A Feynman path integral-like method of quantization on Riemannian manifolds and related problems

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March 8, 2015

§1 Introduction. Feynman path integrals (heuristics)

§1.1 Classical Mechanics

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(Hamiltonian formulation)

$$\begin{cases} H(x,p) = \frac{1}{2}|p|^2 + V(x) \in C^\infty(\mathbf{R}^n_x \times \mathbf{R}^n_p) : \text{Hamiltonian} \\ \frac{dx}{dt} = \frac{\partial H}{\partial p}, \ \ \frac{dp}{dt} = -\frac{\partial H}{\partial x} : \text{Hamitlonian flow} \end{cases}$$

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Legendre transform gives

(Lagrangian formulation)

For $X(\tau) \in \mathbf{R}^n$ and $\dot{X} = \frac{dX}{d\tau}$

$$\begin{cases} L(X, \dot{X}) = \frac{1}{2}\dot{X}^2 - V(x) : \text{Lagrangian} \\ \frac{d}{d\tau}\frac{\partial L}{\partial \dot{X}} - \frac{\partial L}{\partial X} = 0 : \text{Euler-Lagrange eq} \end{cases}$$

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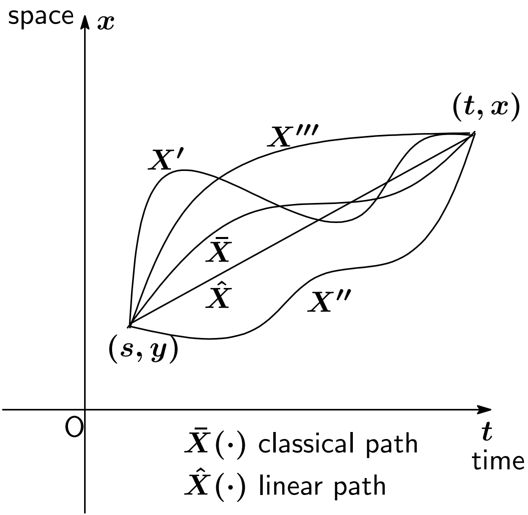
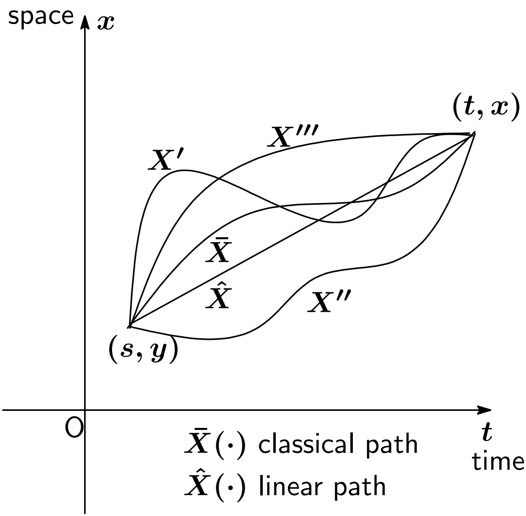


Action integral

$$S(X) = \int_s^t \frac{1}{2} L(X, \dot{X}) d\tau$$



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$$\int_\Omega e^{\frac{i}{h} S(0,t,x,y)} f(y) \mathcal{D}[X] = e^{\frac{-it}{h} \hat{H}} f(x)$$

Ω is the path space connecting $(0, y)$ and (t, x).
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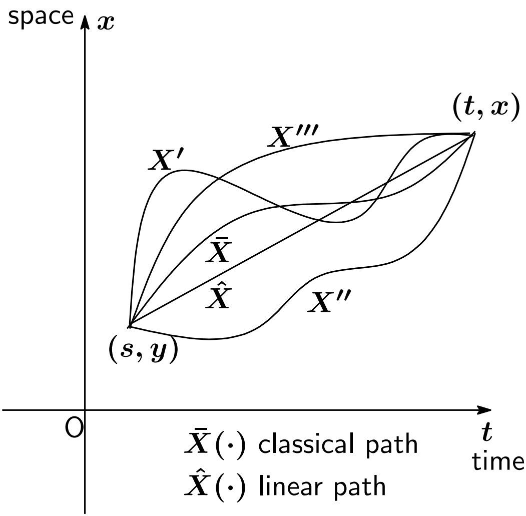

