Spectrum of the sub-Laplacian and Laplacian on nilmanifolds attached to Clifford modules and isospectral, non-diffeomorphic manifolds

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Kenro Furutani（古谷賢朗）

Tokyo University of Science(東京理科大学)

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Isospectral Problem

Notations:

1. (M, g) : a closed Riemannian manifold, Δ : its Laplacian.
2. $Spec(M, g) = \left\{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots\right\}$ the set of eigenvalues of the Laplacian with non-zero eigenfunctions $f_k \in C^\infty(M)$.
3. Heat kernel $K = K(t, x, y) \in C^\infty(\mathbb{R}_+ \times M \times M)$, i.e., the solution to the heat equation

$$\left(\frac{\partial}{\partial t} + \Delta\right) K = 0, \ \lim_{t \downarrow 0} K(t, x, *) = \delta_x$$

4. Wave kernel $W = W(t, x, y) \in \mathcal{D}'(\mathbb{R} \times M \times M)$, i.e., the solution to the wave equation

$$\left(\frac{\partial}{\partial t^2} + \Delta\right) W = \left(\frac{\partial}{\partial t} \pm i \sqrt{\Delta}\right) W = 0, \ \lim_{t \to 0} W(t, x, *) = \delta_x$$

By the method of the separation of variables these kernels are expressed as

$$K(t,x,y) = \sum e^{-t\lambda_k} f_k(x) f_k(y)$$

and

$$W(t,x,y) = \sum e^{\pm i t \sqrt{\lambda_k}} f_k(x) f_k(y)$$

with the eigenvalues λ_k and corresponding orthonormal eigen-functions f_k of the Laplacian Δ. In principle, from these expressions we can determine the heat kernel and wave kernel if we can solve the eigenvalue problem completely:

$$\Delta f = \lambda f.$$

At least, if we know the heat trace

$$\int_M K(t,x,x) dg = \sum e^{-t\lambda_k}$$

we can determine the eigenvalues completely.

So the wave trace

$$\sum e^{i\,t\,\sqrt{\lambda_k}} = \sum \cos\ t\,\sqrt{\lambda_k}$$

is determined.
With these facts in mind I explain the problem asked by Mark Kac in the paper

“Can one hear the shape of a drum?”
American Mathematical Monthly(1966).

This paper can be considered as the initiation to the spectral geometry (although its origin goes back to Hermann Weyl).
In this paper he showed an asymptotic expansion of the heat trace for polygonal domains D under Dirichlet condition:

$$\sum e^{-t\lambda_k} \sim \frac{|D|}{2\pi t} - \frac{1-g}{6} + O(\sqrt{t}\,) \text{ as } t \downarrow 0.$$

Problem

$$Spec(M_1, g_1) = Spec(M_2, g_2) \implies M_1 \overset{\text{isometric ? diffeo ?}}{\cong} M_2$$

This problem can be illustrated in the picture:

It must have been believed from the beginning that the answer is negative (especially in the higher dimensions) and one of the main problem is to find pairs of isospectral but non-isometric or even non-diffeomorphic manifolds. In fact, soon after this paper was published J. Milnor pointed out that there exist such pairs among the $\mathbf{16}$-dimensional flat tori in the paper
Eigenvalues of the Laplace operator on certain manifolds,Proc. Nat. Acad. Sci. U.S.A. **51** *(1964), 542.*
Nowadays, many examples of pairs of such manifolds were found, for example,

A. IKEDA, *On lens spaces which which are isospectral but not isometric*, Ann. Sci. École Norm Sup. (4) 13:3 (1980), 303-315.
A. IKEDA, *On spherical space forms which are isospectral but not isometric*, J. Math. Soc. of Japan, **35** 1983), 437-444.

T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. (2) **121**, no. 1 (1985), 169-186.

C. Gordon, D. Webb, S. Welpert, *Isospectral plane domains and surfaces via Riemanian orbifolds*, Invent. Math. **110**, no. 1 (1992), 1-22.

R. Gorneth, *A new construction of isospectral Riemannian nilmanifolds with examples*, Michigan Math. J. 43 (1996), 159-188.

Even so, in Riemannian geometry it must be still an interesting problem to find such pairs or family of manifolds in a systematic way.

So one of our purpose is to find a new class of isospectral (with respect to sub-Laplacian), but non-diffeomorphic manifolds among compact nilmanifolds.

(1) First, we introduce a class of nilpotent Lie groups with a lattice, and also Laplacian and sub-Laplacian on these groups.

(2) Secondly, we determined eigenvalues of the sub-Laplacian and Laplacian on this class of nilmanifolds explicitly.

(3) Then, we tried to find out isospecral, but non-diffeomorphic pairs in the class.

As a result we know that there are infinitely many isospectral, but non-diffeomorphic nilmanifolds among them with respect to the sub-Laplacians and Laplacians.

