Spectral shift function for the magnetic Schrödinger operators

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Spectral Shift Function

Definition 1

Let H and H_0 be self-adjoint operators on some Hilbert space $\mathscr{H}$. If there exists a function $\xi(\lambda) = \xi(\lambda; H, H_0)$ on $\mathbb{R}$ such that

$$\mathrm{tr}[f(H) - f(H_0)] = \int_{\mathbb{R}} f'(\lambda)\xi(\lambda)d\lambda \left(= - \int_{\mathbb{R}} f(\lambda)\xi'(\lambda)d\lambda \right) \quad (1)$$

for every $f \in C_0^\infty(\mathbb{R})$, we call $\xi(\lambda; H, H_0)$ the spectral shift function (SSF) for the pair (H, H_0).

If both H and H_0 are lower semi-bounded, $\xi(\lambda)$ is uniquely determined under the condition $\lim_{\lambda \to -\infty} \xi(\lambda) = 0$. If the spectrum of H and H_0 are pure point, then ξ is a step function.

Existence of SSF

Theorem 2 (Kreĭn '53)

Let H and H_0 be self-adjoint operators on a Hilbert space $\mathscr{H}$. Assume that $V = H - H_0$ belongs to the class of the trace operators $\mathscr{I}_1$. Then, there exists a spectral shift function $\xi(\lambda) = \xi(\lambda; H, H_0)$ for the pair (H, H_0) satisfying the following properties.

(1) *ξ is integrable on $\mathbb{R}$ and*

$$\int_{\mathbb{R}} \xi(\lambda) d\lambda = \operatorname{tr} V.$$

(2) *If $\pm V \geq 0$, then $\pm\xi(\lambda) \geq 0$.*

Invariance principle

If H and H_0 satisfy

$$(H+c)^{-m} - (H_0+c)^{-m} \in \mathscr{I}_1 \tag{2}$$

for some positive numbers c and m, we can construct SSF for the pair (H, H_0) by the relation

$$\begin{aligned} &\xi(\lambda; H, H_0) \\ = &\begin{cases} -\xi\left((\lambda+c)^{-m}; (H+c)^{-m}, (H_0+c)^{-m}\right) & (\lambda > -c), \\ 0 & (\lambda \leq -c). \end{cases} \end{aligned}$$

This method is quite useful when H and H_0 are the differential operators.

Scattering matrix and SSF

Let H and H_0 be self-adjoint operators on a Hilbert space $\mathscr{H}$. Suppose that the spectrum $\sigma(H_0)$ of H_0 is purely absotutely continuous and $V = H - H_0 \in \mathscr{I}_1$. By the Kato-Rosenblum theorem, the wave operators

$$W_\pm = \mathrm{s} - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}$$

exist and are complete, i.e., $\mathrm{Ran}\, W_\pm = \sigma_{\mathrm{ac}}(H)$. The scattering operator S is defined by $S = W_+^* W_-$.

Scattering matrix and SSF

Moreover, suppose there exists a unitary operator $\mathscr{F}$ from $\mathscr{H}$ to $L^2(\sigma(H_0) \to \mathfrak{h}; d\mu)$ ($\mathfrak{h}$ is another Hilbert space) such that $\mathscr{F} H_0 \mathscr{F}^*$ acts as multiplication by λ ($\lambda \in \sigma(H_0)$). Then, $\mathscr{F} S \mathscr{F}^*$ acts as multiplication by a unitary operator-valued function $S(\lambda)$. The operator $S(\lambda)$ is called the scattering matrix (SM).

Theorem 3 (Birman–Kreĭn '62)

Under the assumptions above, we have $S(\lambda) - I \in \mathscr{I}_1$ for almost every $\lambda \in \sigma(H_0)$ and

$$\det S(\lambda) = \exp(-2\pi i \xi(\lambda)). \tag{3}$$

SSF for Schrödinger operators

The following results are taken from D. Yafaev's review paper.

Theorem 4

Let $H_0 = -\Delta$ *on* $\mathbb{R}^d$ *and* $H = H_0 + V$, *where* V *is a multiplication operator by a real-valued function* $V(x)$. *Assume*

$$|V(x)| \leq C\langle x\rangle^{-\rho}, \quad \rho > d, \tag{4}$$

where we denote $\langle x\rangle = \sqrt{1 + |x|^2}$. *Let* λ_j $(j = 1, \ldots, n)$ *be the* j*-th smallest eigenvalue of* H *counted with multiplicity and put* $\lambda_0 = -\infty$, $\lambda_{n+1} = 0$.

