On stability of solitary waves of the nonlinear Dirac equation in the non-relativistic limit

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A nonlinear Dirac equation

We consider the spectral stability of stationary solutions $\phi_\omega(x)e^{-i\omega t}$ to a nonlinear Dirac equation of the form

$$\mathrm{i}\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t)\in\mathbb{C}^N, \quad x\in\mathbb{R}^n, \quad \text{(NLD)}$$

where N is even, $f(0)=0$, and D_m is the free Dirac operator:

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The $N\times N$ Dirac matrices are hermitian and satisfy $1\le \jmath, k\le n$

$$(\alpha^\jmath)^2 = \beta^2 = I_N, \qquad \alpha^\jmath\alpha^k + \alpha^k\alpha^\jmath = 2\delta_{\jmath k}I_N, \qquad \alpha^\jmath\beta + \beta\alpha^\jmath = 0.$$

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where N is even, $f(0)=0$, and D_m is the free Dirac operator:

$$D_m = -\mathrm{i}\alpha\cdot\nabla + \beta m = \sum_{\jmath=1}^{n} -\mathrm{i}\alpha^\jmath\partial_{x_\jmath} + \beta m \qquad m>0.$$

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Its spectrum is purely absolutely continuous and given by

$$\mathbb{R}\setminus(-m,m).$$

Hypothesis

$f \in C(\mathbb{R})$ and there exists $k > 0$ such that

$$|f(s) - |s|^k| = o_{s\to 0}(|s|^k).$$

If $n \geq 3$ then $k < 2/(n-2)$.

Consider the matrix β in the form:

$$\beta = \begin{bmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{bmatrix}$$

the matrices $(\alpha^j)_{1\leq j\leq n}$ are of the form

$$\alpha^j = \begin{bmatrix} 0 & \sigma_j^* \\ \sigma_j & 0 \end{bmatrix}, \qquad 1 \leq j \leq n,$$

where the $(\sigma_j)_{1\leq j\leq n}$ are hermitian and satisfies

$$\sigma_j\sigma_k^* + \sigma_k\sigma_j^* = 2\delta_{jk}, \qquad 1 \leq j, k \leq n.$$

We consider the existence of solitary waves of Soler or Wakano type $\phi_\omega(x)e^{-i\omega t}$ with

$$\phi_\omega = \begin{bmatrix} v(r)n_1 \\ u(r)\,(e_r\cdot\sigma)\,n_1 \end{bmatrix}.$$

The profiles v and u are real and

$$n_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{N/2}, \qquad e_r = \frac{x}{r} \in \mathbb{R}^n, \qquad \sigma = (\sigma_J)_{1\le J\le n}.$$

From (NLD), we deduce

$$\begin{cases} \partial_r u + \frac{n-1}{r}u + (m-\omega)v = f(v^2-u^2)v, \\ \partial_r v + (m+\omega)u = f(v^2-u^2)u, \end{cases} \qquad r>0.$$

Theorem

$\hat{V}(t) = u_k(|t|)$ *is even, positive, exponentially decreasing and* C^2 *with*

$$-\frac{1}{2m}\hat{V} = -\frac{1}{2m}\left(\partial_t^2 + \frac{n-1}{t}\partial_t\right)\hat{V} - \hat{V}^{2k+1},$$

and $\hat{U}(t) = -\hat{V}'(t)/(2m)$.

Theorem

There exist ω_0 *and, for* $\omega \in (\omega_0, m)$*, a solution of the form:*

$$v(r,\omega) = \epsilon^{\frac{1}{k}}\left[\hat{V}(\epsilon r) + \tilde{V}(\epsilon r, \epsilon)\right], u(r,\omega) = \epsilon^{1+\frac{1}{k}}\left[\hat{U}(\epsilon r) + \tilde{U}(\epsilon r, \epsilon)\right],$$

where ϵ *and* ω *verify* $\epsilon = \sqrt{m^2 - \omega^2}$*,* $\hat{V}(t) = u_k(|t|)$ *is even, positive, exponentially decreasing and* C^2 *with*