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Remark. We can not construct Feynman path measure (Cameron)

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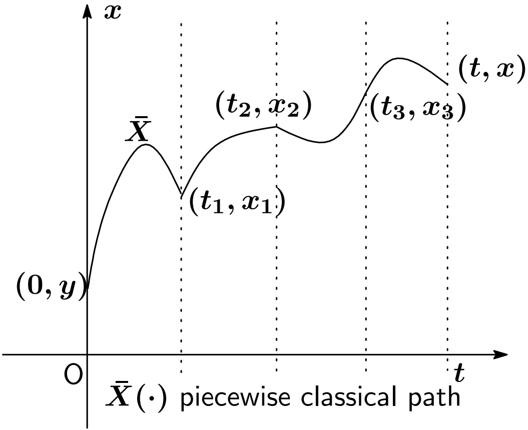
x
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$(0, y)$
O
t
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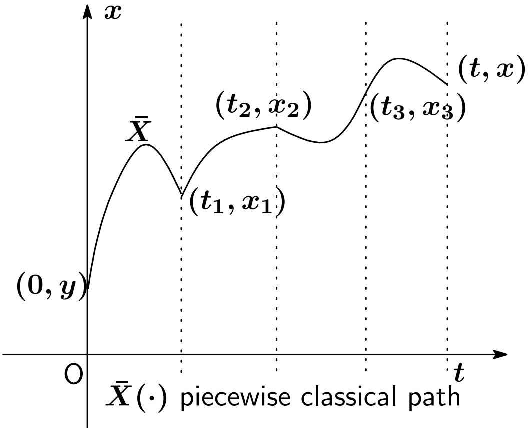

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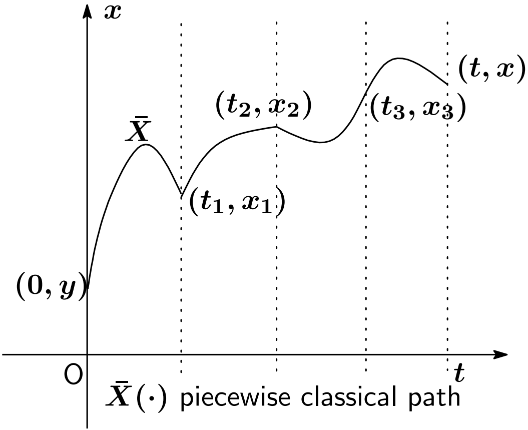

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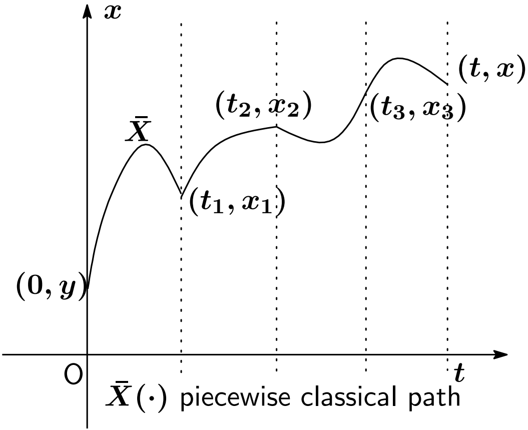

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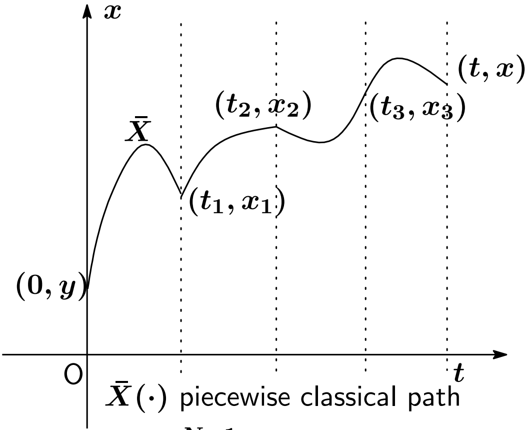
(small time evolution op.)