SSF for Schrödinger operators

Theorem 4 (continued)

Then, (2) holds for $m > (d/2) - 1$ and the SSF $\xi(\lambda; H, H_0)$ ($\lambda \in \mathbb{R}$) and the SM $S(\lambda)$ ($\lambda > 0$) exist. Moreover,

$$\xi(\lambda) = -j \quad (\lambda_j < \lambda < \lambda_{j+1}),$$

for $j = 0, \ldots, n$, and

$$\det S(\lambda) = \exp(-2\pi i \xi(\lambda)) \quad (\lambda > 0).$$

If $\pm V(\lambda) \geq 0$ a.e., then $\pm\xi(\lambda) \geq 0$ a.e.

Phase shift and Scattering Matrix

Let us assume V is radial and decays sufficiently fast at infinity. By the partial wave decomposition in the radial coordinate, the operators H_0 and H are decomposed as

$$\begin{aligned} H_0 &\simeq \bigoplus_n h_{0,n}, \quad h_{0,n} = -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + \frac{\mu_n}{r^2}, \\ H &\simeq \bigoplus_n h_n, \quad h_n = h_{0,n} + V(r), \end{aligned}$$

where h and $h_{0,n}$ act on $L^2((0,\infty); r^{d-1}dr)$, n is the index for the d-dimensional spherical harmonics (eigenfunctions for $-\Delta_{S^{d-1}}$) and μ_n is the corresponding eigenvalue.

Phase shift and Scattering Matrix

The generalized eigenfunction for $h_{0,n}$ with energy k^2 $(k > 0)$ is

$$\begin{aligned} u_{0,n,k}(r) &= r^{-(d-2)/2} J_\nu(kr) \\ &\sim \sqrt{\frac{2}{\pi k}} r^{-(d-1)/2} \cos\left(kr - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \quad (r \to \infty), \end{aligned}$$

where $\nu = \sqrt{\mu_n + (d-2)^2/4}$. If V decays sufficiently fast at infinity, then we can construct the generalized eigenfunction $u_{n,k}$ for h_n with energy k^2 regular at $r = 0$ and

$$u_{n,k}(r) \sim \sqrt{\frac{2}{\pi k}} r^{-(d-1)/2} \cos\left(kr - \frac{\nu\pi}{2} - \frac{\pi}{4} + \delta_{n,k}\right) \quad (r \to \infty)$$

for some $\delta_{n,k} \in \mathbb{R}$. The number $\delta_{n,k}$ is called the scattering phase shift or the time delay.

Phase shift and Scattering Matrix

Then the partial wave SM for the pair $(h_n, h_{0,n})$ is the multiplication operator by a complex number

$$S(k^2; h_n, h_{0,n}) = e^{2i\delta_{n,k}}$$

for any $k > 0$. Thus the determinant of the total SM is

$$\det S(k^2; H, H_0) = \prod_n e^{2i\delta_{n,k}}.$$

Comparing this formula with the Birman–Kreĭn formula, we know

$$\xi(k^2; H, H_0) = -\frac{1}{\pi}\sum_n \delta_{n,k} \tag{5}$$

for any $k > 0$, though we have to choose the branches of $\delta_{n,k}$ appropriately.

Example (point interaction on the half-line)

Let $I = (0, \infty)$, $\alpha \in \mathbb{R}$ and

$$H_0 = -\frac{d^2}{dx^2}, \quad D(H_0) = \{u \in H^2(I) \mid u'(0) = 0\},$$
$$H_\alpha = -\frac{d^2}{dx^2}, \quad D(H_\alpha) = \{u \in H^2(I) \mid u'(0) = \alpha u(0)\},$$
$$H_\infty = -\frac{d^2}{dx^2}, \quad D(H_\infty) = \{u \in H^2(I) \mid u(0) = 0\}.$$

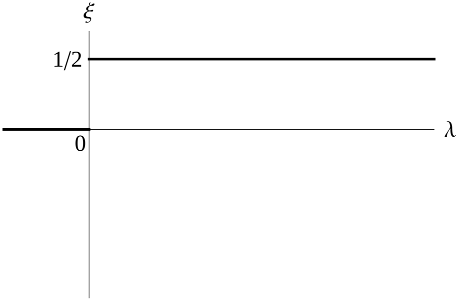
A simple calculation shows that

$$\sigma(H_\alpha) = \begin{cases} [0, \infty) & (0 \le \alpha \le \infty), \\ \{-\alpha^2\} \cup [0, \infty) & (-\infty < \alpha < 0). \end{cases}$$