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and $\hat{U}(t) = -\hat{V}'(t)/(2m)$*.*
There exists $\tau > 0$ *such that* $\tilde{V}$ *and* $\tilde{U}$ *verify*

$$\|e^{\tau\langle r\rangle}\tilde{V}\|_{H^1} + \|e^{\tau\langle r\rangle}\tilde{U}\|_{H^1} = O_{\epsilon\to 0}(1).$$

Some remarks

- We have, for $U = \hat{U} + \tilde{U}$ and $V = \hat{V} + \tilde{V}$,

$$\phi_\omega^*(x)\beta\phi_\omega(x) = |V(|x|)|^2 - |U(|x|)|^2 \geq \frac{|V(|x|)|^2 + |U(|x|)|^2}{2} := \frac{Q(\omega)}{2}.$$

Hypothesis

$f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$ and that there are $k > 0$ and $K > k$ such that

$$|f(\tau) - |\tau|^k| = O(|\tau|^K), \qquad |\tau| \leq 1;$$
$$|\tau f'(\tau) - k|\tau|^k| = O(|\tau|^K), \qquad |\tau| \leq 1.$$

-
  - If $k < 2/n$, or $k = 2/n$ and $K > 4/n$. Then there is $\omega_1 < m$ such that

$$\partial_\omega Q(\omega) < 0$$

    for all $\omega \in (\omega_1, m)$.
  - If instead $k > 2/n$, then there is $\omega_1 < m$ such that

$$\partial_\omega Q(\omega) > 0$$

    for all $\omega \in (\omega_1, m)$.

Linearization and stability

We consider the solution to the nonlinear Dirac equation in the form

$$\psi(x,t) = (\phi_\omega(x) + \rho(x,t))e^{-i\omega t},$$

where ϕ_ω satisfies the stationary equation

$$\omega\phi_\omega = D_m\phi_\omega - f(\phi_\omega^*\beta\phi_\omega)\beta\phi_\omega,$$

so that $\rho(x,t) \in \mathbb{C}^N$ is a "small" perturbation of $\phi_\omega(x)e^{-i\omega t}$. The linearization at a solitary wave (the linearized equation on ρ) is given by

$$\partial_t\rho = \mathrm{J}\mathrm{L}(\omega)\rho,$$

where $\mathrm{J} = 1/i$,

$$\mathrm{L}(\omega) = D_m - \omega - f(\phi_\omega^*\beta\phi_\omega)\beta - 2\Re(\phi_\omega^*\beta\ \cdot)f'(\phi_\omega^*\beta\phi_\omega)\beta\phi_\omega.$$

The the solitary wave is called

- *spectrally stable* if $\sigma(JL(\omega)) \subset \mathrm{i}\mathbb{R}$,
- *spectrally unstable* if $\sigma(JL(\omega)) \not\subset \mathrm{i}\mathbb{R}$.

The nonlinear Schrödinger equation

For the ground state solution $\phi_\omega(x)e^{-i\omega t}$ of the a nonlinear Schrödinger equation

$$\mathrm{i}\partial_t\psi = -\Delta\psi - |\psi|^{2k}\psi, \qquad \psi(x,t)\in\mathbb{C}, \quad x\in\mathbb{R}^n, \qquad \text{(NLS)}$$

where $k>0$, the linearization is given by

$$\partial_t\rho = \mathrm{jl}(\omega)\rho,$$

where

$$\mathrm{jl}(\omega) := \begin{pmatrix} 0 & \mathrm{l}_-(\omega) \\ -\mathrm{l}_+(\omega) & 0 \end{pmatrix}$$

where $\mathrm{j}\sim 1/\mathrm{i}$,

$$\mathrm{l}_+(\omega) = \mathrm{l}_-(\omega) - 2k\Re(\phi_\omega{}^* \cdot)|\phi_\omega|^{2(k-1)}\phi_\omega \quad \mathrm{l}_-(\omega) = -\Delta - \omega - |\phi_\omega|^{2k}$$

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For some $c>0$, we have

$$\mathrm{l}_-(\omega)\phi_\omega = 0 \quad \mathrm{l}_-(\omega) > cI_{\phi_\omega^\perp},\ c>0.$$

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$$\begin{aligned} \mathrm{jl}(\omega)\rho = \lambda\rho &\Rightarrow \mathrm{l}_+(\omega)\mathrm{l}_-(\omega)\rho_2 = -\lambda^2\rho_2 \\ &\Rightarrow \sqrt{\mathrm{l}_-(\omega)}\mathrm{l}_+(\omega)\sqrt{\mathrm{l}_-(\omega)}R = -\lambda^2 R \end{aligned}$$

for $R = \sqrt{\mathrm{l}_-(\omega)}\rho_2$ where ρ_2 is the second component of ρ.

