 0$.
Then there exists a complete asymptotic expansion given by

$$F(z) = \int_0^\infty f(\tau) e^{-\frac{\tau}{z}} d\tau \sim \sum_{m=1}^{\infty} a_m \Gamma\big(\frac{m}{r}\big) z^{\frac{m}{r}}.$$

Remark 3.1

The method just illustrated (by using Watson's lemma) has two technical difficulties.

(1). *The path* Γ *is not easy to find even is hard to know whether it exists.*

(2). *Computations regarding change of variable in* (3.5) *is tedious which makes the calculation is not so easy.*

To overcome the difficulties, a modified steepest descent method is introduced.
We now split the action function $f(\tau)$ into two terms $f(\tau) = f_m(\tau) + (\tau - \tau_1)^\ell f_\ell(\tau)$ where

$$f_m(\tau) = f(\tau_1) + \frac{f^{(m)}(\tau_1)}{m!}(\tau - \tau_1)^m = f(\tau_1) + a_m e^{i\phi}(\tau - \tau_1)^m$$

with $a_m := \left|\frac{f^{(m)}(\tau_1)}{m!}\right|$, $\phi := \arg(f^{(m)}(\tau_1))$.

Here

$$f_\ell(\tau) = \frac{f(\tau) - f_m(\tau)}{(\tau - \tau_1)^\ell} = \frac{f^{(\ell)}(\tau_1)}{\ell!} + \frac{f^{(\ell+1)}(\tau_1)}{(\ell+1)!}(\tau - \tau_1) + \cdots$$

where

$$m = \min\{k > 1\,;\, f^{(k)}(\tau_1) \neq 0\}, \quad \ell = \min\{k > m\,;\, f^{(k)}(\tau_1) \neq 0\}. \tag{3.6}$$

The steepest descent method now can be simplified by considering the steepest descent paths of $f_m(\tau)$. The steepest descent paths Γ_k of $f_m(\tau)$ are those paths in $\mathbf{C}$ satisfying $\mathrm{Im}(f_m(\tau)) = \mathrm{Im}(f_m(\tau_1)) = 0$, and τ_1 is a minimum of $\mathrm{Re}(f(\tau))$ through

$$\Gamma_k := \left\{\tau \in \mathbf{C} : \tau = \tau_1 + re^{i\theta_k},\, \theta_k = \frac{2k\pi - \phi}{m},\, r \geq 0\right\},\ k = 0, 1, \ldots, m-1. \tag{3.7}$$

Consider the disk of convergence $D_R(\tau_1)$ and choose the two most appropriate semi-infinite straight lines Γ_{ℓ_i} and Γ_{ℓ_j} according to the original path $\mathcal{C}$.

Then we deform the original integration path $\mathcal{C}$ to $\widetilde{\Gamma} = \widehat{\Gamma} \cup \Gamma_\varepsilon$ where

$$\begin{aligned}\widehat{\Gamma} &= \left\{\tau \in \mathbf{C} : \ \tau = \tau_1 + re^{i\theta_{\ell_i}}, \ 0 \le r \le R\right\} \\ &\quad \cup \left\{\tau \in \mathbf{C} : \ \tau = \tau_1 + re^{i\theta_{\ell_j}}, \ 0 \le r \le R\right\}\end{aligned}$$

The path Γ_ε is glued at extreme points of $\widehat{\Gamma}$. In $\widehat{I}(t)$, comparing to the integration on $\widehat{\Gamma}$, the integration on Γ_ε is negligible. As a result,

$$\begin{aligned}\widehat{I}(t) &= \int_{\widehat{\Gamma}} e^{-\lambda f(\tau)} V(\tau) d\tau \sim \int_{\widetilde{\Gamma}} e^{-\lambda f(\tau)} V(\tau) d\tau \\ &\sim \int_{\Gamma_{\ell_i} \cup \Gamma_{\ell_j}} e^{-\lambda f(\tau)} V(\tau) d\tau.\end{aligned}$$

This idea was first used by *López-Pagola (2011)*.