§.2 Time slicing approximations (Summary. Euclidean case)



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Time slicing approximations are defined by

$$\int_{\mathbf{R}^n} \cdots \int_{\mathbf{R}^n} \prod_{j=0}^{N-1} a(t_j, t_{j+1}, x_j, x_{j+1}) e^{\frac{i}{\hbar} S(t_j, t_{j+1}, x_j, x_{j+1})} f(y) \prod_{j=0}^{N-1} dx_j$$

$$= [\prod_{j=0}^{N-1} U(t_{j+1} - t_j)] f(x) \to \int_\Omega e^{\frac{i}{\hbar} S(0,t,x,y)} f(y) \mathcal{D}[X] \ (N \to \infty).$$

Known results (Fujiwara Theory) (Euclidean case)

Assumption $\boldsymbol{V(x) \in C^\infty(\mathbf{R}^n)}$, $\boldsymbol{|\partial^\alpha V(x)| < C_\alpha}$ for $\boldsymbol{|\alpha| \geqq 2}$.

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Theorem (D. Fujiwara)

For $t \neq 0$,

$$\lim_{|\Delta|\to 0} \prod_i [U(t_i - t_{i-1})] = \exp \frac{-it}{h}[-\frac{h^2}{2}\triangle + V(x)] \text{ (Operator norm)}$$

Other different alternative definitions of Feynman path integrals

Other alternative methods for path integrals.

1. Trotter Kato forumulas.
2. Analytic continuation of Wiener measure
   by using complex Planch constant h, m or t
3. An improper integral on Hilbert spaces.
   (K.Ito, Albeverio)
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Here, we employ the time slicing products.
to derive the curvature from action integrals.

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Definition (Shortest path approximations on manifolds)

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(Remark. For the simplicity, let $h = 1$.)

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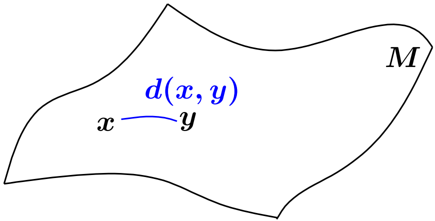

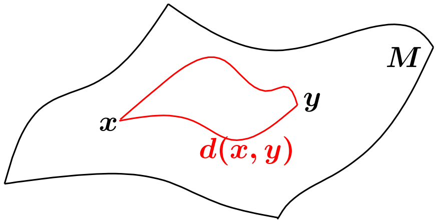

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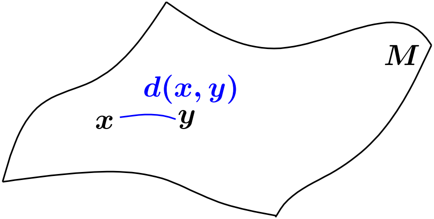

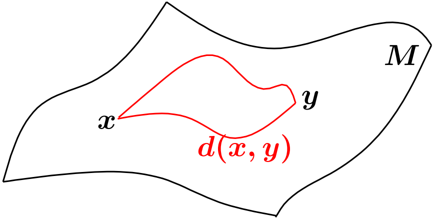

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Problem. $\lim_{N\to\infty} [U(t/N)]^N f(x) = \exp(-it\hat{H}) f(x)$?

What is the $\hat{H}$?

Assumtions for scattering case

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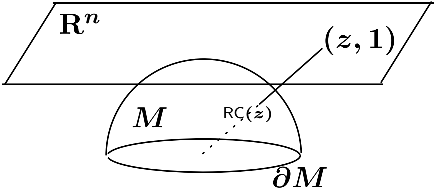
Example(The radial compactification map)



$$\text{RC} : (z, 1) \in \mathbf{R}^n \to \left(\frac{z}{\sqrt{1+|z|^2}}, \frac{1}{\sqrt{1+|z|^2}}\right) \in S^n_+$$

§4 Asymptotically conic non-trapping case

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$\rho(E) : L^2(M) \to L^2(M)$: spectral projector