Clifford module and Pseudo H-type algebras

I introduce a class of nilmanifolds for which we can calculate the spectrum of sub-Laplacians and Laplacians explicitly.

Let $\mathbb{R}^{r,s}$ be the Euclidean space $\mathbb{R}^{r+s}$ with the non-degenerate scalar product

$$< x, y >_{r,s} := \sum_{i=1}^{r} x_i y_i - \sum_{j=1}^{s} x_{r+j} y_{r+j}.$$

By $C\ell_{r,s} \cong \mathcal{T}(\mathbb{R}^{r+s})/I_{r,s}$: the Clifford algebra generated by $\mathbb{R}^{r,s}$.
It is a quotient algebra of the tensor algebra

$$\mathcal{T}(\mathbb{R}^{r,s}) = \mathbb{R} \oplus \mathbb{R}^{r+s} \oplus \left(\bigotimes^{2} \mathbb{R}^{r+s}\right) \oplus \cdots \oplus \left(\bigotimes^{k} \mathbb{R}^{r+s}\right) \oplus \cdots$$

devided by the two sided ideal $I_{r,s}$ generated by the elements of the form $x \otimes x + < x, x >_{r,s}$, $x \in \mathbb{R}^{r,s}$.

Table of Clifford algebras

8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128) \oplus \mathbb{R}(128)$	$\mathbb{R}(256)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64) \oplus \mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
5	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32) \oplus \mathbb{H}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$	$\mathbb{H}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$
s= 0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
s/r	r=0	1	2	3	4	5	6	7	8

$\mathbb{H}(x)$ is $x \times x$ matrix algebra of quaternions and so on.

Let V be a module of $C\ell_{r,s}$. It is called usually **Clifford module**. We can see what are the Clifford modules from the table of the Clifford algebras.

Let V be a Clifford module. We denote the module action

$$J : C\ell_{r,s} \times V \longrightarrow V, \text{ that is}$$

$$J : (z, X) \mapsto J(z, X) =: J_z(X)$$

$$J_z{}^2 = - < z, z >_{r,s}, \ {}^\forall z \in \mathbb{R}^{r,s}.$$

Definition

We call a module V **admissible**, if there is a non-degenerate scalar product $< \bullet, \bullet >_V$ on V satisfying the following condition:

$$< J_z(X), Y >_V + < X, J_z(Y) >_V = 0 \text{ for } {}^\forall z \in \mathbb{R}^{r,s}, \ {}^\forall X, {}^\forall Y \in V.$$

Pseudo H-type algebra

Let $\{ C\ell_{r,s}, V, J, < \bullet, \bullet >_V \}$ be an admissible module. We define an antisymmetric bilinear map $[\bullet, \bullet]$

$$V \times V \longrightarrow \mathbb{R}^{r,s}, \ \ (X, Y) \longmapsto [X, Y],$$

by the relation

$$< J_z(X), Y >_V = < z, [X, Y] >_{r,s} .$$

With this bilinear map we can define a Lie algebra structure on $V \oplus \mathbb{R}^{r,s} \cong \mathcal{N}_{r,s}(V)$, which we call pseudo H-type algebra,

$$\mathcal{N}_{r,s}(V) \times \mathcal{N}_{r,s}(V) \ni ((X, z), (Y, w)) \longmapsto (0, [X, Y]) \in \mathbb{R}^{r+s} \cong \text{center.}$$

It is a two step nilpotent Lie algebra and so we denote the corresponding simply connected Lie group by $\mathbb{G}_{r,s}(V)$.

We identify the group $\mathbb{G}_{r,s}(V)$ with $\mathcal{N}_{r,s}(V)$ through the exponential map. The Campbell-Hausdroff formula gives us the group structure $*$ on $\mathbb{G}_{r,s}(V)$ by

$$(X, z) \ * \ (Y, w) = \left(X + Y,\ z + w + \frac{1}{2}[X,\ Y]\right).$$

$s = 0$ case was studied by A. Kaplan in the paper

A. KAPLAN, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc. **258** (1980), 147-153 and also,

A. KAPLAN, *On the geometry of groups of Heisenberg type*, Bull. Lond. Math. Soc. **15**(1) (1983), 35-42.

In this case, every irreducible module can be an admissible module and the scalar product $< \bullet, \bullet >_V$ on the Clifford module V is taken to be positive definite, that is an inner product.

For $s > 0$ cases, it was proved in the paper

P. CIATTI, *Scalar products on Clifford modules and pseudo-H-type Lie algebras*, Ann. Mat. Pura Appl. (4) **178**, 1-31(2000) that

Theorem

Irreducible modules need not be an admissible module, but in such a case a double of an irreducible module (or sum of two non-equivalent irreducible modules) can be an admissible module. In any case of $s > 0$, the scalar product $< \bullet, \bullet >_V$ on the module V has the same dimensional positive and negative subspaces. So the dimension $\dim V$ of the module space V is even always. In fact it can be devided by 4 (except three lowest "dim" cases).

