Stationary scattering theory on manifold with ends (LAP and RC)

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Prototype Riemannian manifold with ends

Let (M, g) be a connected Riemannian manifold.

An open subset $E \subset M$ is an *end*, if the closure $\bar{E}$ is of the form

$$\bar{E} \cong [1, \infty) \times S$$

for some connected manifold S.

The metric g is of *warped-product type* on E, if

$$g(r, \sigma) = dr \otimes dr + f(r) h_{\alpha\beta}(\sigma)\, d\sigma^{\alpha} \otimes d\sigma^{\beta}; \qquad (r, \sigma) \in (1, \infty) \times S$$

for some Riemannian metric h on S. In particular, the geometry of $S_r := \{r\} \times S \subset M$ is similar to (S, h) with scale factor $\sqrt{f(r)}$.

We mainly consider the following models of growing ends:

- $f(r) = r^{\theta}$ with $\theta > 0$. (Asympt. Euclidean)

- $f(r) = \exp(\delta r^{\theta})$ with $\delta > 0$, $0 < \theta < 1$. (Asympt. Euclidean)

- $f(r) = \exp(\kappa r)$ with $\kappa > 0$. (Asympt. hyperbolic)

Remarks *1. Typical examples are $\mathbb{R}^d$ and $\mathbb{H}^d$.*

2. We do not assume anything on (S, h). Hence the half-spaces of $\mathbb{R}^d$ and $\mathbb{H}^d$ are included. Their sectors are also included.

3. In the following abstraction, more general exterior domains such as the outside of a parabola or a cylinder are included.

Abstraction: Existence of ends

Let (M, g) be a connected Riemannian manifold.

Assumption (A) *There exist* $r \in C^\infty(M)$ *with* $r(M) = [1, \infty)$ *and* $c > 0$ *and* $r_0 \geq 2$ *such that:*

1. *The gradient vector field* $\nabla r \in \mathfrak{X}(M)$ *is forward complete.*

2. *The bound* $|\nabla r| \geq c$ *holds on* $\{x \in M \,|\, r(x) > r_0/2\}$*.*

We call each component of $E = \{x \in M \,|\, r(x) > r_0\}$ an end of M.

Construction of spherical coordinates

It is obvious that E is the union of r-spheres

$$S_R = \{x \in M \,|\, r(x) = R\}; \quad R > r_0.$$

Denote by $y\colon [0,\infty) \times M \ni (t,x) \mapsto y(t,x) \in M$ the flow generated by the *normalized* gradient vector field

$$\tilde{X} = \eta |\nabla r|^{-2} \nabla r \in \mathfrak{X}(M); \quad \eta = 1 - \chi(2r/r_0),$$

It clearly satisfies $r(y(t,x)) = r(x) + t$ for any $x \in E$ and $t \geq 0$, and hence the flow y induces a family of diffeomorphic embeddings

$$\iota_{R,R'} = y(R' - R, \,\cdot\,)|_{S_R} \colon S_R \to S_{R'}; \quad R \leq R'$$

satisfying

$$\iota_{R',R''} \circ \iota_{R,R'} = \iota_{R,R''}; \quad R \leq R' \leq R''.$$

Through the above embeddings we may regard

$$S_R \subset S_{R'} \quad \text{for } R \le R' \text{ in a well-defined manner.}$$

This naturally induces a manifold structure on the union

$$S = \bigcup_{R > r_0} S_R.$$

Let σ be any local coordinates on S, and define $\sigma(x)$ for $x \in E$ by considering $x \in S_{r(x)} \subset S$. The spherical coordinates of $x \in E$ are

$$(r, \sigma) = (r(x), \sigma(x)) \in (r_0, \infty) \times S.$$

Note that in such coordinates the ends E are identified with an open subset of the half-infinite cylinder $(r_0, \infty) \times S$ whose r-sections are monotonically increasing and exploiting S.

Abstract assumption on geometry: Growing ends

We introduce the tensor ℓ and the differential operator L:

$$\ell = g - \eta|\nabla r|^{-2} dr \otimes dr, \qquad L = p_i^* \ell^{ij} p_j.$$

In the spherical coordinates ℓ may be identified with the pull-back of g to the r-spheres, and L with the spherical part of $-\Delta$.

Assumption (B) *There exist* $\sigma, \tau, C > 0$ *such that*

$$r\nabla^2 r \geq \frac{\sigma}{2}|\nabla r|^2 \ell - Cr^{-\tau} g,$$

and for $\alpha = 0, 1$

$$\left|\nabla^\alpha |\nabla r|^2\right| \leq Cr^{-\alpha(1+\tau)}, \quad |\nabla^\alpha \Delta r| \leq C, \quad |L\Delta r| \leq Cr^{-1-\tau}.$$

Note that $\nabla^2 r$ is the geometric Hessian of r.