Example (point interaction on the half-line)

The SSFs are given as follows. First,

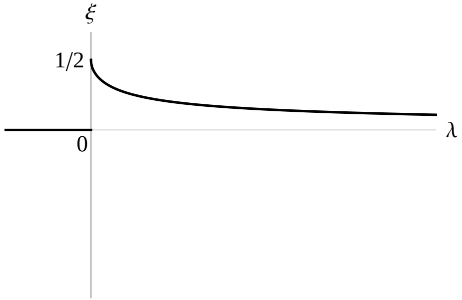
$$\xi(\lambda; H_\infty, H_0) = \begin{cases} 0 & (\lambda < 0), \\ \dfrac{1}{2} & (\lambda > 0). \end{cases}$$



Example (point interaction on the half-line)

For $\alpha > 0$,

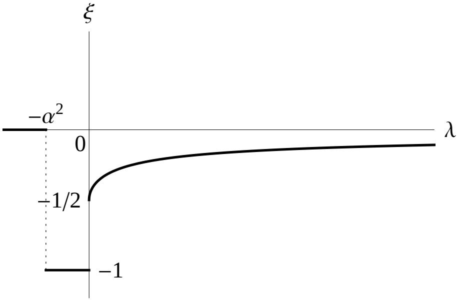
$$\xi(\lambda; H_\alpha, H_0) = \begin{cases} 0 & (\lambda < 0), \\ \dfrac{1}{\pi} \arctan\left(\dfrac{\alpha}{\sqrt{\lambda}}\right) & (\lambda > 0). \end{cases}$$



Example (point interaction on the half-line)

For $\alpha < 0$,

$$\xi(\lambda; H_\alpha, H_0) = \begin{cases} 0 & (\lambda < -\alpha^2), \\ -1 & (-\alpha^2 < \lambda < 0), \\ \dfrac{1}{\pi}\arctan\left(\dfrac{\alpha}{\sqrt{\lambda}}\right) & (\lambda > 0). \end{cases}$$



Example (2-dimensional point interaction)

Let $\alpha \in \mathbb{R}$ and

$$\begin{aligned} H_\infty &= -\Delta, \quad D(H_\infty) = H^2(\mathbb{R}^2), \\ H_\alpha &= -\Delta, \quad D(H_\alpha) = \{u \in L^2(\mathbb{R}^2) \cap H^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\}) \mid \\ & \qquad \Delta|_{\mathbb{R}^2\setminus\{0\}} u \in L^2,\ u_0 = \alpha u_1\}, \end{aligned}$$

where u_0 and u_1 are defined via the asymptotics of u

$$u = u_0 + u_1 \log\frac{1}{r} + o(1) \quad (r \to +0).$$

H_α is a self-adjoint operator on $L^2(\mathbb{R}^2)$ and its spectrum is given by

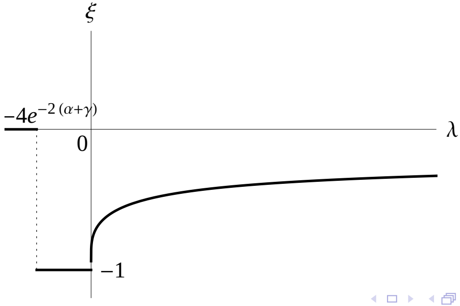
$$\sigma(H_\alpha) = \{\lambda_\alpha\} \cup [0, \infty), \quad \lambda_\alpha = -4e^{-2(\alpha+\gamma)},$$

where γ is the Euler constant.

Example (2-dimensional point interaction)

The SSF for the pair (H_α, H_∞) is given by

$$\xi(\lambda; H_\alpha, H_\infty) = \begin{cases} 0 & (\lambda < \lambda_\alpha), \\ -1 & (\lambda_\alpha < \lambda < 0), \\ -\dfrac{1}{\pi} \operatorname{arccot}\left(\dfrac{2}{\pi}(\alpha + \gamma + \log(\sqrt{\lambda}/2))\right) & (\lambda > 0). \end{cases}$$



SSF for magnetic Schrödinger operators

Let $d = 2$, $H_0 = -\Delta$, and $H(A)$ be the magnetic Schrödinger operator

$$H(A) = \left(\frac{1}{i}\nabla - A\right)^2 = \sum_{j=1}^{2}\left(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j(x)\right)^2.$$

