Integrable PDEs and Nonlinear Steepest Descent

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Rieman-Hilbert factorization

The so-called *nonlinear stationary − phase − steepest − descent* method for the asymptotic analysis of Rieman-Hilbert factorization problems has been very successful in providing
(i) rigorous results on long time, long range and semiclassical asymptotics for solutions of completely integrable equations and correlation functions of exactly solvable models,
(ii) asymptotics for orthogonal polynomials of large degree,
(iii) the limiting eigenvalue distribution of random matrices of large dimension (and thus universality results, i.e. independence of the exact distribution of the original entries, under some conditions),
(iv) proofs of important results in combinatorial probability (e.g. the limiting distribution of the length of longest increasing subsequence of a permutation, under uniform distribution).

Stationary phase

Even though the stationary phase idea was first applied to a Riemann-Hilbert problem and a nonlinear integrable equation by *Alexander Its* (1982) the method became systematic and rigorous in the work of *Deift and Zhou* (1993).
In analogy to the linear stationary-phase and steepest-descent methods, where one asymptotically reduces the given exponential integral to an exactly solvable one, in the nonlinear case one asymptotically reduces the given Riemann-Hilbert problem to an exactly solvable one.

Non-self-adjointness

Our aim here is to comment on the distinction between the stationary-phase idea and the steepest-descent idea, stressing the importance of actual *steepest descent contours* in some problems. We claim that the distinction partly mirrors the *self − adjoint/non − self − adjoint* dichotomy of the underlying Lax operator. To this aim we first have to review some of the main ideas appearing in the self-adjoint case. We mostly use the *defocusing/focusing nonlinear Schrödinger* equation as our working model, but we also digress to the KdV at some point. We stress both here and in the main text that an extra feature appearing only in the nonlinear asymptotic theory is the *Lax − Levermore variational problem*, discovered in 1979, before the work of Its, Deift and Zhou, but reappearing here in the guise of the so-called *g − function* which is catalytic in the process of deforming Riemann-Hilbert factorization problems to exactly solvable ones.

THE LINEAR METHOD

Suppose one considers the Cauchy problem for, say, the linearized KdV

$u_t - u_{xxx} = 0.$

It can off course be solved via Fourier transforms. The end result of the Fourier method is an exponential integral. To understand the long time asymptotic behavior of the integral one needs to apply the stationary-phase method (see e.g. Erdelyi). The underlying principle, going back to Stokes and Kelvin, is that the dominating contribution comes from the vicinity of the stationary phase points. Through a local change of variables at each stationary phase point and using integration by parts we can calculate each contributing integral asymptotically to all orders with exponential error. It is essential here that the phase $x\xi - \xi^3 t$ is real and that the stationary phase points are real.

Airy integral

On the other hand, suppose we have something like the Airy exponential integral

$$Ai(z) = \frac{1}{\pi} \int_0^\infty cos(s^3/3 + zs)ds$$

and we are interested in $z \to \infty$. Set $s = z^{1/2}t$ and $x = z^{3/2}$. So $Ai(x^{2/3}) = \frac{x^{1/3}}{2\pi} \int_{-\infty}^{\infty} exp(ix(t^3/3 + t))dt$.
The phase is $h(t) = \frac{t^3}{3} + t$ and the zeros of $h'(t) = (t^2 + 1)$ are $\pm i$. As they are not real, before we apply any stationary-phase method, we have to deform the integral off the real line and along particular paths: these are the *steepest descent paths*. They are given by the simple characterization $Imh(t) = constant$. In our particular example, the curves of steepest descent are the imaginary axis and the two branches of a hyperbola. By deforming to one of these branches, we finally end up with Laplace type integrals and then apply the same method as above (local change of variables plus integration by parts) to recover valid asymptotics to all orders.

THE NONLINEAR METHOD

The nonlinear method generalizes the ideas above, but also employs new ones.
Consider the initial value problem for the defocusing nonlinear Schrödinger equation

$$i\partial_t\psi + \partial_x^2\psi - |\psi|^2\psi = 0, \quad \textit{under} \quad \psi(x,0) = \psi_0(x),$$

where the initial data function is, say, Schwartz.
The analog of the Fourier transform is the scattering coefficient $r(\xi)$ for the Dirac operator $L = \begin{pmatrix} i\partial_x & i\psi_0(x) \\ -i\psi_0^*(x) & -i\partial_x \end{pmatrix}$.
Suppose we are now interested in the long time behavior of the solution to (4). The inverse scattering problem can be posed in terms of a Riemann-Hilbert factorization problem.