By results of *Beals-Gaveau-Greiner (2000), Calin-Chang-Greiner (2007)*, we known that the unique critical point of $f(\tau)$ on the strip $0 < \mathrm{Im}(\tau) < \pi$ is the point $\tau_1 = i\theta_1$, where θ_1 is the solution of

$$y \;=\; \frac{1}{2}\|x\|^2\Big(\frac{\phi}{\sin^2\phi} - \cot\phi\Big) := \frac{1}{2}\|x\|^2\mu(\phi).$$

A simple calculation yields

$$f'(\tau_1) = -iy + i\frac{1}{2}\mu(-i\tau_1)\|x\|^2$$

and

$$f^{(2)}(\tau_1) = \frac{1}{2}\mu'(-i\tau_1)\|x\|^2, \qquad f^{(3)}(\tau_1) = \frac{1}{2}\mu^{(2)}(-i\tau_1)\|x\|^2$$

where $\mu(\varphi) = \frac{\varphi}{\sin^2\varphi} - \cot\varphi$. See **Figure 3**.

We know that $\mu'(\varphi) > 0$ for $0 < \varphi < \pi$, and hence $f^{(2)}(\tau_1) > 0$ which implies that $\tau = \tau_1$ is a critical point of order one. Moreover, $\mu^{(2)}(\varphi) > 0$ for $\varphi \in (0,\pi)$ and $\mu^{(2)}(\varphi) < 0$ for $\varphi \in (-\pi, 0)$ $\Rightarrow f^{(3)}(\tau_1) > 0$.

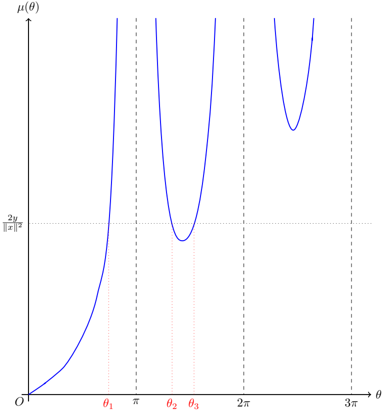

Figure 3. The function $\mu(\theta)$: the solutions of $\mu(\theta) = 2y/\|x\|^2$ give the saddle points $\tau_j = i\theta_j$.

Parameters: $\|x\|^2 = 2,\ y = 5$.

Thus,

$$f(\tau) = f(\tau_1) + f^{(2)}(\tau_1)\frac{(\tau-\tau_1)^2}{2} + f^{(3)}(\tau_1)\frac{(\tau-\tau_1)^3}{6} + \cdots .$$

Now, we split the phase function into two terms

$$f(\tau) = f_2(\tau) + (\tau-\tau_1)^3 f_3(\tau),$$

with

$$\begin{aligned} f_2(\tau) &=: f(\tau_1) + \frac{f^{(2)}(\tau_1)}{2!}(\tau-\tau_1)^2, \\ f_3(\tau) &=: \frac{f(\tau)-f_2(\tau)}{\tau^3} = \frac{f^{(3)}(\tau_1)}{3!} + \frac{f^{(4)}(\tau_1)}{4!}(\tau-\tau_1) + \cdots . \end{aligned}$$

Notice that $f_2(\tau)$ is a polynomial of degree two, and $\mathrm{Im} f_2(\tau_1) = \mathrm{Im} f(\tau_1) = 0$, so the steepest descent path for $f_2(\tau)$ is the horizontal line, which is plotted in **Figure 4** with a red line. This contour is divided into two semi-infinite lines

$$\Gamma_k = \left\{\tau \in \mathbb{C}; \quad \tau = \tau_1 + re^{ik\pi}, \quad r \geq 0\right\}, \qquad k = 0, 1.$$

Then the integral in (3.4) becomes an Laplace integral over $\widehat{\Gamma} = \Gamma_0 \cup \Gamma_1$, and any method for computing the coefficients of Laplace's methods can be applied. Hence,

$$\widehat{I}(t) = \int_\Gamma e^{-\frac{f(\tau)}{t}} V(\tau) d\tau = I_0(t) + I_1(t)$$

where

$$\begin{aligned} I_0(t) &\sim e^{-\frac{f(\tau_1)}{t}} \sum_{n=0}^{\infty} \Gamma\Big(\frac{n+1}{2}\Big) c_n(0) t^{\frac{n+1}{2}}, \\ I_1(t) &\sim e^{-\frac{f(\tau_1)}{t}} \sum_{n=0}^{\infty} \Gamma\Big(\frac{n+1}{2}\Big) c_n(1) t^{\frac{n+1}{2}}. \end{aligned} \tag{3.8}$$