Existence of the lattice in $\mathbb{G}_{r,s}(V)$

So we proved in
Existence of Lattices in General H-type Groups, in Journal of Lie Theory, Vol. 24, No. 4, 979-1011(2014) that

Theorem(K. F. and Irina Markina)

Let $C\ell_{r,s}$ be a Clifford algebra and assume V is an admissible module. Then for each orthonormal basis $\{Z_k\}$ in $\mathbb{R}^{r,s}$ there exists an orthonormal basis $\{X_j\}$ in the Clifford module V such that for each pair of the basis elements (X_i, X_j) there exists "at most" one basis vector $Z_k = Z_{k(i,j)} \in \mathbb{R}^{r,s}$ such that

$$[X_i, X_j] = \sigma_{i,j} Z_{k(i,j)}$$

with $\sigma_{i,j}$ being 1 or -1.

We call this type of basis $\{X_i,\ Z_k\}$ as the *integral basis* on $\mathcal{N}_{r,s}(V)$.

Then by *Malćev's Theorem and the Campbell-Hausdorff formula* we know that the subgroup

$$\Gamma_{r,s}(V) = \left\{ \sum m_i X_i + \frac{1}{2} \sum n_k Z_k \,\middle|\, m_i, n_k \in \mathbb{Z} \right\}$$

is a lattice (= uniform discrete subgroup) of $\mathbb{G}_{r,s}(V)$, which we call an integral lattice.

Remark

The basis vector $\{X_i\}$ corrsponding to the orthonormal vector $\{Z_k\}$ in $\mathbb{R}^{r,s}$ is not unique, but can be mapped to each other by a Lie algebra automorphism, fixing the basis vectors $\{Z_k\}$.

This is an extension of the property proved for the positive definite cases $\mathcal{N}_{r,0}(V)$ by

G. Crandall, J. Dodziuk, *Integral structures on H-type Lie algebras*, J. Lie Theory **12** (2002), no. 1, 69-79.

Remark

Even the algebra $C\ell_{r,s} \cong C\ell_{r',s'}$, the pseudo H-type algebras $\mathcal{N}_{r,s}(V)$ and $\mathcal{N}_{r',s'}(V)$ need not be isomorphic, also even if the algebras $C\ell_{r,s}$ and $C\ell_{r',s'}$ are not isomorphic, in some cases $\mathcal{N}_{r,s}(V)$ and $\mathcal{N}_{r',s'}(V')$ are isomorphic.

We have a partial classification theorem among the pseudo H-type algebras:

Theorem (Christian Autenred, K.F. and Irina Markina

(1) If $r + s \neq r' + s'$, then $\mathcal{N}_{r,s}(V)$ and $\mathcal{N}_{r',s'}(V')$ are never isomorphic.
(2) Let $r + s = r' + s'$. Assume one of the admissible modules V of $C\ell_{r,s}$ or V' of $C\ell_{r',s'}$ are minimal and $\mathbf{dim}\, V = \mathbf{dim}\, V'$. Then only for the cases $(r', s') \in \{(r, s), (s, r)\}$, $\mathcal{N}_{r,s}(V)$ and $\mathcal{N}_{r',s'}(V')$ **can** be isomorphic.

So there is possibility of examples of isospectral, non-diffeomorphic manifolds among pairs $\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)$ and $\Gamma_{s,r}(U)\backslash\mathbb{G}_{s,r}(U)$.

Laplacian and sub-Laplacian on $\mathbb{G}_{r,s}(V)$

Let $\mathbb{G}_{r,s}(V)$ be a pseudo H-type group and we fix an integral lattice

$$\Gamma_{r,s}(V) = \left\{ \sum m_i X_i + \frac{1}{2} \sum k_k Z_k \right\}$$

associated with the orthonormal basis $\{Z_k\}$ in the center $\mathbb{R}^{r,s}$.We denote the left invariant vector field defined by each vector X_i (and also Z_k) by $\tilde{X}_i$ ($\tilde{Z}_k$).
In the definition of admissible modules the scalar product in the module space V need not be positive definite, so we have two types of invariant differential operators on $\mathbb{G}_{r,s}(V)$,

Laplacian (sub-Laplacian) and ultra-hyperbolic operators.

Both come from Id map in Hom$(\mathcal{N}_{r,s}(V),\ \mathcal{N}_{r,s}(V))$.