The Riemann-Hilbert problem for dNLS

THEOREM. There exists a 2x2 matrix Q with analytic entries in the upper and lower open half-planes, such that the normal limits Q_+, Q_-, as ξ approaches the real line from above or below respectively, exist and satisfy

$$Q_+(\xi) = Q_-(\xi) \begin{pmatrix} 1 - |r(\xi)|^2 & -r^*(\xi)e^{-2i\xi x - 4i\xi^2 t} \\ r(\xi)e^{2i\xi x + 4i\xi^2 t} & 1 \end{pmatrix}, \xi \; real$$

and $\quad lim_{\xi \to \infty} Q(\xi) = I.$

The solution to dNLS is recovered via $\psi(x, t) = -2 lim_{\xi \to \infty} \xi Q_{12}$.

The stationary phase idea

It was first realized by Alexander Its (1982) that the long time asymptotics for the solution of dNLS can be extracted by reducing the problem above to a "local" Riemann-Hilbert problem located in a small neighborhood of the stationary phase point ξ_0 such that $\Theta'(\xi_0) = 0$ where $\Theta = \xi x + 2\xi^2 t$. The deformation method has been made rigorous and systematic by Deift-Zhou (1993). Here are the basic ideas, which have been used in all works on the stationary-phase-steepest-descent-method since.

Equivalent singular integral operator

This goes back to *Mushkelishvilli* and the Georgian school and provides a nice way to show that under some conditions, small changes in the jump data result in small changes in the solution.

$$Q = I + \int_{\Sigma} \frac{(I - C_w)^{-1}(I)(s)w(s)}{2\pi i(s-z)} ds$$

where w encodes the jump data, C_w is a sum of weighted Cauchy operators and Σ is the jump contour.

Auxiliary Factorization

Suppose $\xi_0 = -\frac{x}{4t}$.
Consider the region $\xi_0 < M$, some positive constant. Note the following upper/lower and lower/upper respectively factorizations.

$$\begin{pmatrix} 1-|r(\xi)|^2 & -r^*(\xi)e^{-2i\Theta} \\ r(\xi)e^{2i\Theta} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -r^* e^{-2i\Theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ re^{2i\Theta} & 1 \end{pmatrix} \quad \mathit{for} \qquad \xi > \xi_0,$$

LU Factorization

and the lower/upper factorization

$$\begin{pmatrix} 1-|r(\xi)|^2 & -r^*(\xi)e^{-2i\Theta} \\ r(\xi)e^{2i\Theta} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} d_-^{-1} & 0 \\ \frac{rd_-^{-1}exp(2i\Theta)}{1-|r|^2} & d_- \end{pmatrix} \begin{pmatrix} d_+ & \frac{-r^* d_+ exp(-2i\Theta)}{1-|r|^2} \\ 0 & d_+^{-1} \end{pmatrix} \textit{ for } \xi < \xi_0,$$

where

Auxiliary Scalar Problem

d is a function analytic in $\mathcal{C} \setminus (-\infty, \xi_s]$ such that

$$d_+(\xi) = d_-(\xi)(1 - |r(\xi)|^2) \qquad \text{for } -\infty < \xi \leq \xi_s,$$

$$d_+(\xi) = d_-(\xi) \qquad \text{for } \xi > \xi_s,$$

$$d \to 1 \quad \text{as } \xi \to \infty.$$

Conjugation

Note that both the JUMP MATRIX and the FACTORS are CONSTANT (in x, t) matrices conjugated by the diagonal $diag[e^{-i\xi x - 2i\xi^2 t}, e^{i\xi x + 2i\xi^2 t}]$. Note that because of the actual conjugation, the exponential decay / increase of the off-diagonal term depends on the triangularity! That is why different factorizations (UL or LU) are appropriate in different areas of the complex spectral plane. These areas meet at the stationary phase point.

Now consider a cross centered on the stationary phase point (see Figure 1 below). The actual position of the branches is not important. The branches do not even have to be straight half-lines. All that matters is that the sign of the crucial quantity $Re(i\Theta)$ is constant along each branch.