The vector-valued function $A(x) = (A_1(x), A_2(x))$ is called the magnetic vector potential, and the corresponding magnetic field perpendicular to the plane is

$$B(x) = \mathrm{curl}\, A(x) = \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)(x).$$

SSF for magnetic Schrödinger operators

Proposition 5

Assume there exist positive constants C and ϵ such that

$$|A(x)| \leq C\langle x\rangle^{-2-\epsilon}, \quad |\operatorname{div} A(x)| \leq C\langle x\rangle^{-2-\epsilon}. \tag{6}$$

Then the SSF $\xi(\lambda; H, H_0)$ exists.

The proof is easily done by using

$$\begin{aligned} & (H-z)^{-2} - (H_0-z)^{-2} \\ = \; & (H-z)^{-2}\left(\frac{1}{i}(2\nabla\cdot A - \operatorname{div} A) - A^2\right)(H_0-z)^{-1} \\ & +(H-z)^{-1}\left(\frac{1}{i}(\operatorname{div} A + 2A\cdot\nabla) - A^2\right)(H_0-z)^{-2}. \end{aligned}$$

SSF for magnetic Schrödinger operators

However, the condition (6) never holds unless the total flux vanishes.

Proposition 6

Suppose $B \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$ *and* $\operatorname{supp} B \subset \{|x| \le R\}$. *Put*

$$\alpha = \frac{1}{2\pi} \int_{\mathbb{R}^2} B(x) dx.$$

(1) Assume $\alpha = 0$. *Then, for any* $\epsilon > 0$, *there exists* $A \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$ *with* $B = \operatorname{curl} A$ *and* $\operatorname{supp} A \subset \{|x| \le R + \epsilon\}$.
(2) Assume $\alpha \neq 0$. *Then, for any* $A \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ *with* $B = \operatorname{curl} A$, *we have*

$$\max_{|x|=r} |A(x)| \ge \frac{|\alpha|}{r} \quad (r > R).$$

SSF for magnetic Schrödinger operators

Proposition 6 is an immediate consequence of the Stokes formula

$$\alpha = \frac{1}{2\pi} \int_{|x|=r} A(x) \cdot dl.$$

Proposition 6 seems to say it is impossible to define SSF $\xi(\lambda; H, H_0)$ if the total flux $\alpha \neq 0$. Surprisingly, Tamura '08 pointed out that it is possible to define the SSF $\xi(\lambda; H, H_0)$ even if $\alpha \neq 0$, by interpreting the trace formula (1) in a bit weak sense.

Weak trace

Definition 7 (Weak trace)

Let $\chi = \chi(r)$ be a smooth function on $\mathbb{R}$ satisfying

$$0 \leq \chi(r) \leq 1, \quad \chi(r) = \begin{cases} 1 & (r \leq 1), \\ 0 & (r \geq 2). \end{cases}$$

For $R > 0$, we denote $\chi_R(x) = \chi(|x|/R)$. For a bounded linear operator X on $L^2(\mathbb{R}^d)$, we denote

$$\widetilde{\mathrm{tr}}\, X = \lim_{R \to \infty} \mathrm{tr}\left[\chi_R X \chi_R\right],$$

if the limit in the right hand side exists and is independent of the choice of χ.

Weak trace

Let us list the basic properties of $\widetilde{\mathrm{tr}}$.

Proposition 8

(1) *If* $X \in \mathscr{I}_1(L^2(\mathbb{R}^d))$, *then* $\widetilde{\mathrm{tr}}\, X$ *exists and* $\widetilde{\mathrm{tr}}\, X = \mathrm{tr}\, X$.

(2) *Let* X_j $(j = 1, 2)$ *be bounded operators on* $L^2(\mathbb{R}^d)$ *such that* $\chi_R X_j \chi_R \in \mathscr{I}_1$ *for any* $j = 1, 2$ *and any* $R > 0$. *Suppose* $\widetilde{\mathrm{tr}}[X_1 - X_2]$ *exists. Then, for any complex-valued functions* $\Phi_j(x)$ $(j = 1, 2)$ *with* $|\Phi_j(x)| = 1$, *we have*

$$\widetilde{\mathrm{tr}}[X_1 - X_2] = \widetilde{\mathrm{tr}}[\Phi_1^* X_1 \Phi_1 - \Phi_2^* X_2 \Phi_2].$$

In the case X_j $(j = 1, 2)$ are the magnetic Schrödinger operators, the property (2) means the weak trace is independent of the choice of the vector potentials.