§4 Asymptotically conic non-trapping case (Results)

(in preparation)

Theorem (strong limits)

Assume (A1)∼ (A4). For $t \neq 0$

$$s\text{-}\lim_{N\to\infty} [U(t/N)]^N \rho(N) f(x) = \exp\left[-it\left(-\frac{1}{2}(\triangle - \frac{R}{6})\right)\right] f(x) \text{ in } L^2$$

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Assume (A1)∼ (A4). If $R(x) \leqq 0$, then injrad$(M) = \infty$.
Moreover we can take $\chi(d(x,y)) = 1$ and $U^*(t)U(t) \in \Psi^0_{sc}$

Theorem (strong limits without cut off and spectral projectors)

Assume (A1) ∼ (A4), If $R(x) \leqq 0$ *and* $\chi(d(x,y)) = 1$. *Then*

$$\underset{N\to\infty}{s\text{-}\lim}[U(t/N)]^N f(x) = \exp\left[-it\left(-\frac{1}{2}(\triangle - \frac{R}{6})\right)\right] f(x) \text{ in } L^2$$

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§5 In the case of a compact manifold (Sphere)

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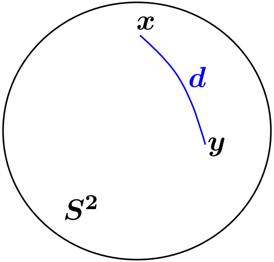
2. $d = d(x, y) = \arccos(\vec{x} \cdot \vec{y})$ (geodesic distance)



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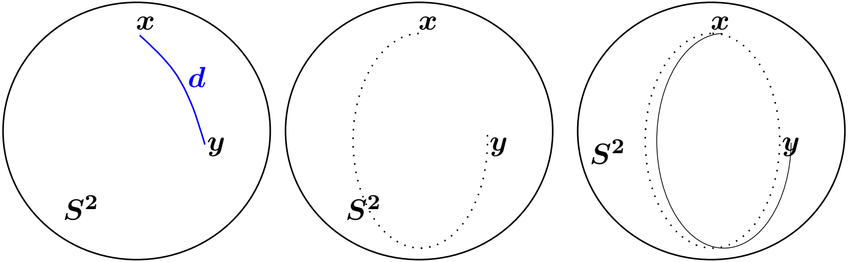

(We don't consider $d \geqq \pi$.)

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4. Van Vleck determinant on manifolds

   $$D(t,x,y) = G^{-1/2}(x)G^{-1/2}(y)\det(\partial^2 S(t,x,y)/\partial x \partial y)$$

   $\chi(d(x,y))$: cut off

   (bump ft. with compact support contained in $d(x,y) < \pi$.)

   $$a(t,x,y) = \chi(d(x,y))D(t,x,y)^{1/2}$$

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Definition (Shortest path approximations on S^2)

$$U(t)f(x) \equiv (2\pi i)^{-1} \int_{S^2} a(t,x,y) e^{iS(t,\ x,\ y)} f(y)\, dy$$

(Remark. For the simplicity, let $h = 1$.)

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§5 Path integrals on the sphere.

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$\rho(E) : L^2(S^2) \to \mathrm{L.h.}\{u_j \mid E_j \leqq E\}$: spectral projector

(Spectral projectors)

§5 Path integrals on the sphere. (Results)

[Adv. Appl. Math. Anal. (2014)]

Theorem (operator norm)

For $t \neq 0$ and small $\varepsilon > 0$,

$$\lim_{N\to\infty} [U(t/N)]^N \rho(N^{1/3-\varepsilon}) = \exp\left[-it\left(-\frac{1}{2}(\triangle - \frac{R}{6})\right)\right] \text{ in } L^2$$

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Corollary

Let u_j *be an eigenfunction of Laplacian. For* $t \neq 0$,

$$s\text{-}\lim_{N\to\infty} [U(t/N)]^N u_j = \exp\left[-it\left(-\frac{1}{2}(\triangle - \frac{R}{6})\right)\right] u_j \text{ in } L^2$$

§5 Path integrals on the sphere (Results)

Remark 1.