A stationary phase point centered on a cross

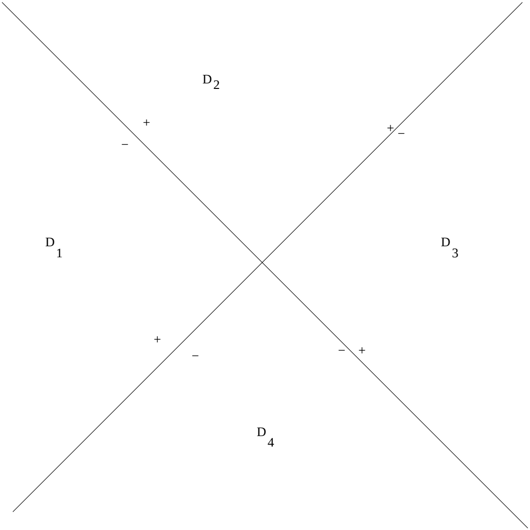

Figure 1. A cross centered at a stationary phase point.

Change of Variable

Define a new matrix by $M = Q, \quad \xi \in D_2 \cup D_4,$

$$M = \begin{pmatrix} d^{-1} & \frac{r^* d e^{2i\Theta}}{1-|r|^2} \\ 0 & d \end{pmatrix} Q, \quad \xi \in D_1 \cap \{Im\xi > 0\},$$

$$M = Q \begin{pmatrix} d^{-1} & 0 \\ \frac{r d^{-1} e^{-2i\Theta}}{1-|r|^2} & d \end{pmatrix}, \quad \xi \in D_1 \cap \{Im\xi < 0\},$$

$$M = Q \begin{pmatrix} 1 & -r^* e^{-2i\Theta} \\ 0 & 1 \end{pmatrix}, \xi \in D_3 \cap \{Im\xi < 0\},$$

$$M = \begin{pmatrix} 1 & 0 \\ -re^{2i\Theta} & 1 \end{pmatrix} Q, \xi \in D_3 \cap \{Im\xi > 0\}.$$

Approximation

One sees immediately that the off-diagonal terms of the resulting jumps are exponentially small away from the center of the cross. So, they can be neglected asymptotically. One ends up with a Rieman-Hilbert problem on a cross centered at ξ_0. Apart from a small cross centered at ξ_0 the jumps are diagonal everywhere. In this sense, the dominating contribution to the solution of the Rieman-Hilbert problem comes from a small neighborhood of the stationary phase point. The Rieman-Hilbert problem can be solved explicitly via parabolic cylinder functions (following Its) and the asymptotics are recovered.

Local RHP

More precisely, using the rescaling $\xi \to -\xi_0 + \xi(-t\xi_0)^{-1/2}$, the Riemann-Hilbert problem is rescaled to a new problem on an infinite cross. After deforming the components of the cross back to the real line, it is equivalent to the following problem on the real line:

$$H_+(\xi) =$$
$$H_-(\xi) exp(-i\xi^2\sigma_3) \begin{pmatrix} 1 - |r(\xi_0)|^2 & -r^*(\xi_0) \\ r(\xi_0) & 1 \end{pmatrix} exp(i\xi^2\sigma_3),$$
$$H(\xi) \sim \xi^{i\nu\sigma_3},$$

where ν is a constant depending only on ξ_0 and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a Pauli matrix.

Final RHP

After conjugating we end up with a jump that is constant.

$$H_+(\xi)exp(-i\xi^2\sigma_3) =$$
$$H_-(\xi)exp(-i\xi^2\sigma_3)\begin{pmatrix} 1-|r(\xi_0)|^2 & -r^*(\xi_0) \\ r(\xi_0) & 1 \end{pmatrix},$$

$$H(\xi)exp(-i\xi^2\sigma_3) \sim exp(-i\xi^2\sigma_3)\xi^{i\nu\sigma_3}, \quad \xi \to \infty.$$

The analogy with the linear case is clear.

The g-function and a "shock" phenomenon with no linear analogue

The full force of the g-function idea and the connection to the *Lax − Levermore* (1979) variational problem is explored in the analysis of the KdV equation by *Deift, Venakides Zhou* (1997).

$u_t - 6uu_x + \epsilon^2 u_{xxx} = 0, \ u(x,0) = u_0(x)$, in the limit as $\epsilon \to 0$.

Assume for simplicity, that the initial data are real analytic, positive and consist of a "hump" of unit height.