We consider here only the Laplacian and sub-Laplacian:

$$\Delta^{\mathbb{G}_{r,s}(V)} = \Delta = -\sum \tilde{X}_j^2 - \sum \tilde{Z}_k^2,$$

and sub-Laplacian is

$$\Delta_{sub}^{\mathbb{G}_{r,s}(V)} = \Delta_{sub} = -\sum \tilde{X}_i^2.$$

These two are descended to the quotient space $\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)$ which are our operators considering here. Then we have

Theorem A

Let V and U be admissible modules of $C\ell_{r,s}$ and $C\ell_{s,r}$, respectively and assume $\dim V = \dim U$, then two nilmanifolds $\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)$ and $\Gamma_{s,r}(U)\backslash\mathbb{G}_{s,r}(U)$ are isospectral with respect to the sub-Laplacians and Laplacians always.

Now we consider two conditions **IR1** and **IR2** in relation with the irreducibility and admissibility of the Clifford module for a pair of Clifford algebras $C\ell_{r+1,s}$ and $C\ell_{s,r+1}$ with $r+1 \neq s$:

IR1: An irreducible module V of $C\ell_{r+1,s}$ is admissible with a scalar product $< \bullet, \bullet >_V$ and also assume the dimension of an irreducible module of $C\ell_{r,s} = \dim V$.

Hence irreducible modules of the algebra $C\ell_{r,s}$ is admissible.

IR2: The dimension of the minimal admissible module of $C\ell_{s,r+1}$ $\leq \dim V$.

Then under these conditions we may take an admissible module $\{U, \tilde{J}\}$ of $C\ell_{s,r+1}$ with $\dim U = \dim V$.

Then we have

Theorem B

The pseudo H-type algebras $\mathcal{N}_{r+1,s}(V)$ and $\mathcal{N}_{s,r+1}(U)$ are not isomorphic.

Hence, Theorem A + Theorem B imply

Main Theorem

For $r+1 \neq s$, we assume that the Clifford algebras $C\ell_{r+1,s}$, $C\ell_{r,s}$ and $C\ell_{s,r+1}$ satisfy the two conditions **IR1** and **IR2**. So let V be an irreducible module of $C\ell_{r+1,s}$ (which is admissible by the conditions) and we take an admissible module U of $C\ell_{s,r+1}$ with $\dim U = \dim V$, then two nilmanifolds

$$\Gamma_{r+1,s}(V)\backslash\mathbb{G}_{r+1,s}(V) \text{ and } \Gamma_{s,r+1}(U)\backslash\mathbb{G}_{s,r+1}(U)$$

are isospectral, but non-diffeomorphic.

A note for the proof of Main Theorem

First we note a general fact in nilpotent Lie group theory:

Lemma

Let G and G' be two simply connected nilpotent Lie groups with lattices Γ and Γ' respectively. If the nilmanifolds $\Gamma\backslash G$ and $\Gamma'\backslash G'$ are diffeomorphic, then G and G' are isomorphic.

This can be seen by the arguments:

The given diffeomorphism between $\Gamma\backslash\mathbb{G}$ *and* $\Gamma^{'}\backslash\mathbb{G}^{'}$ *induces an isomorphism of the fundamental groups* $\pi_1(\Gamma\backslash\mathbb{G}) \cong \Gamma$ *and* $\pi_1(\Gamma^{'}\backslash\mathbb{G}^{'}) \cong \Gamma'$, *which can be extended to the whole groups.*

Hence this lemma together with Theorem B implies the non-diffeomorphy assertion in the Main Theorem .

Table of dimensions of minimal admissible modules

Dimensions of minimal admissible modules of $C\ell_{r,s}$ for $r, s \leq 8$

8	16	32	64	$64_{\times 2}$	128	128	128	$128_{\times 2}$	256
7	16	32	64	64	128	128	128	128	256
6	16	$^{\flat}16_{\times 2}$	32	32	64	$64_{\times 2}$	128	128	256
5	16	16	$^{\#}16$	16	32	64	128	128	256
4	8	8	8	$\mathbf{8_{\times 2}}$	**16**	32	64	$64_{\times 2}$	128
3	8	8	8	**8**	**16**	**32**	64	**64**	128
2	4	$\mathbf{4_{\times 2}}$	**8**	**8**	**16**	$\mathbf{^{\#}16_{\times 2}}$	**32**	**32**	64
1	**2**	**4**	**8**	**8**	**16**	**16**	$\mathbf{^{\flat}16}$	**16**	32
0	**1**	**2**	**4**	$\mathbf{4_{\times 2}}$	**8**	**8**	**8**	$\mathbf{8_{\times 2}}$	**16**
s/r	0	1	2	3	4	5	6	7	8

black = irreducible, red = double of irreducible,

$*_{\times 2}$ = two minimal dimensional admissible modules

Low dimensional examples

Now we have examples of isospectral, but non-diffeomorphic nilmanifolds from the table of the dimensions of minimal admissible modules.