For $t \neq 0$,

$$\lim_{N\to\infty} \|[U(t/N)]^N - \exp\Big[-it\Big(-\frac{1}{2}(\triangle - \frac{R}{6})\Big)\Big]\|_{L^2} \neq 0.$$

(Time slicing products does not converge in operator norm without spectral projector,) Remark 1.

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Remark 2.

$f(x) \in G_{1/6}(S^2)$ (Gevrey class). For $t \neq 0$,

$$s\text{-}\lim_{N\to\infty}[U(t/N)]^N f(x) = \exp\Big[-it\Big(-\frac{1}{2}(\triangle - \frac{R}{6})\Big)\Big]f(x) \text{ in } L^2.$$

(The convergence for low energy functions)

High energy functions cannot be captured by shortest path approximations.

§5 Path integrals on the sphere (Related results)

Let $f(x) \in C^{\infty}(S^2)$ and $t = \frac{8\pi m}{k} \in \mathbf{Q}$ (k and m are relatively prime.)

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$$\begin{aligned}
& s\text{-}\lim_{N\to\infty} \{U(8\pi m/kN)\}^N \rho(N) f(x) \times e^{iRt/12} \\
&= \int_{S^2} \sum_{l=0}^{\infty} \left(\frac{2l+1}{4\pi}\right) e^{-4\pi i\{3ml(l+1)+1\}/3k} C_l^{1/2}(\cos d(x,y))\ f(y)dy \\
&= \left\{ e^{2\pi i m/3k} \sum_{j=0}^{2k-1} \Gamma(m,k,j) \cos \frac{2\pi j}{k} A \right\} f(x)
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$$= \left\{ e^{2\pi i m/3k} \sum_{j=0}^{2k-1} \Gamma(m,k,j) \cos \frac{2\pi j}{k} A \right\} f(x)$$

where $\Gamma(m,k,j) = \frac{1}{2\pi} \sum_{l=0}^{2k-1} e^{\pi i (l^2 m + lj)/k}$ is a Gaussian sum,

$$A = \sqrt{-\triangle + \frac{1}{4}},$$

C_l : Gegenbauer polynomials are defined by

$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{l=0}^{\infty} C_l^{1/2}(x)\ t^l.$$

For the case of Sphere S^2 We find

$$D(t,x,y) = \frac{d(x,y)}{t^2 \sin d(x,y)} \quad \text{for } 0 \leqq d < \pi.$$

For $\chi(d)K(t,x,y) = \chi(d)D(t,x,y)^{1/2}e^{iS}$, we obtain

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2}\triangle_x - \frac{R}{12}\right)(\chi(d)K(t,x,y))$$
$$= \left[\chi\left(\frac{d^2 - \sin^2 d}{8d^2 \sin^2 d} + \frac{1}{8} - \frac{R}{12}\right) + \frac{1}{2}(\triangle_x \chi)\right]K(t,x,y)$$
$$+ \frac{\partial \chi}{\partial d}\left(\frac{\sin d - d\cos d}{2d \sin d}\right)K(t,x,y)$$
$$+ \frac{\partial \chi}{\partial d}\left(\frac{id}{t}\right)K(t,x,y).$$

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By using stationary phase methods and integral equations,

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$$\|U(t)f\|_{L^2} \leqq (1 + C_1 t)\|f\|_{L^2} + C_2 t^2 \|(-\triangle_{S^2} + 1)f\|_{L^2} \quad \cdots (3)$$

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$$\|U(t)f\|_{L^2} \leqq (1 + C_1 t)\|f\|_{L^2} + C_2 t^2 \|f\|_{\mathcal{H}_l} \text{ for } l \geqq 2 \quad \cdots (3)'$$

$$\|U(t)f - \exp(-it\hat{H})f\|_{L^2} \leqq \frac{C_3 t^2}{2} \|f\|_{\mathcal{H}_m} \text{ for } m > [n/2] + 1 \quad \cdots (4)'$$

Here $\mathcal{H}_k = \bigcap_{s=0}^{k} x^{k-s} H^s_{sc}(M)$: weighted scattering Sobolev sp.