The RHP for KdV

The associated RH problem is $S_+(z) = S_-(z) \begin{pmatrix} 1-|r(z)|^2 & -r^*(z)e^{(-izx-4iz^3t)/\epsilon} \\ r(z)e^{(izx+4iz^3t)/\epsilon} & 1 \end{pmatrix}$, z *real*,
and $lim_{z\to\infty} S(z) = (1,1)$.

The solution of KdV is given by

$$u(x,t;\epsilon) = -2i\epsilon \frac{\partial}{\partial x} S_1^1(x,t;\epsilon).$$

Scattering data for small ϵ

The reflection coefficient r also depends on ϵ. In fact, the WKB approximation is

$$r(z) \sim -ie^{-2i\rho(z)/\epsilon}\chi_{[0,1]}(z)$$
$$1 - |r(z)|^2 \sim e^{-2\tau(z)/\epsilon},$$

where $\rho(z) = x_+ z + \int_{x_+}^{\infty}[z - (z^2 - u_0(x))^{1/2}]dx,$
$\tau(z) = Re \int (u_0(x) - z^2)^{1/2} dx$

and $x_+(z)$ is the largest solution of $u_0(x_+) = z^2$.

The Lax-Levermore Variational Problem

THEOREM [LL]. The weak limit of KdV exists and satisfies
$w - lim_{\epsilon \to 0} u(x, t, \epsilon) = \partial_{xx}[min_A Q(\psi; x, t)]$,

where $A = \{\psi \in L^1[0, 1] : 0 \leq \psi \leq \phi\}$

$Q(\psi; x, t) = (4/\pi) \int_0^1 a(\eta, x, t)\psi(\eta)d\eta -$
$(1/\pi^2) \int_0^1 \int_0^1 log(\frac{\eta-\mu}{\eta+\mu})^2 \psi(\eta)d\eta\psi(\mu)d\mu$,

$a(\eta, x, t) = \eta x - 4\eta^3 t - \eta x_+(\eta) - \int_{x_+(\eta)}^{\infty} (\eta - (u(y) + \eta^2)^{1/2})dy$.
$\phi(\eta, x, t) = \int_{x_-(\eta)}^{x_+(\eta)} (u(y) + \eta^2)^{-1/2} dy$.

The g-function; the finite band ansatz

Let $g(z) = \int log(z - \eta) d\mu(\eta)$

where $d\mu = \psi dx$.
Further assume that $supp(d\mu)$ is supported in $\cup_j^N I_j$, where
$I_j = I_j(x,t), \quad N = N(x,t)$.

The conditions for the g-function

Following Deift, Venakides, Zhou (1997) introduce the transformation $\hat{S}(z) = S(z) exp(ig(z)\sigma_3/h)$

(i) g is analytic off the interval $[0,1]$ and vanishes at infinity.
(ii) "Finite gap ansatz". There exists a finite set of disjoint open real intervals $I_j \in [0,1]$ such that the normal limits g_+, g_- of g exist along these intervals and,

denoting $h(z) = g_+(z) + g_-(z) - 2\rho + 4tz^3 + xz$,

(iia) For $z \in \cup_j I_j$, we have $-\tau < (g_+ - g_-)/2i < 0$ and $h' = 0$.
(iib) For $z \in [0,1] \setminus \cup_j I_j$, we have $2i\tau = g_+ - g_-$ and $h' < 0$.

Justification of the finite band ansatz

In general (for any data u_0) it is not true that the above conditions can be satisfied. It is believed however that under the condition of analyticity a g-function satisfying the "finite gap ansatz" exists. (In fact *Kuijlaars* (2000) gave a proof of the "finite gap ansatz" in the analogous problem of the continuum Toda equations.)

Consequence of the finite band ansatz

Assuming the "finite gap ansatz" one can show that the RH problem reduces to one supported on the bands I_j with jumps of the form

$$\begin{pmatrix} 0 & -ie^{-ih(z)/\epsilon} \\ -ie^{ih(z)/\epsilon} & 0 \end{pmatrix}$$

and in fact, because of (iib), $h(z)$ is a real constant on each band I_j. This RH problem can be solved explicitly via theta functions. The details in [DVZ97] involve the so-called "lens"-argument. Auxiliary contours are introduced near pieces of the real line, and appropriate factorizations and analytic extensions are used, similarly to the discussion above.