The Schrödinger operator

On such (M, g) we study the Schrödinger operator

$$H = H_0 + V \quad \text{on } \mathcal{H} = L^2\Big(M, \sqrt{\det g}\, dx\Big),$$

where H_0 is the free Schrödinger operator

$$H_0 = -\frac{1}{2}\Delta = \frac{1}{2} p_i^* g^{ij} p_j = -\frac{1}{2\sqrt{\det g}} p_i g^{ij} \sqrt{\det g}\, p_j; \quad p_j = -i\partial_j.$$

The operator Δ is called the Laplace–Beltrami operator.

Under the assumptions below we consider the Dirichlet self-adjoint realization.

Joint potential

Define the effective *joint potential* by

$$q = V + \frac{1}{8}\eta|\nabla r|^{-2}\Big[(\Delta r)^2 + 2\nabla^r \Delta r\Big]; \qquad \nabla^r = g^{ij}(\nabla_i r)\nabla_j.$$

Assumption (C) *There exists a splitting by real-valued functions:*

$$q = q_1 + q_2; \qquad q_1 \in C^1(M) \cap L^\infty(M),\ \ q_2 \in L^\infty(M),$$

such that for some $\epsilon, C > 0$

$$|\nabla q_1| \le Cr^{-1-\epsilon}, \qquad |q_2| \le Cr^{-1-\epsilon}.$$

Critical energy

Define the *critical energy* $\lambda_H \in \mathbb{R}$ by

$$\lambda_H = \limsup_{r\to\infty} q = \lim_{R\to\infty} \Big(\sup\{q(x);\ r(x) \geq R\}\Big).$$

Remarks *1. Kumura ('97) proved that* $[\lambda_H, \infty) \subset \sigma_{ess}(H)$.

2. If $f(r) = \exp(\kappa r)$ *and* $V_1 \equiv 0$, *we have*

$$\lambda_H = \frac{(d-1)^2\kappa^2}{32}.$$

Recall for $\mathbb{H}^d$ *we have*

$$f(r) = (\sinh r)^2 \sim \exp(2r), \qquad \sigma(H_0) = \Big[\frac{(d-1)^2}{8}, \infty\Big).$$

Weighted spaces

Define the weighted spaces

$$\mathcal{H}^s = (H_0 + 1)^{-s/2}\mathcal{H}, \quad \mathcal{H}_s = r^{-s}\mathcal{H}; \quad s \in \mathbb{R}.$$

Set the dyadic annuli $\Omega_\nu = \{x \in M;\ 2^\nu \leq r(x) < 2^{\nu+1}\}$ for $\nu \geq 0$, and define the associated Besov spaces by

$$B = \Big\{\psi \in L^2_{\mathrm{loc}}(M);\ \|\psi\|_B = \sum_{\nu=0}^{\infty} 2^{\nu/2}\|\chi_{\Omega_\nu}\psi\|_{\mathcal{H}} < \infty\Big\},$$

$$B^* = \Big\{\psi \in L^2_{\mathrm{loc}}(M);\ \|\psi\|_{B^*} = \sup_{\nu\geq 0} 2^{-\nu/2}\|\chi_{\Omega_\nu}\psi\|_{\mathcal{H}} < \infty\Big\},$$

$$B^*_0 = \overline{C^\infty_0(M)} \text{ in } B^*.$$

Recall the nesting holding for any $s > 1/2$:

$$\mathcal{H}_s \subsetneq B \subsetneq \mathcal{H}_{1/2} \subsetneq \mathcal{H} \subsetneq \mathcal{H}_{-1/2} \subsetneq B^*_0 \subsetneq B^* \subsetneq \mathcal{H}_{-s}.$$

Functions satisfying the Dirichlet boundary condition

Choose a non-negative $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1 & \text{for } t \leq 1, \\ 0 & \text{for } t \geq 2, \end{cases}$$

and define $\chi_N, \bar{\chi}_N \in C^\infty(M)$ for $N \geq 0$ by

$$\chi_N = \chi(r/2^N), \quad \bar{\chi}_N = 1 - \chi_N.$$

Let us introduce an auxiliary space:

$$\mathcal{N} = \{\psi \in L^2_{\mathrm{loc}}(M);\ \chi_N \psi \in \mathcal{H}^1 \text{ for any } N \geq 0\}.$$

This is the space of the functions that locally satisfy the Dirichlet boundary condition, possibly with infinite $\mathcal{H}^1$-norm. Recall:

$$\mathcal{D}(H_0) = \{\psi \in \mathcal{H}^1;\ \Delta\psi \in \mathcal{H} \text{ in the distributional sense}\}.$$

Boundedness outside the ends

Assumption (D) *The embeddings*

$$r^{-s}\mathcal{H}^1 \hookrightarrow \mathcal{H}$$

are compact for any $s > 0$*, or equivalently, the mapping*

$$\chi_N (H_0 + 1)^{-1/2} \colon \mathcal{H} \to \mathcal{H}.$$

is compact for all for all $N \geq 0$.

Assumption (D) says the "boundedness" of $M \setminus E$.

Limiting absorption principle (LAP)

We set the resolvent $R(z) = (H - z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$.