Aharonov-Bohm magnetic field

For $\alpha \in \mathbb{R}$, we introduce a singular magnetic vector potential A_α by

$$A_\alpha = \alpha\Lambda, \quad \Lambda(x) = \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right),$$

and put $H_\alpha = H(A_\alpha)$. The corresponding magnetic field is

$$\operatorname{curl} A_\alpha(x) = 2\pi\alpha\delta(x),$$

where δ is the Dirac function supported on 0. This magnetic field is called the Aharonov–Bohm magnetic field. The operator H_α is self-adjoint on $L^2(\mathbb{R}^2)$ with the domain

$$D(H_\alpha) = \{u \in L^2(\mathbb{R}^2) \cap H^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\}) \mid (-i\nabla - A_\alpha)^2 u \in L^2, \ \limsup_{x\to 0} |u(x)| < \infty\}.$$

SSF for the magnetic Schrödinger operator

Theorem 9 (Tamura '08)

Let $B \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$. *Let* α *be the total flux, that is,*

$$\alpha = \frac{1}{2\pi}\int_{\mathbb{R}^2} B(x)dx,$$

and $\beta = \alpha - [\alpha]$ *(the fractional part of* α*). Let* $A \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ *such that* $B = \operatorname{curl} A$ *on* $\mathbb{R}^2$ *and* $A(x) = A_\alpha(x)$ *for sufficiently large* $|x|$. *Then, for any* $f \in C_0^\infty(\mathbb{R})$ *with* $f'(\lambda) = 0$ *near* $\lambda = 0$,

$$\begin{aligned}\widetilde{\operatorname{tr}}[f(H(A)) - f(H_0)] &= \frac{\beta(\beta-1)}{2}f(0) \\ &\quad + \operatorname{tr}[f(H(A)) - f(H_\alpha)].\end{aligned}$$

SSF for the Aharonov-Bohm magnetic field

The key of the proof of Theorem 9 is the following fact.

Lemma 10

Let f satisfy the same assumption in Theorem 9. Then,

$$\widetilde{\mathrm{tr}}[f(H_\alpha) - f(H_0)] = \frac{\beta(\beta-1)}{2} f(0). \tag{7}$$

(7) implies

$$\begin{aligned} \widetilde{\mathrm{tr}}[f(H_\alpha) - f(H_0)] &= \int_{\mathbb{R}} f'(\lambda)\xi(\lambda; H_\alpha, H_0)d\lambda, \\ \xi(\lambda; H_\alpha, H_0) &= \begin{cases} \beta(1-\beta)/2 & (\lambda > 0), \\ 0 & (\lambda < 0). \end{cases} \end{aligned}$$

Thus the SSF exists in this sense, though the perturbation A_α has the long-range nature.

Partial wave decomposition

Though the result is very simple, the proof of (7) is far from trivial. Without loss of generality, we assume $0 < \alpha = \beta < 1$. In the radial coordinate, H_0 and H_α are decomposed as

$$H_0 \simeq \bigoplus_{n\in\mathbb{Z}} h_{|n|}, \quad H_\alpha \simeq \bigoplus_{n\in\mathbb{Z}} h_{|n-\alpha|},$$

$$h_\nu = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{\nu^2}{r^2}.$$

The generalized e.f. of h_ν for the energy k^2 $(k > 0)$ is $J_\nu(kr)$, so

$$J_{|n|}(kr) \sim \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{|n|\pi}{2} - \frac{\pi}{4}\right),$$

$$J_{|n-\alpha|}(kr) \sim \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{|n-\alpha|\pi}{2} - \frac{\pi}{4}\right).$$

Partial wave decomposition

The phase shifts $\delta_{n,k}$ for the pair $(h_{|n-\alpha|}, h_{|n|})$ are

$$\delta_{n,k} = \begin{cases} \pi\alpha/2 & (n = 1, 2, \ldots), \\ -\pi\alpha/2 & (n = 0, -1, -2, \ldots), \end{cases}$$

which are independent of k. This means the series

$$\begin{aligned} -\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \delta_{n,k} \quad = \quad & \quad -\frac{\alpha}{2} - \frac{\alpha}{2} - \cdots \\ & +\frac{\alpha}{2} + \frac{\alpha}{2} + \frac{\alpha}{2} + \cdots \end{aligned}$$

does not converge at all! Actually, we cannot change the order of the limit $R \to \infty$ and the sum $\sum_n$ in the calculation of the weak trace.