Steepest Descent Contours

Having reviewed some essential ideas in the previous sections, we are ready to consider the focusing NLS equation, following *Kamvissis, K.McLaughlin, P.Miller* (2003).

$i\hbar\partial_t\psi + \frac{\hbar^2}{2}\partial_x^2\psi + |\psi|^2\psi = 0$,
under $\psi(x,0) = \psi_0(x)$.

$L = \begin{pmatrix} ih\partial_x & -i\psi_0(x) \\ -i\psi_0^*(x) & -ih\partial_x \end{pmatrix}$ is a non-self-adjoint Lax operator.

We shall see that the deformation of the semiclassical RH problem can be no more confined to a small neighborhood of the real axis but is instead fully two-dimensional. A *steepest descent contour* needs to be discovered!

Spectral Data for small $\hbar$

By the way, in the long time asymptotics for the above with $\hbar = 1$ a collisionless shock phenomenon is also present; for x, t in the shock region the deformed RH problem is supported on a vertical imaginary slit. (K. (1996)) But here, we rather focus on the semiclassical problem $\hbar \to 0$ which is far more complicated.

For simplicity consider the very specific data $\psi_0(x) = A sech x$ where $A > 0$. Let $x_-(\eta) < x_+(\eta)$ be the two solutions of $sech^2(x) + \eta^2 = 0$. Also assume that $\hbar = A/N$ and consider the limit $N \to \infty$. It is known that the reflection coefficient is identically zero and that the eigenvalues of L lie uniformly placed on the imaginary segment $[-iA, iA]$. In fact the eigenvalues are the points $\lambda_j = i\hbar(j + 1/2), j = 0, ..., N-1$ and their conjugates. The norming constants oscillate between -1 and 1.

Riemann-Hilbert for focusing NLS

The associated RH problem is a meromorphic problem with no jump: to find a rational function with prescribed residues at the poles λ_j and their conjugates. It can be turned into a holomorphic problem by constructing two loops, one encircling the λ_j which we denote by C and one C^* encircling their conjugates. We redefine the unknown 2x2 matrix inside the loops so that the poles vanish (there is actually a discrete infinity of choices, corresponding to an infinity of analytic interpolants of the norming constants, see below) and thus arrive at a nontrivial jump across the two loops, encircling the segments $[0, iA]$ and $[-iA, 0]$ respectively.
THEOREM. Let $d\mu = (\rho^0(\eta) + (\rho^0)^*(\eta^*))d\eta$, where $\rho^0 = i$ is the asymptotic density of eigenvalues supported on the linear segment $[0, iA]$. Set $X(\lambda) = \pi(\lambda - iA)$.

Riemann-Hilbert for focusing NLS

Letting M_+ and M_- denote the limits of M on $\Sigma = C \cup C^*$ from left and right respectively, we define the Riemann-Hilbert factorization problem $M_+(\lambda) = M_-(\lambda)J(\lambda)$,

where $J(\lambda) = v(\lambda), \lambda \in C, \quad J(\lambda) = \sigma_2 v(\lambda^*)^* \sigma_2, \lambda \in C^*$,

$lim_{\lambda \to \infty} M(\lambda) = I$,
and $v(\lambda) =$

$$\begin{pmatrix} 1 & -i \; exp(1/h[\int log(\lambda - \eta) d\mu(\eta)) - (2i\lambda x + 2i\lambda^2 t - X(\lambda))]) \\ 0 & 1 \end{pmatrix}$$

Then the solution of fNLS is given by $\psi(x,t) = 2i \; lim_{\lambda \to \infty}(\lambda M_{12})$.

Asymptotic analysis of the Riemann-Hilbert

Our analysis in [KMM] makes use of all the ideas described in the previous sections (factorization, lenses, the singular integral operator, an auxiliary scalar problem), but it also takes care of the fact that while the loops can be deformed anywhere away from the poles as long as h is not small, they have to be eventually located at a very specific position in order to asymptotically simplify the RH problem, as $h \to 0$. Appropriately, the definition of a g-function has to be generalized. Not only will it introduce the division of the loop into arcs, called "bands" and "gaps", but it must implicitly select a contour, the *steepest descent contour*. Rather than giving the complicated set of equations and inequalities defining the g-function, we will rather focus on the associated variational problem; it is not a maximization problem but rather a maximin problem. Here's the setting.

