Commutator criteria for strong mixing

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Strong mixing

A unitary operator U in a Hilbert space $\mathcal{H}$ is a surjective isometry:

$$U^*U = UU^* = 1.$$

Example (Discrete group of unitary operators)

If U is a unitary operator in a Hilbert space $\mathcal{H}$,

$$U_n := U^n, \quad n \in \mathbb{Z},$$

defines a discrete 1-parameter group of unitary operators.

Example (Continuous group of unitary operators)

If H is a self-adjoint operator in a Hilbert space $\mathcal{H}$, then

$$U_t := \mathrm{e}^{-itH}, \quad t \in \mathbb{R},$$

defines a strongly continuous 1-parameter group of unitary operators.

Example (Koopman operator)

If $T : X \to X$ is an automorphism of a probability space (X, μ), then the Koopman operator

$$U_T : \mathrm{L}^2(X, \mu) \to \mathrm{L}^2(X, \mu), \quad \varphi \mapsto \varphi \circ T,$$

is a unitary operator.

Ergodicity, weak mixing and strong mixing of an automorphism $T : X \to X$ are expressible in terms of the Koopman operator U_T:

- T is ergodic iff 1 is a simple eigenvalue of U_T.
- T is weakly mixing iff U_T has purely continuous spectrum in $\{\mathbb{C} \cdot \mathbf{1}\}^\perp$.
- T is strongly mixing iff

$$\lim_{N\to\infty} \langle \varphi, (U_T)^N \varphi \rangle = 0 \quad \text{for all } \varphi \in \{\mathbb{C} \cdot \mathbf{1}\}^\perp.$$

a.c. spectrum in $\{\mathbb{C} \cdot \mathbf{1}\}^\perp$ $\Rightarrow$ strong mixing $\Rightarrow$ weak mixing $\Rightarrow$ ergodicity

Commutators

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, bounded linear operators on $\mathcal{H}$
- A, self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$

Definition

An operator $S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

$S \in C^1(A)$ if and only if

$$\big|\langle A\varphi, S\varphi\rangle - \langle \varphi, SA\varphi\rangle\big| \leq \mathrm{Const.}\, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The operator corresponding to the continuous extension of the quadratic form is denoted by $[S, A]$, and one has

$$[iS, A] = \text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H}).$$

Definition

A self-adjoint operator H in $\mathcal{H}$ is of class $C^k(A)$ if $(H-z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$.

If H is of class $C^1(A)$, then

$$\left[A, (H-z)^{-1}\right] = (H-z)^{-1} [H, A] (H-z)^{-1},$$

with $[H, A] \in \mathscr{B}\big(\mathcal{D}(H), \mathcal{D}(H)^*\big)$ the operator corresponding to the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi\rangle - \langle A\varphi, H\varphi\rangle \in \mathbb{C}.$$

Discrete groups

Theorem (Strong mixing for discrete groups)

Let U and A be a unitary and a self-adjoint operator in $\mathcal{H}$, with $U \in C^1(A)$. Assume that

$$D := \underset{N\to\infty}{\text{s-lim}} \frac{1}{N} \sum_{n=0}^{N-1} U^n\big([A, U]U^{-1}\big)U^{-n}$$

exists, and suppose that $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$ for each $\eta \in C_c^\infty(\mathbb{R} \setminus \{0\})$. Then,

(a) $\lim_{N\to\infty} \langle \varphi, U^N \psi \rangle = 0$ *for each* $\varphi \in \ker(D)^\perp$ *and* $\psi \in \mathcal{H}$,

(b) $U|_{\ker(D)^\perp}$ *has purely continuous spectrum.*

- D is bounded and self-adjoint because it is the strong limit of bounded self-adjoint operators.
- $\eta(D)$ with $\eta \in C_c^\infty(\mathbb{R} \setminus \{0\})$ is well-defined by functional calculus.
- $DU^n = U^n D$ for each $n \in \mathbb{Z}$. So, $\ker(D)^\perp$ is a reducing subspace for U, and $U|_{\ker(D)^\perp}$ is a unitary operator.
- Point (b) is a simple consequence of point (a).
- Point (b) could be compared with the Virial theorem for unitary operators ("$C^1(A) + \cdots$" $\Rightarrow$ continuous spectrum).