Partial wave decomposition

By the Fourier-Bessel transform, we have

$$\mathrm{tr}[\chi_R f(h_\nu)\chi_R] = \int_0^M \int_0^{2R} f(\lambda)\chi(r/R) J_\nu(\sqrt{\lambda}r)^2\, rdr\, d\lambda,$$

where $M = \max \operatorname{supp} f$. By using this expression and various formulas for the Bessel functions, Tamura analyzes the cancellation mechanism in the following sum, and then takes the limit $R \to \infty$.

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \left(\mathrm{tr}[\chi_R f(h_{|n-\alpha|})\chi_R] - \mathrm{tr}[\chi_R f(h_{|n|})\chi_R]\right) \\
= &\sum_{n=1}^{\infty} \mathrm{tr}\left[\chi_R \left(f(h_{n+\alpha}) - 2f(h_n) + f(h_{n-\alpha})\right)\chi_R\right] \\
&+ \mathrm{tr}\left[\chi_R \left(f(h_\alpha) - f(h_0)\chi_R\right)\right].
\end{aligned}$$

J. L. Borg's result

A similar statement is proved in the Ph. D thesis of J. L. Borg.

Theorem 11 (Borg '06)

Let $\alpha \in \mathbb{R}$, and β the fractional part of α. Then, we have for $t > 0$

$$\widetilde{\mathrm{tr}}[\exp(-tH_\alpha) - \exp(-tH_0)] = \frac{\beta(\beta-1)}{2}. \tag{8}$$

(8) seems to be obtained by putting $f = e^{-t\lambda}$ in (7), but in fact $e^{-t\lambda}$ does not satisfy the assumption in Theorem 9. Borg proves the above result by using the Feynman-Kac-Itô formula for the integral kernel of H_α.

Feynman-Kac-Itô formula

The Feynman-Kac-Itô formula for H_α becomes as follows.

$$e^{-tH_\alpha}(x,x') = \frac{1}{4\pi t} e^{-|x-x'|^2/(4t)} \mathbb{E}_{0,x,t,x'}\left[e^{-i\alpha\Theta_w}\right].$$

Here, $w = (w_s)_{0\le x\le t}$ is a sample path for the Brownian bridge process starting from x at time 0 and ending at x' at time t, and $\mathbb{E}_{0,x,t,x'}$ denotes the expectation for the Brownian bridge process. Θ_w is the winding angle of the path w from the origin 0, i.e. $\Theta_w = \arg w_t - \arg w_0$. Borg calculates $\widetilde{\mathrm{tr}}[e^{-tH_\alpha} - e^{-tH_0}]$ using the above formula and the known probability distribution of Θ_w (see e.g. Itô-McKean's textbook).

Two-point AB magnetic field

Let $\alpha_j \in \mathbb{R}$ $(j = 1, 2)$ and $d > 0$. Put

$$A_{\alpha_1,\alpha_2,d}(x) = \alpha_1 \Lambda(x_1 + d, x_2) + \alpha_2 \Lambda(x_1 - d, x_2)$$

and $H_{\alpha_1,\alpha_2,d} = H(A_{\alpha_1,\alpha_2,d})$. The corresponding magnetic field is

$$\mathrm{curl}\, A_{\alpha_1,\alpha_2,d}(x) = 2\pi\alpha_1 \delta(x_1 + d, x_2) + 2\pi\alpha_2 \delta(x_1 - d, x_2).$$

Tamura '08 considers the case $\alpha_2 = -\alpha_1 = \alpha$ and studies the asymptotic behavior of SSF $\xi(\lambda; H_{-\alpha,\alpha,d}, H_0)$ as $d \to \infty$, which is equivalent to the semi-classical limit via a scaling. Notice that the ordinary SSF exists in this case, because the total flux is 0.