The variational problem

Let the complex upper-half plane $\{z : Imz > 0\}$ be denoted by $\mathcal{H}$ and its closure $\{z : Imz \geq 0\} \cup \{\infty\}$ be denoted by $\bar{\mathcal{H}}$. Let also $\{z : Imz > 0\} \setminus \{z : Rez = 0, 0 < Imz \leq A\}$ be denoted by $\mathcal{K}$. In the closure of this space, $\bar{\mathcal{K}}$, we consider the points ix_+ and ix_-, where $0 \leq x < A$ as distinct. In other words, we cut a slit in the upper half-plane along the segment $(0, iA)$ and distinguish between the two sides of the slit. The point infinity belongs to $\bar{\mathcal{K}}$, but not $\mathcal{K}$. Let $G(z;\eta) = log \frac{|z-\eta^*|}{|z-\eta|}$ be the Green's function for the upper half-plane and let $d\mu^0(\eta)$ be the nonnegative measure $-id\eta$ on the segment $[0, iA]$ oriented from 0 to iA. The star denotes complex conjugation. Let the "external field" ϕ be defined by

$$\phi(z) = -\int G(z;\eta)d\mu^0(\eta) - Re(\pi(iA - z) + 2i(zx + z^2 t)),$$

where, wlog $x > 0$.

Free and weighted energy

Let $\mathcal{M}$ be the set of all positive Borel measures μ on $\bar{\mathcal{K}}$, such that both the free energy

$$E(\mu) = \int \int G(x, y) d\mu(x) d\mu(y)$$

and $\int \phi d\mu$ are finite.
The weighted energy of the field ϕ is

$$E_{\phi}(\mu) = E(\mu) + 2 \int \phi d\mu, \ \mu \in \mathcal{M}.$$

Now, given any curve F in $\bar{\mathcal{K}}$, the equilibrium measure λ^F supported in F is defined by $E_{\phi}(\lambda^F) = min_{\mu \in M(F)} E_{\phi}(\mu)$, where $M(F)$ is the set of measures in $\mathcal{M}$ which are supported in F, provided such a measure exists.

S-curves

The finite gap ansatz is equivalent to the existence of a so-called *$S - curve$* joining the points 0_+ and 0_- and lying entirely in $\bar{\mathcal{K}}$. By S-curve we mean an oriented curve F such that the equilibrium measure λ^F exists, its support consists of a finite union of analytic arcs and at any interior point of $supp \mu$ the so called S-property is satisfied

$$\frac{d}{dn_+}(\phi + V^{\lambda^F}) = \frac{d}{dn_-}(\phi + V^{\lambda^F}).$$

The analytic arcs are actually trajectories of *quadratic differentials*

A maximin problem

The appropriate variational problem is: seek a a "continuum" (a compact connected set containing $0_+, 0-$) C such that

$$E_\phi(\lambda^C) = max_{continua} E_\phi(\lambda^F) = max_{continua} min_{\mu \in M(F)} E_\phi(\mu).$$

The existence of a nice S-curve follows from the existence of a continuum C maximizing the equilibrium measure, in particular the associated Euler-Lagrange equations and inequalities.

Justification of the steepest descent method

EXISTENCE THEOREM [*Kamvissis*, *Rakhmanov* 2005]
For the external field ϕ, there exists a continuum $F \in \mathcal{F}$ such that the equilibrium measure λ^F exists and
$E_\phi[F](= E_\phi(\lambda^F)) = max_{F \in \mathcal{F}} min_{\mu \in M(F)} E_\phi(\mu)$.

REGULARITY THEOREM [Kamvissis, Rakhmanov 2005]. The continuum F is in fact an S-curve, so long as it does not touch the spike $[0, iA]$ at more than a finite number of points.

If F touches the spike $[0, iA]$ at more than a finite number of points, more work is required
$(K.2009, Kamvissis - Papadimitropoulos(2015))$.
The S-curve lives in an infinite-sheeted Riemann surface!

Main ideas of the proofs: compactness

Let ρ_0 be the distance between compact sets E, F in $\bar{\mathcal{K}}$: $\rho_0(E, F) = max_{z \in E} min_{\zeta \in F} \rho_0(z, \zeta)$. Introduce the Hausdorff metric on the set I ($\bar{\mathcal{K}}$) of closed non-empty subsets of $\bar{\mathcal{K}}$: $\rho_{\mathcal{K}}(A, B) = sup(\rho_0(A, B), \rho_0(B, A))$.