Sketch of the proof of (a). Since $U \in C^1(A)$, one has $U^N \in C^1(A)$ with

$$\begin{aligned}
\left[A, U^N\right] = \sum_{n=0}^{N-1} U^{N-1-n}[A, U]U^n &= \left(\sum_{n=0}^{N-1} U^{N-1-n}\big([A, U]U^{-1}\big)U^{n+1-N}\right) U^N \\
&= \left(\sum_{n=0}^{N-1} U^n\big([A, U]U^{-1}\big)U^{-n}\right) U^N \\
&= N D_N\, U^N
\end{aligned}$$

and

$$D_N := \frac{1}{N}\sum_{n=0}^{N-1} U^n\big([A, U]U^{-1}\big)U^{-n}.$$

So, we have for $\varphi \equiv \eta(D)\varphi$ with $\varphi \in \mathcal{D}(A)$ and $\psi \in \mathcal{D}(A)$

$$
\begin{aligned}
&\left|\left\langle \varphi, U^N \psi \right\rangle\right| \\
&= \left|\left\langle DD^{-1}\eta(D)\varphi, U^N \psi \right\rangle\right| \\
&\leq \left|\left\langle (D_N - D)D^{-1}\eta(D)\varphi, U^N \psi \right\rangle\right| + \left|\left\langle D^{-1}\eta(D)\varphi, D_N U^N \psi \right\rangle\right| \\
&\leq \left\|(D_N - D)D^{-1}\eta(D)\varphi\right\| \cdot \|\psi\| + \frac{1}{N}\left|\left\langle D^{-1}\eta(D)\varphi, \left[A, U^N\right]\psi \right\rangle\right| \\
&= \left\|(D_N - D)D^{-1}\eta(D)\varphi\right\| \cdot \|\psi\| \\
&\quad + \frac{1}{N}\left|\left\langle AD^{-1}\eta(D)\varphi, U^N \psi \right\rangle - \left\langle D^{-1}\eta(D)\varphi, U^N A\psi \right\rangle\right| \\
&\leq \left\|(D_N - D)D^{-1}\eta(D)\varphi\right\| \cdot \|\psi\| \\
&\quad + \frac{1}{N}\Big(\left\|AD^{-1}\eta(D)\varphi\right\| \cdot \|\psi\| + \left\|D^{-1}\eta(D)\varphi\right\| \cdot \|A\psi\|\Big),
\end{aligned}
$$

which goes to 0 as $N \to \infty$.

- If $\sum_{N\geq 1}\|(D_N-D)\varphi\|^2<\infty$ for suitable $\varphi\in\mathcal{H}$, then $\sum_{N\geq 1}|\langle\varphi,U^N\varphi\rangle|^2<\infty$ for all $\varphi\in\ker(D)^\perp$, and $U|_{\ker(D)^\perp}$ has purely a.c. spectrum.
- D can be seen as a topological degree of the map $N\mapsto U^N$ in $\mathrm{U}(\mathcal{H})$. Indeed, one has

$$D=\operatorname*{s-lim}_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}U^n\big([A,U]U^{-1}\big)U^{-n}=\operatorname*{s-lim}_{N\to\infty}\frac{1}{N}\big[A,U^N\big]U^{-N}.$$

  So, if one considers $[A,\,\cdot\,]$ as a derivation on $\{U^N\}_{N\in\mathbb{Z}}\subset\mathrm{U}(\mathcal{H})$, then D can be seen as a renormalised winding number of the map $N\mapsto U^N$ in $\mathrm{U}(\mathcal{H})$ (the logaritmic derivative $\frac{\mathrm{d}z}{z}$ in the usual definition of winding number is replaced by the "logaritmic derivative" $\big[A,U^N\big]U^{-N}$ associated to $[A,\,\cdot\,]$).