§5.1 Outline of proof (We mainly discuss the case of sphere)

The binomical coefficients bounds $\binom{N}{k}\frac{1}{N^k} < \frac{1}{k!}$ yields the following estimates

$$
\begin{aligned}
&\|\{e^{-it\hat{H}} - U_\chi(t/N)^n\}f(x)\|_{L^2} \\
&\quad = \|\left[e^{-it\hat{H}} - \{e^{-it\hat{H}/N}(1+\tilde{E}(t/N))\}^N\right] f(x)\|_{L^2} \\
&\quad \leqq \sum_{k=1}^{N} \binom{N}{k} \|\{e^{-i(N-k)t\hat{H}/N}\tilde{E}(t/N)^k\}f(x)\|_{L^2} \\
&\quad \leqq \sum_{k=1}^{N} \binom{N}{k} \left(\frac{\tilde{C}}{2}\right)^k \left(\frac{t}{N}\right)^{2k} \|(-\triangle+1)^{3k}f(x)\|_{L^2} \\
&\quad \leqq \sum_{k=1}^{N} \frac{1}{k!}\left(\frac{\tilde{C}t^2}{2N}\right)^k \|(-\triangle+1)^{3k}f(x)\|_{L^2}.
\end{aligned}
$$

§6 Summary

(Case 1) $M = \mathbf{R}^n$, $H(x,p) = \frac{1}{2}|p|^2 + V(x) \in C^\infty(T^*M)$

Classical mechanics	Canonical quantization	Feynman quantization
$V(x) = O(\lvert x\rvert^2)$+error. (Fujiwara theory)	$\hat{H} = -\frac{h^2}{2}\triangle + V(x)$	$\lim_{N\to\infty}[U(\frac{t}{N})]^N$ $= \exp\left(\frac{-it}{h}\hat{H}\right)$

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(Integral kernel) $e^{\frac{-it}{h}\hat{H}}f(x) = \int_{\mathbf{R}^n} K(t,x,y)f(y)\,dy$.

Classical mechanics	Orbits of CM	integral kernel
$V(x) = O(\lvert x\rvert^2)$+error.	time locally global diffeo on config. space	$K(t,x,y) \in C^\infty((0,t) \times \mathbf{R}^{2n})$
$V(x) = C\lvert x\rvert^n$ ($C > 0$, $n \geqq 4$)	infinite many small periodic curves	If $n = 1$, $K(t,x,y)$ is nowhere C^1

§6 Summary

(Case 2) (M, g) : Riem. mfd., $H(x,p) = \frac{1}{2}|p|^2 \in C^\infty(T^*M)$

Manifolds	Feynman quantization
Asymptotically conic + cutoff	$\lim_{N\to\infty} [U(\frac{t}{N})]^N \rho(N) = \exp\left[-it\left(-\frac{1}{2}(\triangle - \frac{R}{6})\right)\right]$
Asymptotically conic $R(x) \leqq 0$ without cutoff	$\lim_{N\to\infty} [U(\frac{t}{N})]^N = \exp\left[-it\left(-\frac{1}{2}(\triangle - \frac{R}{6})\right)\right]$

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Manifolds	Feynman quantization
S^2 (sphere) + cutoff	$\lim_{N\to\infty} [U(\frac{t}{N})]^N \rho(N) = \exp\left[-it\left(-\frac{1}{2}(\triangle - \frac{R}{6})\right)\right]$
S^2 without cutoff	$\lim_{N\to\infty} [U(\frac{t}{N})]^N$ How to define rigorously ?

Prospects

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(3) Application? Spectral geometry $\triangle + \beta R$
- $\beta = 0$ (well-known)
- $\beta = 1/6$ (geometry of spectrum clustering)

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§3 Appendix

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- Quantization (manifolds)
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§7 Appendix

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$$\sum_\alpha \tilde{L}_\alpha^2 \pi^* Y_{l,m} = \frac{h^2}{2}(l(l+1) + \frac{1}{8})\pi^* Y_{l,m}$$

Thank you for your attention.