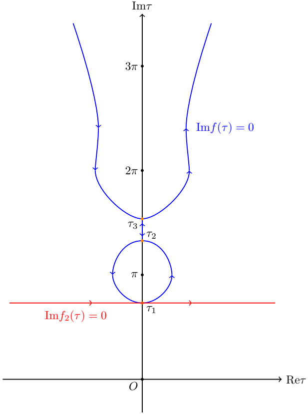

Figure 4. Complex τ plane: the blue curve is the steepest descent path for $f(\tau)$; the red line is the steepest descent path for $f_2(\tau)$.

Hence,

$$\begin{aligned}\mathcal{P}_t(x_1,x_2,y) &\sim \frac{e^{-\frac{d_{cc}^2}{2t}}}{(2\pi t)^2}\sum_{n=0}^{\infty} C_n\Gamma\Big(\frac{n+1}{2}\Big)t^{\frac{n+1}{2}} \\ &\sim \frac{e^{-\frac{d_{cc}^2}{2t}}}{(2\pi)^2 t^{\frac{3}{2}}}\sum_{n=0}^{\infty} C_n\Gamma\Big(\frac{n+1}{2}\Big)t^{\frac{n}{2}},\end{aligned} \tag{3.9}$$

Coefficients C_n in the expansion (3.9) are given by $C_n = c_n(0) + c_n(1)$ with $c_n(1) = (-1)^n c_n(0)$.
The coefficients $c_n(0)$ can be calculated recursively by *Wojdylo's method*.

Assume that

$$f(\tau+\tau_1)=\sum_{k=0}^{\infty}a_k\tau^k, \qquad V(\tau+\tau_1)=\sum_{k=0}^{\infty}b_k\tau^{k+\alpha-1}.$$

The first four coefficients for the expansion of $I_0(t)$ are given by

$$\begin{aligned}
c_0(0) &= \frac{1}{2}\Big(\frac{b_0}{a_2^{1/2}}\Big)\\
c_1(0) &= \frac{1}{2}\Big(\frac{b_1}{a_2}+a_3\frac{b_0}{a_2^2}\Big)\\
c_2(0) &= \frac{1}{2}\Big(\frac{b_2}{a_2^{3/2}}-\frac{3a_3}{2}\frac{b_1}{a_2^{5/2}}-\frac{3a_4}{2}\frac{b_0}{a_2^{5/2}}+\frac{15a_3^2}{8}\frac{b_0}{a_2^{7/2}}\Big)\\
c_3(0) &= \frac{1}{2}\Big(\frac{b_3}{a_2^2}-2a_3\frac{b_2}{a_2^3}-2a_4\frac{b_1}{a_2^3}+6a_3^2\frac{b_1}{a_2^4}-2a_5\frac{b_0}{a_2^3}+12a_3a_4\frac{b_0}{a_2^4}-4a_3^3\frac{b_0}{a_2^5}\Big)
\end{aligned}$$

Assume that

$$f(-\tau+\tau_1)=\sum_{k=0}^{\infty}a_k(-\tau)^k, \qquad V(-\tau+\tau_1)=\sum_{k=0}^{\infty}b_k(-\tau)^{k+\alpha-1}.$$

The first four coefficients for the expansion of $I_1(t)$ are given by

$$\begin{aligned}
c_0(1) &= \frac{1}{2}\Big(\frac{b_0}{a_2^{1/2}}\Big)\\
c_1(1) &= \frac{-1}{2}\Big(\frac{b_1}{a_2}+a_3\frac{b_0}{a_2^2}\Big)\\
c_2(1) &= \frac{1}{2}\Big(\frac{b_2}{a_2^{3/2}}-\frac{3a_3}{2}\frac{b_1}{a_2^{5/2}}-\frac{3a_4}{2}\frac{b_0}{a_2^{5/2}}+\frac{15a_3^2}{8}\frac{b_0}{a_2^{7/2}}\Big)\\
c_3(1) &= \frac{-1}{2}\Big(\frac{b_3}{a_2^2}-2a_3\frac{b_2}{a_2^3}-2a_4\frac{b_1}{a_2^3}+6a_3^2\frac{b_1}{a_2^4}-2a_5\frac{b_0}{a_2^3}+12a_3a_4\frac{b_0}{a_2^4}-4a_3^3\frac{b_0}{a_2^5}\Big)
\end{aligned}$$