Theorem *For any* $I \Subset (\lambda_H, \infty)$ *set*

$$I_\pm = \{z = \lambda \pm i\Gamma \in \mathbb{C};\ \lambda \in I,\ \Gamma \in (0, 1)\}.$$

Then there exists $C > 0$ *such that for any* $\psi \in B$ *and* $z = \lambda \pm i\Gamma \in I_\pm$

$$\|R(z)\psi\|_{B^*} \leq C\|\psi\|_B.$$

Moreover, $R(z)$ *extend for* $z \in I \cup I_\pm$ *continuously in the norm topology of* $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$ *for any* $s > 1/2$*, and the limits*

$$R(\lambda \pm i0) := \lim_{I_\pm \ni z \to \lambda} R(z); \quad \lambda \in I,$$

belong to $\mathcal{B}(B, B^*)$*.*

An interpretation by forced oscillation

Let $z = \lambda + i\Gamma \in I_+$, and consider a system with forced oscillation:

$$u'(t) = -iHu(t) + ie^{-itz}f; \quad f \in B.$$

Obviously we have an explicit solution:

$$u(t) = e^{-itz}R(z)f + e^{-itH}\Big(u_0 - R(z)f\Big).$$

- In general, $R(z)f$ is a response to the external force $e^{-itz}f$.
- On entering into the system wave escapes toward infinity. When the initial wave front gets far away, the amplitude of oscillator is larger. Hence $R(z)f$ must have a certain decay.
- The smaller $\Gamma \approx 0$ gets, the milder the decay rate would be. $R(\lambda + i0)f$ would be a response to $e^{-it\lambda}f$ after a long time.
- If f is well localized, the waves escape in the spherical shape, and thus the natural target of $R(\lambda + i0)$ is B^*.

Radiation condition bound

Let us set

$$A = \mathrm{i}[H_0, r] = \mathrm{Re}\, p^r = \frac{1}{2}\Big(p^r + (p^r)^*\Big), \qquad a = \eta_\lambda |\nabla r| \sqrt{2(\lambda - q_1)},$$

where $p^r = (\nabla r)^i \nabla_i$ and $\eta_\lambda = \bar{\chi}(2r/r_\lambda)$.

Theorem *Let* $I \Subset (\lambda_H, \infty)$. *Then there exist* $\delta, C > 0$ *such that for any* $\psi \in r^{-\delta}B$ *and* $z \in I \cup I_\pm$

$$\Big\| r^\delta (A \mp a) R(z) \psi \Big\|_{B^*} \le C \| r^\delta \psi \|_B.$$

Remark *The radial operator* $A \mp a$ *eliminates the leading spherical wave of* $R(\lambda \pm \mathrm{i}0)$ *at infinity.*

Sommerfeld's uniqueness

Theorem *Let* $\lambda > \lambda_H$, $\phi \in L^2_{loc}(M)$ *and* $\psi \in B$. *Then* $\phi = R(\lambda \pm i0)\psi$ *holds if and only if both of the following holds:*

1. $\phi \in B^* \cap \mathcal{N}$ *and* $(A \mp a)\phi \in B^*_0$.
2. $(H - \lambda)\phi = \psi$ *in the distributional sense.*

Remark *Let* ψ *be given, and suppose* ϕ *solves* $(H - \lambda)\phi = \psi$. *Then* $u(t) = e^{-it\lambda}\phi$ *is a stationary solution to*

$$u'(t) = -iHu(t) + ie^{-it\lambda}\psi. \qquad (\Diamond)$$

To any solution $u(t)$ *of* $(\Diamond)$ *we can freely add a solution* $v(t)$ *of* $v'(t) = -iHv(t)$ *as background. But note such* $v(t)$ *has source and sink only at inifinity. If* $u(t)$ *is purely outgoing, the only source is the external force* $ie^{-it\lambda}\psi$, *and hence the uniquenss follows.*

Commutator theory

We construct a commutator theory with respect to the conjugate operator: $A = \mathrm{i}[H_0, r] = \mathrm{Re}\, p^r$, $\mathcal{H}^1 \subset \mathcal{D}(A)$.

Lemma *As quadratic forms on* $C_0^\infty(M)$,

$$[H, \mathrm{i}A] = p_i^*(\nabla^2 r)^{ij} p_j - (\nabla^r q_1) + \frac{1}{4}(L\Delta r) + \frac{1}{8}(\nabla^r \eta |dr|^{-2})(\Delta r)^2 - 2\,\mathrm{Im}(q_2 A),$$

and hence $[H, \mathrm{i}A]$ *extends as a quadratic form on* $\mathcal{H}^1$*. Moreover, for any* $\psi \in \mathcal{D}(H)$

$$\langle \psi, [H, \mathrm{i}A]\psi \rangle \le \langle H\psi, \mathrm{i}A\psi \rangle + \langle \mathrm{i}A\psi, H\psi \rangle.$$

Remark *The missing positivity of* $\nabla^2 r$ *in the radial direction is recovered from Carleman's weight.*