Example (Skew products of compact Lie groups)

$$\cdots$$

Continuous groups

Theorem (Strong mixing for continuous groups)

Let H and A be self-adjoint operators in $\mathcal{H}$, with $(H-i)^{-1} \in C^1(A)$. Assume that

$$D := \underset{t\to\infty}{\text{s-lim}} \frac{1}{t} \int_0^t \mathrm{d}s \, \mathrm{e}^{isH} (H+i)^{-1} [iH, A] (H-i)^{-1} \, \mathrm{e}^{-isH}$$

exists, and suppose that $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$ for each $\eta \in C_{\mathrm{c}}^\infty(\mathbb{R} \setminus \{0\})$. Then,

(a) $\lim_{t\to\infty} \langle \varphi, \mathrm{e}^{-itH} \psi \rangle = 0$ *for each* $\varphi \in \mathcal{H}$ *and* $\psi \in \ker(D)^\perp$,

(b) $H|_{\ker(D)^\perp}$ *has purely continuous spectrum.*

Example (Canonical commutation relation)

Assume that $(H-i)^{-1} \in C^1(A)$ *with* $[iH, A] = 1$*. Then, for all* $t > 0$

$$D_t = \frac{1}{t}\int_0^t \mathrm{d}s \; \mathrm{e}^{isH}(H+i)^{-1}\,[iH,A]\,(H-i)^{-1}\,\mathrm{e}^{-isH} = (H^2+1)^{-1} = D$$

and $\ker(D) = \{0\}$*. So, the theorem implies that* H *has purely a.c. spectrum. In fact, we have in this case the Weyl commutation relation*

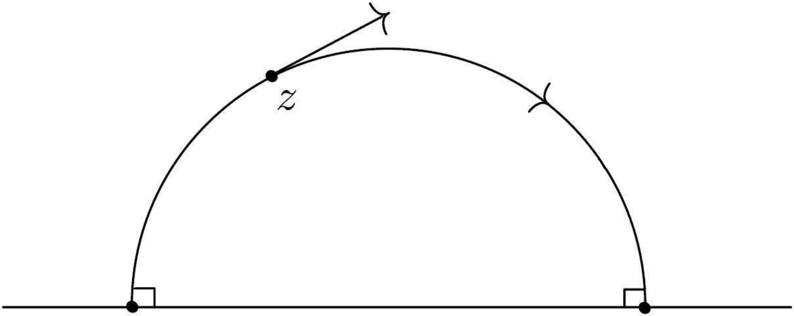
$$\mathrm{e}^{-itA}\,\mathrm{e}^{isH}\,\mathrm{e}^{itA} = \mathrm{e}^{ist}\,\mathrm{e}^{isH}, \quad s,t \in \mathbb{R}.$$

Thus, Stone-von Neumann theorem implies that H *has Lebesgue spectrum with uniform multiplicity.*

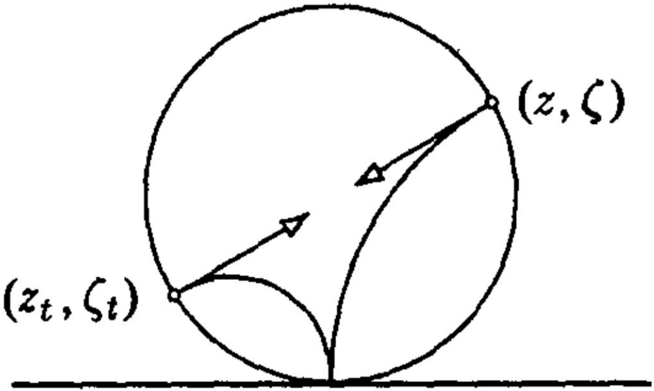
Example (Time changes of horocycle flows).

- Σ, compact Riemannian surface of constant negative curvature
- $M := T^1\Sigma$, unit tangent bundle of Σ
  (M is a compact 3-manifold with probability measure μ,
  $M \simeq \Gamma \setminus \mathsf{PSL}(2;\mathbb{R})$ for some cocompact lattice Γ in $\mathsf{PSL}(2;\mathbb{R})$)
- $F_1 \equiv \{F_{1,t}\}_{t\in\mathbb{R}}$, horocycle flow on M
- $F_2 \equiv \{F_{2,t}\}_{t\in\mathbb{R}}$, geodesic flow on M

The flows F_1, F_2 are one-parameter groups of diffeomorphisms preserving the measure μ.