Calculations show that

$$a_2 = \frac{\sin\theta_1 - \theta_1\cos\theta_1}{2\sin^3\theta_1}\|x\|^2, \quad a_3 = \frac{\theta_1 + 2\theta_1\cos^2\theta_1 + 3\sin\theta_1\cos\theta_1}{6\sin^4\theta_1}\|x\|^2,$$

$$b_0 = \frac{\theta_1}{\sin\theta_1}, \quad b_1 = \frac{\sin\theta_1 - \theta_1\cos\theta_1}{\sin^2\theta_1}, \quad b_2 = \frac{\theta_1(1+\cos^2\theta_1) - 2\sin\theta_1\cos\theta_1}{\sin^3\theta_1}.$$

It follows that

$$C_0 = \frac{b_0}{a_2^{1/2}} = \frac{\tau_1}{\sinh\tau_1}\frac{2}{\|x\|}\sqrt{\frac{1}{\mu'(-i\tau_1)}} = \frac{\theta_1}{\|x\|}\sqrt{\frac{2\sin\theta_1}{\sin\theta_1 - \theta_1\cos\theta_1}},$$

$$C_2 = \frac{b_2 - \frac{3}{2}b_1 - \frac{3}{2}\frac{a_3}{a_2}b_0 + \frac{15}{8}b_0}{a_2^{3/2}}, \ldots,$$

and

$$C_{2j+1} = 0, \qquad j = 0, 1, 2, 3, \ldots.$$

Remark 3.2

There are infinitely many critical points of $f(\tau)$*, and the points are of the form* $\tau_j = i\theta_j$ *for* $j = 1, 2, \cdots, N$*. Since* $y > 0$*, we have* $\theta_1 < \theta_2 < \cdots < \theta_N$*. The values* $f(\tau_j)$ *provide the length of geodesics. Explicitly, for* $1 \le j \le N$*,*

$$f(\tau_j) = \frac{1}{2}\ell_j^2 = \frac{1}{2}\nu(\theta_j)\big(2|y| + \|x\|^2\big),$$

where

$$\nu(\theta_j) = \frac{\theta_j^2}{\theta_j + \sin^2\theta_j - \sin\theta_j\cos\theta_j}$$

and ℓ_j *is the length of* j*th geodesics joining* (x_1, x_2, y) *and the origin. Of course,* $f(\tau_1) = d_{cc}((0,0,0);(x_1,x_2,y))$*. Other critical values are larger than the first one, the contributions of these critical points are exponentially small compared with the first critical point. We ignore them.*

Remark 3.3

We have the following form of asymptotics for the heat kernel

$$\mathcal{P}_t(x_1, x_2, y) \sim \frac{C}{t^{\frac{Q}{2}}} e^{-\frac{d_{cc}^2}{2t}},$$

where C and Q are constants and d_{cc} is the Carnot-Carathéodory distance between (x_1, x_2, y) and the origin. We note that the power α of t varies. Namely,

$$2\alpha = \begin{cases} 4 = Q > n, & \text{when } x = 0,\ y = 0,\ \text{diagonal;} \\ 4 = n + 1, & \text{when } x = 0,\ y \neq 0,\ \text{off-diagonal, cut-conjugate;} \\ 3 = n, & \text{when } x \neq 0,\ \text{off-diagonal, not cut-conjugate.} \end{cases}$$

Here, $n = 3$ is the topological dimension and Q is the Hausdorff dimension. This agrees with the previous result on the asymptotics for the heat kernels on the diagonal, i.e. when the $(x_1, x_2, y) = (0, 0, 0)$.