Compactness of $\mathcal{F}$ is the necessary first ingredient to prove existence of a maximizing contour.

Main ideas of the proofs: upper semicontinuity

The second ingredient is semicontinuity of the energy functional that takes a given continuum F to the equilibrium energy on this continuum:

$$\mathcal{E} : \mathcal{F} \to \mathcal{E}_\psi[\mathcal{F}] = \mathcal{E}_\psi(\lambda^{\mathcal{F}}) = \inf_{\mu \in \mathcal{M}(\mathcal{F})} \mathcal{E}_\psi(\mu).$$

The proof involves *balayage*.

Main ideas of the proofs: regularity

For regularity, the crucial step is
THEOREM. Let F be the maximizing continuum of and λ^F be the equilibrium measure. Let μ be the extension of λ^F to the lower complex plane via $\mu(z^*) = -\mu(z)$. Then

$$Re(\int \frac{d\mu(u)}{u-z} + V'(z))^2 = Re(V'(z))^2 - 2Re \int \frac{V'(z)-V'(u)}{z-u} d\mu(u)$$
$$+ \text{Re} [z^{-2} \int 2(u+z)V'(u) \ d\mu(u)].$$

Here V is the logarithmic potential of μ.

The proof is achieved by taking *Schiffer variations* of the energy functional.

$$d\mu^\tau(z^\tau) = d\mu(z), \quad z^\tau = z + \tau h(z)$$

where

$$h(u) = \frac{u^2}{u-z}, \quad \frac{h(u)-h(v)}{u-v} = 1 - \frac{z^2}{(u-z)(v-z)}$$

Main ideas of the proofs: regularity

But the support of the equilibrium measure of the maximizing continuum is characterized by $\int log \frac{1}{|u-z|} d\mu(u) + V_R(z) = 0$. By differentiating and using the identity above we see that it is also characterized by

$Re \int^z (R_\mu)^{1/2} dz = 0$, where

$$R_\mu(z) = (V'(z))^2 - 2 \int_{supp\mu} \frac{V'(z)-V'(u)}{z-u} d\mu(u)$$

$$+ \, z^{-2}(\int_{supp\mu} 2(u+z)V'(u) \; d\mu(u)).$$

The S-property follows easily and this proves the Regularity Theorem.

The Theorem

Let x_0, t_0 be given. The solution of NLS is described (locally) as a slowly modulated $G+1$ phase wavetrain. Setting $x = x_0 + \hbar\hat{x}$ and $t = t_0 + \hbar\hat{t}$, so that x_0, t_0 are "slow" variables while $\hat{x}, \hat{t}$ are "fast" variables, there exist parameters
$a, U = (U_0, U_1, ..., U_G)^T$, $k = (k_0, k_1, ..., k_G)^T$,
$w = (w_0, w_1, ..., w_G)^T$, $Y = (Y_0, Y_1, ..., Y_G)^T$,
$Z = (Z_0, Z_1, ..., Z_G)^T$ depending on the slow variables x_0 and t_0 (but not $\hat{x}, \hat{t}$) such that

$$\psi(x,t) = \psi(x_0 + \hbar\hat{x}, t_0 + \hbar\hat{t}) \sim A(X_0, t_0) e^{iU_0(x_0,t_0)/\hbar} e^{i(k_0(x_0,t_0)\hat{x} - w_0(x_0,t_0)\hat{t})}$$
$$\cdot \frac{\Theta(Y(x_0,t_0) + iU(x_0,t_0)/\hbar + i(k(x_0,t_0)\hat{x} - w(x_0,t_0)\hat{t}))}{\Theta(Z(x_0,t_0) + iU(x_0,t_0)/\hbar + i(k(x_0,t_0)\hat{x} - w(x_0,t_0)\hat{t}))}.$$

All parameters can be defined in terms of an underlying Riemann surface X.

Conclusion

I expect that these results concerning the nonlinear steepest contour method applied to integrable systems with a non-self-adjoint Lax operator, may be useful in the treatment of Riemann-Hilbert problems arising in the analysis of general complex or normal random matrices and the Hele-Shaw problem. There is also a conjecture of Kuijlaars that some recent problems where a vector equilibrium measure problem appears may be justified by proving existence and regularity for a solution of a maximin variational problem in two dimensions.

References

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