Two-point AB magnetic field

Theorem 12 (Tamura '08)

Let $\alpha \in \mathbb{R}$ *and* β *the fractional part of* α. *Then,*

$$\begin{aligned} & \xi'(\lambda; H_{-\alpha,\alpha,d}, H_0) \\ = \ & 2(2\pi)^{-2}\lambda^{-1}\sin^2(\beta\pi)\sin(4\lambda^{1/2}d) + O(d^{-1/3+\delta}) \end{aligned}$$

as $d \to \infty$, *locally uniformly in* $\lambda \in (0, \infty)$ *for any* δ *with* $0 < \delta < 1/3$, *and*

$$\begin{aligned} & \xi(\lambda; H_{-\alpha,\alpha,d}, H_0) \\ = \ & \beta(1-\beta) - (2\pi)^{-2}\lambda^{-1/2}d^{-1}\sin^2(\beta\pi)\cos(4\lambda^{1/2}d) \\ & +o(d^{-1}) \end{aligned}$$

as $d \to \infty$, *locally uniformly in* $\lambda \in (0, \infty)$.

Two-point AB magnetic field

The leading term comes from

$$\xi(\lambda; H_{-\alpha}, H_0) + \xi(\lambda; H_\alpha, H_0) = \frac{(1-\beta)\beta}{2} + \frac{\beta(1-\beta)}{2} = \beta(1-\beta).$$

The subleading term comes from the formula

$$\begin{aligned} & \xi'(\lambda; H_{-\alpha,\alpha,d}, H_0) \\ \sim\ & \xi'(\lambda; H_{-\alpha}, H_0) + \xi'(\lambda; H_\alpha, H_0) \\ & -\frac{\lambda^{-1/2}}{\pi} \operatorname{Re}\left[\exp(4i\lambda^{1/2}d) a_{-\alpha}(\pi \to 0; \lambda) a_\alpha(0 \to \pi; \lambda)\right], \end{aligned}$$

where $a_{\pm\alpha}(\theta \to \theta'; \lambda)$ is the scattering amplitude for the pair $(H_{\pm\alpha}, H_0)$ with the initial direction θ, the final direction θ' and the energy λ. A similar result for the scalar potential perturbation is obtained by Kostrykin–Schrader '98.

Two-point AB magnetic field

J. L. Borg also studies this case. His result is as follows.

Theorem 13 (Borg '06)

Let $\alpha_j \in \mathbb{R}$ *(*$j = 1, 2$*),* β_j *the fractional part of* α_j*,* γ *the fractional part of* $\alpha_1 + \alpha_2$*, and* $t > 0$*. Then, we have*

$$\lim_{d\to\infty} \widetilde{\mathrm{tr}}[\exp(-tH_{\alpha_1,\alpha_2,d}) - \exp(-tH_0)] = \frac{\beta_1(\beta_1 - 1)}{2} + \frac{\beta_2(\beta_2 - 1)}{2}, \tag{9}$$

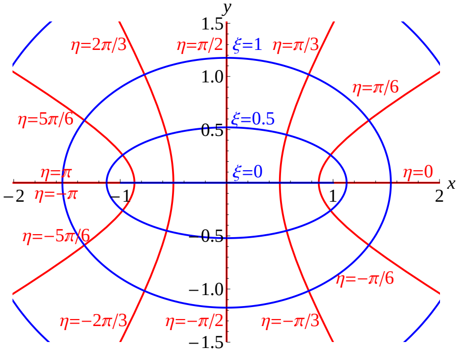
$$\lim_{d\to 0} \widetilde{\mathrm{tr}}[\exp(-tH_{\alpha_1,\alpha_2,d}) - \exp(-tH_0)] = \frac{\gamma(\gamma - 1)}{2}. \tag{10}$$

(9) is consistent with the leading term of Tamura's result.

Quantized two-point AB magnetic field

Let us consider the special case $\alpha_1 = -1/2$ and $\alpha_2 = 1/2$, which means the magnetic fluxes are equal to the magnetic flux quantum. In this case, the Hamiltonian $H_{-1/2,1/2,d}$ is explicitly solvable by the separation of variables in the elliptic coordinate

$$(x_1, x_2) = (d \cosh \xi \cos \eta, d \sinh \xi \sin \eta).$$



Quantized two-point AB magnetic field

The generalized e.f. for $H_0 = -\Delta$ with energy λ are

$$\mathrm{Ce}_n(\xi, q)\mathrm{ce}_n(\eta, q) \quad (n = 0, 1, 2, \ldots),$$
$$\mathrm{Se}_n(\xi, q)\mathrm{se}_n(\eta, q) \quad (n = 1, 2, \ldots),$$

where $q = d^2\lambda/4$. The functions ce_n and se_n are called the Mathieu functions, and Ce_n and Se_n the modified Mathieu functions. These functions are special solutions of

$$-\frac{d^2 y}{d\eta^2} + (-\mu + 2q\cos(2\eta))y = 0 \quad \text{(Mathieu equation)},$$
$$\frac{d^2 y}{d\xi^2} + (-\mu + 2q\cosh(2\xi))y = 0 \quad \text{(modified Mathieu equation)},$$

respectively. The function ce_n is even and periodic, se_n odd and periodic, and $\mathrm{Ce}_n(\xi, q) = \mathrm{ce}_n(i\xi, q)$, $\mathrm{Se}_n(\xi, q) = \mathrm{se}_n(i\xi, q)/i$.