The second example of subelliptic operators is introduced by Grushin. Consider the following vector fields on $\mathbf{R}^{m+1} = \{(x_1, x_2, \cdots, x_m, y)\}$

$$X_j = \frac{\partial}{\partial x_j}, \qquad Y_j = x_j \frac{\partial}{\partial y}, \qquad 1 \leq j \leq m. \tag{4.1}$$

These vector fields give all $(m+1)$ directions on $\mathbb{R}^{m+1}$ except on y-axis, where their bracket $[X_j, Y_j] = \frac{\partial}{\partial y}$ gives the missing direction. The *Grushin operator*

$$\Delta_G = \frac{1}{2}\sum_{j=1}^{m}(X_j^2 + Y_j^2) = \frac{1}{2}\sum_{j=1}^{m}\frac{\partial^2}{\partial x_j^2} + \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_m^2)\frac{\partial^2}{\partial y^2} \tag{4.2}$$

is therefore subelliptic by Hörmander's theorem. When $m = 1$, Δ_G is the classical Grushin operator and its heat kernel is constructed in *e.g.*, *Calin-Chang-Furutani-Iwasaki (2011)*, the case of $m \geq 1$ is studied in *Calin-Chang-Hu-Li (2011)*.

The heat kernel of Δ_G has the following form

$$\mathcal{P}_t(q_0, q) = \frac{1}{(2\pi t)^{\frac{m+2}{2}}} \int_{-\infty}^{\infty} e^{-\frac{g(\tau)}{t}} W(\tau)\, d\tau, \tag{4.3}$$

where $q_0(x_0, y_0)$ and $q(x, y)$ are two points in $\mathbb{R}^{m+1}$. The phase function (the modified complex action function) is

$$\begin{aligned} g(\tau) &= -i\tau(y - y_0) + \frac{\tau}{2\sinh\tau}\left[(\|x\|^2 + \|x_0\|^2)\cosh\tau - 2\langle x, x_0\rangle\right] \\ &= -i\tau(y - y_0) + \frac{\tau}{4}\left[(x + x_0)^2 \tanh\frac{\tau}{2} + (x - x_0)^2 \coth\frac{\tau}{2}\right]. \end{aligned} \tag{4.4}$$

and the amplitude function (the volume element) is

$$W(\tau) = \left(\frac{\tau}{\sinh\tau}\right)^{\frac{m}{2}}. \tag{4.5}$$

The heat kernel of Δ_G has the following form

$$\mathcal{P}_t(q_0, q) = \frac{1}{(2\pi t)^{\frac{m+2}{2}}} \int_{-\infty}^{\infty} e^{-\frac{g(\tau)}{t}} W(\tau)\, d\tau, \tag{4.6}$$

where $q_0(x_0, y_0)$ and $q(x, y)$ are two points in $\mathbb{R}^{m+1}$. The phase function (the modified complex action function) is

$$\begin{aligned} g(\tau) &= -i\tau(y - y_0) + \frac{\tau}{2\sinh\tau}\left[(\|x\|^2 + \|x_0\|^2)\cosh\tau - 2\langle x, x_0\rangle\right] \\ &= -i\tau(y - y_0) + \frac{\tau}{4}\left[(x + x_0)^2 \tanh\frac{\tau}{2} + (x - x_0)^2 \coth\frac{\tau}{2}\right]. \end{aligned} \tag{4.7}$$

and the amplitude function (the volume element) is

$$W(\tau) = \left(\frac{\tau}{\sinh\tau}\right)^{\frac{m}{2}}. \tag{4.8}$$

Theorem 4.1

The heat kernel $\mathcal{P}_t(x_0, y_0; x, y)$ of the Grushin operator given in (4.6) with $m = 1$ has the following asymptotic expansion as $t \to 0^+$:

(1). $x = x_0 = 0$ *and* $y = y_0$,

$$\mathcal{P}_t(x, y; x, y) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \left(\frac{\tau}{\sinh \tau}\right)^{\frac{1}{2}} d\tau < \infty;$$