Geodesic in the Poincaré half plane



(Positive) horocycle flow in the Poincaré half plane

Each flow has an essentially self-adjoint generator

$$H_j\varphi := -iX_j\varphi, \quad \varphi \in C^\infty(M) \subset \mathsf{L}^2(M,\mu),$$

with X_j the vector field associated to F_j. H_1 is of class $C^1(H_2)$ with

$$[iH_1, H_2] = H_1.$$

A C^1-time change of X_1 is a vector field fX_1 with $f \in C^1\big(M;(0,\infty)\big)$. fX_1 has a complete flow $\widetilde{F}_1 \equiv \{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ and a generator $H := fH_1$ essentially self-adjoint on $C^1(M) \subset \mathcal{H} := \mathsf{L}^2(M, f^{-1}\mu)$.

The operator $A := f^{1/2} H_2 f^{-1/2}$ is self-adjoint in $\mathcal{H}$, and $(H - i)^{-1} \in C^1(A)$ with

$$(H+i)^{-1}[iH, A](H-i)^{-1} = (H+i)^{-1}\big(H\xi + \xi H\big)(H-i)^{-1}$$

and

$$\xi := \frac{1}{2} - \frac{1}{2} f^{-1} X_2(f).$$

So,

$$\begin{aligned} D_t &:= \frac{1}{t}\int_0^t \mathrm{d}s \; \mathrm{e}^{isH}(H+i)^{-1}[iH, A](H-i)^{-1}\,\mathrm{e}^{-isH} \\ &= (H+i)^{-1}\big(H\xi_t + \xi_t H\big)(H-i)^{-1} \end{aligned}$$

with

$$\xi_t := \frac{1}{t}\int_0^t \mathrm{d}s \; \mathrm{e}^{isH}\,\xi\,\mathrm{e}^{-isH} = \frac{1}{t}\int_0^t \mathrm{d}s \,\big(\xi \circ \widetilde{F}_{1,-s}\big).$$

Since F_1 is uniquely ergodic with respect to μ, $\widetilde{F}_1$ is uniquely ergodic with respect to $\widetilde{\mu} := \frac{f^{-1}\mu}{\int_M f^{-1}\mathrm{d}\mu}$. Thus,

$$\lim_{t\to\infty} \xi_t = \frac{1}{2} - \frac{1}{2}\int_M \mathrm{d}\widetilde{\mu}\, f^{-1} X_2(f) = \frac{1}{2} + \frac{1}{2\int_M f^{-1}\mathrm{d}\mu}\int_M \mathrm{d}\mu\, X_2\big(f^{-1}\big) = \frac{1}{2}$$

uniformly on M, and

$$D := \underset{t\to\infty}{\text{s-lim}}\, D_t = (H+i)^{-1}\left(H\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot H\right)(H-i)^{-1} = H\left(H^2+1\right)^{-1}.$$

So, $\ker(D) = \ker(H)$, and the theorem implies that

$$\lim_{t\to\infty}\left\langle \varphi, \mathrm{e}^{-itH}\,\psi\right\rangle = 0 \quad \text{for all } \varphi\in\mathcal{H} \text{ and } \psi\in\ker(H)^\perp.$$

Therefore, C^1-time changes of horocycle flows are strongly mixing.

Two questions

Strong mixing for Schrödinger operators:
Given $H := -\triangle + V$ in $\mathsf{L}^2(\mathbb{R}^d)$, can we find conditions on V such that the theorem applies with A the generator of dilations? Can we prove the continuity of $\sigma(H)$ in some interval in our set-up? Can we add a compact error term in the theorems?

Strong mixing for C^*-dynamical systems:
Given $\{\mathcal{A}, G, \alpha\}$ with $\mathcal{A}$ a C^*-algebra, G a locally compact group and α a strongly continuous representation of G in $\mathrm{Aut}(\mathcal{A})$, can we find conditions on α in terms of an auxiliary map on $\mathcal{A}$ (maybe a derivation of $\mathcal{A}$ replacing the map $[A, \cdot\,]$ appearing in our set-up) guaranteeing that α is strong mixing?

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