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The generalized e.f. for $H_{-1/2,1/2,d}$ are also written as

$$e^{i(-\theta_1+\theta_2)/2}\mathrm{Fe}_n(\xi,q)\mathrm{ce}_n(\eta,q), \quad (n=0,1,2,\dots)$$
$$e^{i(-\theta_1+\theta_2)/2}\mathrm{Ge}_n(\xi,q)\mathrm{se}_n(\eta,q) \quad (n=1,2,\dots),$$

where $\theta_1=\arg(x_1+d+ix_2)$, $\theta_2=\arg(x_1-d+ix_2)$, and $\mathrm{Fe}(\xi,\mathrm{q})$ and $\mathrm{Ge}(\xi,\mathrm{q})$ are the solutions to the modified Mathieu equations having the opposite parities to those of Ce_n and Se_n, respectively. Thus, the method of the partial wave decomposition works, and the phase shifts and the SSF can be calculated explicitly.

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Theorem 14 (Main result)

We have

$$\xi(\lambda; H_{-1/2,1/2,d}, H_0) = \begin{cases} 0 & (\lambda \le 0), \\ -\dfrac{1}{\pi}\left(\displaystyle\sum_{n=0}^{\infty} \delta_{n,\sqrt{\lambda}} + \sum_{n=1}^{\infty} \epsilon_{n,\sqrt{\lambda}}\right) & (\lambda > 0), \end{cases} \tag{11}$$

$$\delta_{n,k} = \arctan \frac{\mathrm{Ce}_n(0,q)}{\mathrm{Fey}_n(0,q)}, \quad \epsilon_{n,k} = \arctan \frac{\mathrm{Se}'_n(0,q)}{\mathrm{Gey}'_n(0,q)},$$

$$q = a^2k^2/4,$$

where Ce_n, Se_n, Fey_n *and* Gey_n *are the modified Mathieu functions. The series converges locally uniformly in* $\lambda \in (0, \infty)$.

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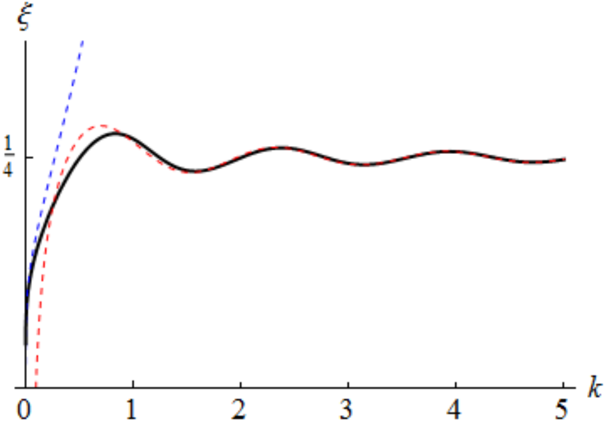
The phase shifts $\delta_{n,k}$ and $\epsilon_{n,k}$ decays very fast as $n \to \infty$, and can be used for the numerical calculation. The result for $d = 1$ is as follows.



Here, the black curve is the exact formula. The red one is the Tamura formula. The blue one is the Low energy formula in the next page. The horizontal axis is the $k = \sqrt{\lambda}$ axis.

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Borg's result suggests

$$\lim_{\lambda \to 0} \xi(\lambda; H_{-1/2,1/2,d}, H_0) = 0$$

(notice that the limit $\lambda \to 0$ is equivalent to $d \to 0$). We can also deduce more detailed asymptotics as $\lambda \to 0$ by the formula (11).

Corollary 15

$$\xi(\lambda; H_{-1/2,1/2,d}, H_0) \sim \frac{1}{2 \log \left(\frac{2}{d\sqrt{\lambda}} \right)} \tag{12}$$

as $\lambda \to 0$.

(12) is proved simply by taking the leading asymptotics of the first term of (11).