(2). $x = x_0 \neq 0$ *and* $y = y_0$,

$$\mathcal{P}_t(x_0, y_0; x, y) \sim \frac{2}{(2\pi)^{\frac{3}{2}} t} \sum_{k=0}^{\infty} \Gamma\Big(k + \frac{1}{2}\Big) c_{2k} t^k,$$

where $c_{2k+1} = 0$ *for* $k = 0, 1, 2, \ldots$ *and*

$$c_0 = \frac{\sqrt{2}}{2|x|}, \quad c_2 = \frac{\sqrt{2}}{24|x|^3}, \quad c_4 = \frac{-\sqrt{2}}{320|x|^5}, \ldots,$$

(3). $x = -x_0 \neq 0$ and $|y - y_0| = \frac{\pi}{2}x^2$,

$$\mathcal{P}_t(x_0, y_0; x, y) \sim \frac{e^{-\frac{d_{cc}^2(q_0,q)}{2t}}}{(2\pi)^{\frac{3}{2}} t^{\frac{5}{4}}} \sum_{k=0}^{\infty} \Gamma\Big(k + \frac{1}{4}\Big) D_k t^k;$$

(4). other cases,

$$\mathcal{P}_t(x_0, y_0; x, y) \sim \frac{e^{-\frac{d^2_{cc}(q_0,q)}{2t}}}{(2\pi)^{\frac{3}{2}} t} \sum_{k=0}^{\infty} \Gamma\Big(k + \frac{1}{2}\Big) D_k t^k;$$

where $D_k = C_{2k}$ for $k = 0, 1, 2, \ldots$. The $C_k = c_k(0) + c_k(1)$ with $c_k(1) = (-1)^k c_k(0)$ and $c_k(0)$ can be computed recursively by Wojdylo's method

$$c_k(0) = \frac{1}{2\Gamma\big(\frac{k+1}{2}\big)} \sum_{n=0}^{k} b_{k-n} \sum_{s=0}^{n} \frac{(-1)^s}{a_2^{\frac{k+1}{2+s}} \frac{B_{n,s}}{s}!} \Gamma\big(\frac{k+1}{2} + s\big)$$

Here b_k is given in $W(\tau) = 1 - \frac{1}{12}\tau^2 + \cdots = \sum_{k=0}^{\infty} b_k \tau^k$ and the partial ordinary Bell polynomials $B_{n,s}$ are defined by $B_{0,0} = 1$, $B_{n,0} = 1$, and

$$B_{n,s} = \sum^{n-1} a_{n+2-j} B_{j,s-1}, \quad n \geq s \geq 1.$$

Fix a point $q_0 \in \mathcal{M}$, q is a *conjugate point* of q_0 if q and q_0 can be connected by a shortest geodesic γ. However, this geodesic is no longer the shortest after this point q. This indicates that $d_{cc}(q_0, q)$ as a function of q is not smooth at a conjugate point.

A point q is a *cut point* of q_0 if: (i) there are more than one shortest geodesic joining q and q_0; or (ii) q is a conjugate point of q_0 along one shortest geodesic.

Take the Grushin plane as an example for illustration. The cut locus of $q_0(0, y_0)$ is $\{(0, y);\ y \neq y_0\}$, but these points are not conjugate points.

For the point $q_0(x_0, y_0)$ with $x_0 \neq 0$, a point $q(x, y)$ is a conjugate point of $q_0 \Leftrightarrow |y - y_0| = \frac{k\pi}{2}x^2$ and $x = (-1)^k x_0$ for some positive integer k. But only the first two conjugate points associated with $k = 1$ are cut points.

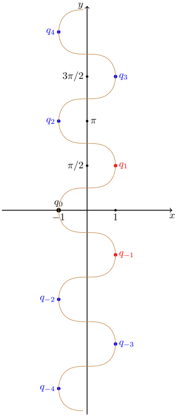

Figure 5. Cut points and conjugate points of q_0 in the Grushin plane.

$$q_0 = (-1, 0)\text{: } \mathrm{Conj}(q_0) = \{q_{\pm 1}, \ldots q_{\pm k}, \ldots\},$$

these points are conjugate with q_0 along the geodesic plotted in the brown curve;

$\mathrm{Cut}(q_0) = \{q_1, q_{-1}\}$, so q_1 and q_{-1} are the only cut-conjugate points of q_0.