Corollary

The following pairs satisfy the conditions **IR1** and **IR2**:

$$C\ell_{4+1,2} \text{ and } C\ell_{2,4+1}, \qquad C\ell_{5+1,1} \text{ and } C\ell_{1,5+1},$$

These are low dimensional cases and we have many isospectral but non-diffeomorphic pairs of nilmanifolds of pseudo H-type groups. In fact

Higher dimensional examples

Theorem

The pairs of nilmanifolds

$$\Gamma_{3+\ell,\ell}\backslash\mathbb{G}_{3+\ell,\ell} \quad \text{and} \quad \Gamma_{\ell,3+\ell}\backslash\mathbb{G}_{\ell,3+\ell}$$

for $\ell = 2 + 4k$ with $k = 0, 1, \ldots,$

and

$$\Gamma_{5+4k+1,4k+1}\backslash\mathbb{G}_{5+4k+1,4k+1} \quad \text{and} \quad \Gamma_{4k+1,5+4k+1}\backslash\mathbb{G}_{4k+1,5+4k+1},$$

$k = 0, 1, \ldots$

are isospectral, but non-diffeomorphic.

Here we consider the minimal dimensional admissible modules of $C\ell_{3+\ell,\ell}$ and $C\ell_{5+4k+1,4k+1}$ respectively and corresponding modules of $C\ell_{\ell,3+\ell}$ and that of $C\ell_{4k+1,5+4k+1}$ with the same dimensions.

Other examples

Let $V \cong \mathbb{R}^{4,4}$ be the irreducible module of $C\ell_{3,1} \cong \mathbb{H}(2)$, then it is admissible and we take the minimal admissible module $U \cong \mathbb{R}^{4,4}$ of $C\ell_{1,3} \cong \mathbb{R}(4)$ which is the double of the irreducible module, then the pairs

$$\Gamma_{3,1}(V)\backslash\mathbb{G}_{3,1}(V) \text{ and } \Gamma_{1,3}(U)\backslash\mathbb{G}_{1,3}(U)$$

are isospectral, but non-diffeomorphic, although they do not satisfy the conditions **IR1**. This case is a minimal dimensional example among these nilmanifolds.
Also similar holds for the pair

$$\Gamma_{3,2}(V)\backslash\mathbb{G}_{3,2}(V) \text{ and } \Gamma_{2,3}(U)\backslash\mathbb{G}_{2,3}(U)$$

with the irreducible module $V \cong \mathbb{R}^{4,4}$ of the algebra $C\ell_{3,2} \cong \mathbb{C}(4)$ and the sum of the non-equivalent irreducible modules of the algebra $C\ell_{2,3} \cong \mathbb{R}(4) \oplus \mathbb{R}(4)$.

Heat kernel on two step nilpotent Lie groups

For the proof of Theorem A we recall an integral formula of the heat kernel for a sub-Laplacian on simply connected two step nilpotent Lie groups.

$\mathbb{G}$: a simply connected two step nilpotent Lie group
$\mathcal{N}$: Lie algebra $\exp : \mathcal{N} \cong \mathbb{G}$. We assume that

$$[\mathcal{N}, \mathcal{N}] = \textit{center of } \mathcal{N}$$

$\{X_i, Z_k\}$: linear basis in $\mathcal{N}$ such that $\{Z_k\}_{k=1}^{d}$ and $\{X_i\}_{i=1}^{N}$ span the center $[\mathcal{N}, \mathcal{N}]$ and a complement of the center, respectively. Moreover, we assume that the basis vectors $\{X_i, Z_k\}$ are orthonormal. Then the Lie algebra $\mathcal{N}$ is decomposed into an orthogonal sum

$$\mathcal{N} = [\{X_i\}_{i=1}^{N}] \oplus_{\perp} [\mathcal{N}, \mathcal{N}] \cong \mathbb{R}^N \oplus_{\perp} \mathbb{R}^d,$$

Put

$$[X_i, X_j] = \sum_{k=1}^{d} c_{ij}^{k} Z_k, \quad c_{ij}^{k} \text{ are structure constants.}$$

Denote by $\Omega(z)$ the skew-symmetric matrix

$$\Omega(z) = \sum_{k=1}^{d} z_k \left(\; c_{ij}^{k} \; \right) \in \mathbb{R}(N) = N \times N \textit{ real matrices},$$

where $z = \sum z_k Z_k \in [\mathcal{N}, \mathcal{N}]$.

Remark

We identify the group $\mathbb{G}$ *with* $\mathbb{R}^N \times \mathbb{R}^d$ *via the coordinates*

$$\mathbb{G} \ni g \to (x_1, \ldots, x_N, z_1, \ldots, z_d) \in \mathbb{R}^N \times \mathbb{R}^d.$$

Then the exponential map $\exp : \mathcal{N} \xrightarrow{\approx} \mathbb{G}$ *is the identity map.*

$\tilde{X}_i$: the left invariant vector field on $\mathbb{G}$

sub-Laplacian : $\Delta^{\mathbb{G}}_{\text{sub}} = -\frac{1}{2}\sum_{i=1}^{N} \tilde{X}_i^2.$

We known that $\Delta^{\mathbb{G}}_{\text{sub}}$ is sub-elliptic and essentially selfadjoint in $L_2(\mathbb{G})$ (with respect to the Haar measure).