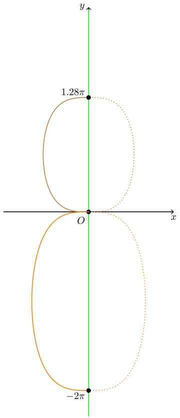

Figure 6. $q_0 = (0,0)$: $\mathrm{Cut}(q_0) = \{(0,y); y \neq 0\}$ and $\mathrm{Conj}(q_0) = \emptyset$,

any point $q(0,y)$ is joined with q_0 by two shortest geodesics plotted in a solid and a dotted curves.

Illustration is done for $q = (0, -2\pi)$ and $q = (0, 1.28\pi)$.

Remark 4.1

The heat kernel of the Grushin operator has the small-time asymtotics in the following form

$$\mathcal{P}_t(x_0, y_0; x, y) \sim \frac{C}{t^\alpha} e^{-\frac{d_{cc}^2}{2t}},$$

where C is a constant and

$$2\alpha = \begin{cases} 3 = Q > n, & x = x_0 = 0, \quad y = y_0, \quad \textit{diagonal}; \\ 2 = Q = n, & x = x_0 \neq 0, \quad y = y_0, \quad \textit{diagonal}; \\ \frac{5}{2} = n + \frac{1}{2}, & x = -x_0 \neq 0, \quad |y - y_0| = \pi x^2/2, \\ & \textit{off-diagonal and cut-conjugate}; \\ 2 = n, & \textit{off-diagonal and not cut-conjugate points}. \end{cases}$$

Here, $n = 2$ is the topological dimension and Q is the Hausdorff dimension. We see that in most of the cases $\alpha = \frac{Q}{2}$. $2\alpha = \frac{5}{2}$ corresponds to the case when q_0 q are cut-conjugate points.

Recall that the scalar curvature is the trace of the Ricci curvature tensor

$$\mathcal{R} = \sum_{i,j} g^{ij} R_{ij},$$

where the Ricci tensor can be calculated via the Christoffel symbols

$$R_{ij} = \sum_{\ell} \left[\frac{\partial \Gamma^{\ell}_{ij}}{\partial x^{\ell}} - \frac{\partial \Gamma^{\ell}_{i\ell}}{\partial x^{j}} + \sum_{m} \left(\Gamma^{m}_{ij} \Gamma^{\ell}_{\ell m} - \Gamma^{m}_{i\ell} \Gamma^{\ell}_{jm} \right) \right]$$

and

$$\Gamma^{m}_{ij} = \frac{1}{2} \sum_{k} g^{mk} \left(\frac{\partial g_{ki}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{k}} + \frac{\partial g_{jk}}{\partial x^{i}} \right).$$

Consider the Grushin vector fields in (4.1) for $m = 1$, and the coordinates (x, y) are understood as (x^1, x^2) in the above formulas. The metric is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & x^{-2} \end{pmatrix}, \qquad (g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & x^{2} \end{pmatrix}.$$

Hence,

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = \Gamma^2_{22} = 0,$$
$$\Gamma^1_{22} = \frac{1}{x^3}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{-1}{x}.$$

$$R_{11} = -\frac{1}{x^2}, \qquad R_{22} = -\frac{5}{x^4}, \qquad R_{12} = R_{21} = 0$$

A slight calculation shows

$$\mathcal{R}(x,y) = -\frac{4}{x^2}, \qquad x \neq 0.$$

Let us look at the diagonal case Case II, for which we have

$$\mathcal{P}_t(x,y;x,y) \sim \frac{1}{2\pi t|x|}\left(1 + \frac{1}{24|x|^2}t + \cdots\right), \qquad \text{as} \quad t \to 0^+.$$

The second term here is related to the scalar curvature in the sense that $\frac{1}{24|x|^2} = -\frac{\mathcal{R}(x,y)}{96}$.