Since the sub-Laplacian $\Delta^{\mathbb{G}}_{\text{sub}}$ is left-invariant it can be shown that the heat kernel K is of the form $K(t, g, h) = k^{\mathbb{G}}(t,\ g^{-1}\ *\ h)$ with a smooth function $k^{\mathbb{G}} \in C^\infty(\mathbb{R}_+ \times \mathbb{G})$.

In the paper by R. Beals, B. Gaveau, P. Greiner,

The Green function of model step two hypoelliptic operators and the analysis of certain tangential Cauchy Riemannian complexes, Adv. in Math. **121** (1996), 288-345,

the explicit integral expression of the heat kernel is given:

Theorem (Beals-Gaveau-Greiner)

$$K(t,g,h) = k^{\mathbb{G}}(t, g^{-1} * h) = \frac{1}{(2\pi t)^{N/2+d}} \int_{\mathbb{R}^d} e^{-\frac{f(\tau, g^{-1} * h)}{t}} W(\tau)\, d\tau,$$

where the functions $f = f(\tau, g) \in C^\infty(\mathbb{R}^d \times \mathbb{G})$ and $W(\tau)$ are given as follows:
put $g = (x, z) \in \mathbb{R}^N \times \mathbb{R}^d$,

$$f(\tau, g) = f(\tau, x, z) = \sqrt{-1} < \tau,\ z > + \frac{1}{2}\langle \Omega(\sqrt{-1}\tau)\coth(\Omega(\sqrt{-1}\tau)) \cdot x, x\rangle$$

$$W(\tau) = \left\{\det \frac{\Omega(\sqrt{-1}\tau)}{\sinh \Omega(\sqrt{-1}\tau)}\right\}^{1/2},$$

where $< z, z' >= \sum_{k=1}^{d} z_k z'_k$ and $\langle \Omega(\sqrt{-1}\tau)\coth(\Omega(\sqrt{-1}\tau)) \cdot x, x\rangle$ are the Euclidean inner product on $\mathbb{R}^d$ and on $\mathbb{R}^N$ respectively.

Outline of the proof of Theorem A

(i) The spectrum of the sub-Laplacian on the nilmanifold $\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)$ is given by the explicit determination of the heat trace

$$\mathrm{Tr}\, e^{-t\Delta_{sub}} = \int_{F_{r,s}(V)} \sum_{\gamma\in\Gamma_{r,s}(V)} K(t,\gamma\cdot(x,z),(x,z))dxdz$$

$$= \sum_{\lambda_k\ :\ \text{eigenvalue}} e^{-t\lambda_k}$$

where $K(t,(x,z),(y,w)) \in C^\infty(\mathbb{R}_+ \times G_{r,s}(V) \times \mathbb{G}_{r,s}(V))$ is the kernel function of the heat operator $e^{-t\Delta_{sub}}$ and $F_{r,s}(V)$ is a fundamental domain of the lattice $\Gamma_{r,s}(V)$.

(ii) We do not employ Selberg trace formula method, instead we decompose the function space $C^\infty\left(\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)\right)$ into Fourier coefficients. The nilmanifold $\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)$ is the total space of a principal bundle

$$\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)$$
$$\downarrow$$
$$(\Gamma_{r,s}(V)\cap V)\backslash V \cong \mathbb{T}^{\dim V}$$

with the structure group $\mathbb{R}^{r+s}/(\Gamma_{r,s}(V)\cap\mathbb{R}^{r,s}) \cong \mathbb{T}^{r+s}$.
So we decompose

$$C^\infty\left(\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)\right) = \sum_{\chi}\mathcal{F}^\chi,$$

where $\chi : \mathbb{T}^{r+s} \to U(1)$ is a unitary character and

$$\mathcal{F}^\chi = \left\{ f \in C^\infty\left(\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)\right) \,\middle|\, f((x,z)*\lambda) = \overline{\chi(\lambda)}\cdot f(x,z)\right\}$$

(iii) Then the sub-Laplacian (also Laplacian) leaves invariant each subspace $\mathcal{F}^{\chi}$. Hence if we denote by $\mathcal{D}^{(\chi)}$ the sub-Laplacian restricted to the subspace $\mathcal{F}^{\chi}$, $\mathcal{D}^{(\chi)} = \Delta_{sub}\big|_{\mathcal{F}^{\chi}}$, then

$$\sum_{\chi} \mathbf{Tr}\, e^{-t\mathcal{D}^{(\chi)}} = \mathbf{Tr}\, e^{-t\Delta_{sub}}$$

and we can determine each term $\mathbf{Tr}\, e^{-t\mathcal{D}^{(\chi)}}$ explicitly, which I shall describe below:

The unitary character χ is identified with integers $\chi := 2(m_1, \ldots, m_r, n_1, \ldots, n_s) = 2(\mu, \nu) \in 2\mathbb{Z}^{r+s}$ and the trace

$$\mathbf{Tr}\, e^{-t\mathcal{D}^{(\chi)}}$$

is expressed as follows (we put $N = \dim V$):

Theorem (Wolfram Bauer, K. F. and Chisato Iwasaki)

(1) If $\mathbf{n} = \mathbf{0}$, then the trace of the operator $e^{-t\mathcal{D}^{(0)}}$ is given by

$$\mathrm{Tr}\left(e^{-t\mathcal{D}^{(0)}}\right) = \frac{1}{(2\pi t)^N} \sum_{\ell \in \mathbb{Z}^{2N}} e^{-\frac{||\ell||^2}{2t}}.$$

(2) Assume that $\mathbf{n} \in [\Gamma_{r,s}(V) \cap \mathbb{A}]^*$ with $\sum_{i=1}^{r} {m_i}^2 = \sum_{j=1}^{s} {n_j}^2$, that is, $||\mu|| = |\nu||$ and let $d_0 > 0$ be the greatest common divisor among non-zero components of the integers $(\mu, \nu) = (m_1, \ldots, m_r, n_1, \ldots, n_s)$. Then

$$\mathrm{Tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{1}{(\pi t)^{N/2}} \sum_{\ell \in \mathbb{Z}^{2N}} e^{-\frac{||\mu||^2 \cdot ||\ell||^2}{{d_0}^2 t}} \left(\frac{2||\mu||}{\sinh(8\pi t||\mu||)}\right)^{N/2}.$$

(3) For $\mathbf{n} = 2(\mu + \nu)$ with $||\mu|| \neq ||\nu||$ the matrix $\mathbf{\Omega(n)}$ is non-singular.

$$\mathrm{Tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right)$$

$$= 2^{2N} \cdot \left(\frac{||\mu||^2 - ||\nu||^2}{\sinh\{4\pi\, t(||\mu|| + |\nu||)\} \sinh\{4\pi\, t(||\mu|| - |\nu||)\}}\right)^{N/2.} .$$

Theorem A

Let V and U be minimal admissible modules of $C\ell_{r,s}$ and $C\ell_{s,r}$ respectively and assume $\dim V = \dim U$, then the nilmanifolds $\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)$ and $\Gamma_{s,r}(U)\backslash\mathbb{G}_{s,r}(U)$ are isospectral with respect to the sub-Laplacian (and the Laplacian, since $\Delta = \Delta_{sub} + C_\chi$ on $\mathcal{F}^\chi$).

Remark

• The pseudo H-type algebra $\mathcal{N}_{r,s}(V_{min})$ with a minimal dimensional admissible module V_{min} does not depend on the chosen minimal admissible module, even if the Clifford algebra has two non-equivalent irreducible modules.

• Moreover, in the above determination we do not explicitly use the assumption that the admissible module is minimal.

• This implies that if V is a sum of k minimal admissible modules, then the heat trace in each of the case of above three is the k-th power of the corresponding heat trace for the manifold $\Gamma_{r,s}(V_{min})\backslash\mathbb{G}_{r,s}(V_{min})$.

Proof of Theorem B

Under the assumption **IR1** and **IR2** first I explain a Proposition:

Let $Z_1, \ldots, Z_{r+1}, Z_{r+2}, \ldots, Z_{r+s+1}$ be the orthonormal generators of the Clifford algebra $C\ell_{r+1,s}$, Z_i for $i = 1, \ldots, r+1$ positive and the last s vectors negative.
Also let $W_1, \ldots, W_r, W_{r+1}, \ldots, W_{r+s}$ be the orthonormal generators of $C\ell_{r,s}$, again the first r vectors being positive and so on.
We define a map $\mathcal{I} : \mathbb{R}^{r,s} \longrightarrow C\ell_{r+1,s}$ in such a way that

$$W_1 \longmapsto Z_1 Z_{r+1}, \ \ldots, \ W_r \longmapsto Z_r Z_{r+1},$$
$$W_{r+1} \longmapsto Z_{r+2} Z_{r+1}, \ldots, W_{r+s} \longmapsto Z_{r+s+1} Z_{r+1},$$

then it satisfies the property

$$\mathcal{I}(W_i)^2 = - < W_i, W_i >_{r,s} .$$

Hence it can be extended to the algebra homomorphism

$$C\ell_{r,s} \longrightarrow C\ell_{r+1,s},$$

which we also denote by $\mathcal{I}$.
Let $\{ V, J, < \bullet, \bullet >_V \}$ be an admissible module of $C\ell_{r+1,s}$. Then

Proposition

The action of $C\ell_{r,s}$ on $\{V, < \bullet, \bullet >_V\}$ by the composition map $\hat{J} := J \circ \mathcal{I}$ also gives us an admissible module.

Proof. This can be seen from

$$\begin{aligned} &< \hat{J}_{W_i}(X), Y >_V = < J_{Z_i} J_{Z_{r+1}}(X), Y >_V \\ &= < X, J_{Z_{r+1}} J_{Z_i}(Y) >_V = - < X, J_{Z_i} J_{Z_{r+1}}(Y) >_V \\ &= - < X, \hat{J}_{W_i}(Y) >_V . \end{aligned}$$

Now let's assume that there exists an algebra isomorphism

$$\Phi = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} : \mathcal{N}_{r+1,s} \cong \begin{matrix} V \\ \oplus_\perp \\ \mathbb{R}^{r+1,s} \end{matrix} \longrightarrow \begin{matrix} U \\ \oplus_\perp \\ \mathbb{R}^{s,r+1} \end{matrix} \cong \mathcal{N}_{s,r+1},$$

then the map

$$\Phi^\tau \circ \Phi = \begin{pmatrix} A^\tau \circ A & 0 \\ 0 & -Id \end{pmatrix} : \mathcal{N}_{r+1,s} \cong \begin{matrix} V \\ \oplus_\perp \\ \mathbb{R}^{r+1,s} \end{matrix} \longrightarrow \begin{matrix} V \\ \oplus_\perp \\ \mathbb{R}^{r+1,s} \end{matrix} \cong \mathcal{N}_{r+1,s},$$

is an algebra automorphism, where A^τ denote the adjoint operator defined by

$$< A(X), Y >_U = < X, A^\tau(Y) >_V$$

and satisfies the conditions

$$A^\tau \circ \hat{J}_w \circ A = J_{C^\tau(w)}, \quad \text{and} \quad C^\tau \circ C = -Id.$$

Then we have

$$(A^\tau \circ A) \circ J_z \circ (A^\tau \circ A) = J_{-z}, \text{ for } z \in \mathbb{R}^{r+1,s}. \qquad (1)$$

By some arguments we know it satisfies

$$(A^\tau \circ A) \circ \hat{J}_W = \hat{J}_W \circ (A^\tau \circ A), \ W \in \mathbb{R}^{r,s}.$$

By the irreducibility assumption of the module $\{C\ell_{r,s}, \hat{J}, < \bullet, \bullet >_V\}$ we know that the intertwining operator $A^\tau \circ A$ must be a constant $= \lambda \cdot \mathrm{Id}$ (Schur's lemma). Then by the relation (1) we have

$$\lambda^2 J_z = -J_z, \ z \in \mathbb{R}^{r+1,s}.$$

Hence $\lambda^2 < 0$ which is a contradiction. So the two algebras $\mathcal{N}_{r+1,s}(V)$ and $\mathcal{N}_{s,r+1}(U)$ are not isomorphic.

Existence of any number of isopsectral, non-diffeomorphic manifolds

($\bullet$) Let denote irreducible modules of $C\ell_{3,0} \cong \mathbb{H} \oplus \mathbb{H}$ by $V_{\pm} \cong \mathbb{R}^{4,0}$. In this case irreducible modules are admissible. Then two algebras $\mathcal{N}_{3,0}(\overset{p}{\oplus} V_{+} \oplus \overset{q}{\oplus} V_{-})$ and $\mathcal{N}_{3,0}(\overset{p'}{\oplus} V_{+} \oplus \overset{q'}{\oplus} V_{-})$ are isomorphic for the cases of $(p', q') \in \{(p, q),\ (q, p)\}$.
Hence the nilmanifolds $\Gamma_{3,0}(\overset{p}{\oplus} V_{+} \oplus \overset{q}{\oplus} V_{-}) \backslash \mathbb{G}_{3,0}(\overset{p}{\oplus} V_{+} \oplus \overset{q}{\oplus} V_{-})$ and $\Gamma_{3,0}(\overset{p}{\oplus} V_{+} \oplus \overset{q}{\oplus} V_{-}) \backslash \mathbb{G}_{3,0}(\overset{p'}{\oplus} V_{+} \oplus \overset{q'}{\oplus} V_{-})$ for $(p', q') \notin \{(p, q), (q, p)\}$ are isospectral, but non-diffeomorphic.

($\bullet$) The same occurs for the case $C\ell_{7,0}$ and accordingly for the cases of $C\ell_{k,0}$ with $k = 3\ mod\ 4$.


Thank you
for your
Attention