We have shown that the leading power of t in the small-time asymptotic expansion of the heat kernel (3.2) for the Heisenberg subLaplacian varies as the point (x_1, x_2, y) varies. To be precise, as (x_1, x_2) approaches $(0, 0)$, the power α of t changes from $\frac{3}{2} \to 2$. From the integral point of view, the discontinuity of α is due to the coalescing of the two saddle points τ_1 and τ_2 to the same value $i\pi$ as $\|x\| \to 0$. Note that $\theta_1 < \pi < \theta_2$ for $\|x\| \neq 0$, and that θ_1 and θ_2 both tend to π as $\|x\| \to 0$, and $\tau = i\pi$ is also a simple pole of the phase function $f(\tau)$. For such a case, *Frenzen and Wong (1988)* derived a uniform asymptotic approximation in terms of Bessel function. Their idea is to introduce a rational mapping $\tau \mapsto u$ by

$$-2f(i\eta) = u - \frac{A^2(\sigma)}{u}, \tag{5.1}$$

where $\sigma = \frac{2y}{\|x\|^2}$ and the function $A(\sigma)$ is determined as follows. Note that

$$-2if_\eta(i\eta)\frac{d\tau}{du} = 1 + \frac{A(\sigma)^2}{u^2}.$$

In order to have a one-to-one mapping in the region of interest, one requires $\frac{d\eta}{du} \neq 0$ or ∞. Note that $f_\eta(i\eta_1) = 0$ and $f_\eta(i\eta_2) = 0$, thus we let τ_1 and τ_2 correspond to $u = iA(\sigma)$ and $u = -iA(\sigma)$, respectively. Therefore,

$$A(\sigma) = if(i\eta_1) = i\frac{d_{cc}^2}{2}.$$

Here, we have made use of the fact that $f(i\eta_1) = \frac{d_{cc}^2}{2}$. By the transformation $\tau \mapsto u$, the heat kernel (3.2) reduces to

$$\mathcal{P}_t(x,y) = \frac{1}{(2\pi t)^3}\int u^{-1}h(u)\exp\left\{\frac{1}{t}\left(u - \frac{A(\sigma)^2}{u}\right)\right\}du, \qquad (5.2)$$

where

$$h(u) = uV\big(\tau(u)\big)\frac{d\tau}{du} \qquad (5.3)$$

is analytic near $u = 0$. Recall Schläfli's integral representation of the Bessel function $J_\nu(z)$:

$$J_\nu(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \int_{-\infty}^{(0+)} t^{-(\nu+1)}\exp\left\{t - \frac{z^2}{4t}\right\}dt.$$

The change of variables $t = \frac{\lambda u}{2}$ leads to

$$J_\nu(\lambda z) = \frac{z^\nu}{2\pi i} \int_{-\infty}^{(0+)} u^{-(\nu+1)} \exp\left\{ \frac{\lambda}{2} \left(u - \frac{z^2}{u} \right) \right\} du.$$

Comparing the last equation with (5.2) yields

$$\mathcal{P}_t(x, y) \sim \frac{1}{t^2} J_0 \left(\frac{A(\sigma)}{t} \right) a_0(x, y) + O\left(\frac{1}{t} \right), \qquad \text{as} \quad t \to 0^+,$$

where $a_0(x, y)$ is a function of x, y. Follow the approach of *Fenzen and Wong (1988)*, we can derive an asympttoic expansion of the form

$$\mathcal{P}_t(x_1, x_2, y) \sim \frac{1}{t^2} \left\{ J_0 \left(A(\sigma)/t \right) \sum_{s=0}^{\infty} a_s t^s + \frac{J_1 \left(A(\sigma)/t \right)}{A(\sigma)} \sum_{s=0}^{\infty} b_s t^{s+1}, \right\}$$

as $t \to 0^+$.

In the small-time asymptotics of the heat kernel for the Grushin operator, we see that the leading power of t, α, changes from $1 \rightarrow \frac{5}{4}$ as q approaches the cut-conjugate point of q_0. From the integral point of view, this discontinuity of α is due to the coalescing of the saddle point τ_1 and the singularity of $W(\tau)$ at $\tau = i\pi$. The treatment of such coalesce of critical points can be found in *Wong (2001)*, and the uniform asymptotic expansion involves parabolic cylinder functions. The discontinuous change of α from $\frac{3}{2}$ to 1 can be smoothed out in a same manner as in the Heisenberg case. Since there is no alternation in the methods, we omit the details here.


Thanks!