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We have

$$\sigma(\mathrm{jl}(\omega)) \subset \mathbb{R}\cup\mathrm{i}\mathbb{R}.$$

Absence of embedded eigenvalues

Lemma

Let $|\lambda| > m + |\omega|$*. There are* C *and* $R_0 > 0$ *such that*

$$\forall u \in H_0^1(B(0, R_0)^c),\ \tau \geq 1,\ \tau^{1/2}\|e^{\tau r} u\| \leq C\|r e^{\tau r}(\mathrm{JL}(\omega) - \lambda)u\|.$$

Lemma

The operator $\mathrm{JL}(\omega)$ *has no embedded eigenvalues* $\lambda \in i\mathbb{R}$ *with* $|\lambda| > m + |\omega|$*.*

Proposition (Limiting absorption principle)

Let $\omega_0 \in [-m, m]$ *and such that for* $\omega_j \in [-m, m]$, $\omega_j \to \omega_0$, *there are* $C < \infty$, *and* $\varepsilon > 0$ *with*

$$\begin{cases} \|\langle r\rangle^{1+\varepsilon} V(\omega_0)\|_{L^\infty(\mathbb{R}^n,\,\mathrm{End}\,(\mathbb{C}^N))} < \infty, \\ \lim_{j\to\infty} \|\langle r\rangle^{1+\varepsilon} \left(V(\omega_j) - V(\omega_0)\right)\|_{L^\infty(\mathbb{R}^n,\,\mathrm{End}\,(\mathbb{C}^N))} = 0, \end{cases}$$

Let

$$\lambda_0 \in \mathrm{i}\mathbb{R}, \qquad |\lambda_0| > m + |\omega_0|, \qquad \lambda_0 \notin \sigma_{\mathrm{p}}\big(J(D_m - \omega + V(\omega_0))\big).$$

Then, for any $s > 1/2$, *there is an open neighborhood* $I \subset [-m, m]$ *of* ω_0 *and an open neighborhood* $U \subset \mathbb{C}$ *of* λ_0 *such that for* $\omega \in I$ *the resolvent of* $J(D_m - \omega + V(\omega))$ *at* $\lambda \in \overline{U} \setminus \mathrm{i}\mathbb{R}$ *extends to a continuous mapping*

$$\big(J(D_m - \omega + V(\omega)) - \lambda\big)^{-1} : \; H_s^{-1/2}(\mathbb{R}^n, \mathbb{C}^N) \to H_{-s}^{1/2}(\mathbb{R}^n, \mathbb{C}^N),$$

which is bounded uniformly in $\lambda \in \overline{U} \setminus \mathrm{i}\mathbb{R}$.

Theorem (Bifurcation of point eigenvalues)

Let $(\omega_j)_{j\in\mathbb{N}}$, $\omega_j \in [-m, m]$, $\omega_j \to \omega_0 \in [-m, m]$, *and assume that* V *is hermitian and that there is* $\varepsilon > 0$ *such that*

$$\begin{cases} \|\langle r\rangle^{1+\varepsilon} V(\omega_0)\|_{L^\infty(\mathbb{R}^n,\,\mathrm{End}\,(\mathbb{C}^N))} < \infty, \\ \lim_{j\to\infty} \|\langle r\rangle^{1+\varepsilon} \left(V(\omega_j) - V(\omega_0)\right)\|_{L^\infty(\mathbb{R}^n,\,\mathrm{End}\,(\mathbb{C}^N))} = 0. \end{cases}$$

Let $\lambda_j \in \sigma_{\mathrm{p}}(JL(\omega_j))$ *be a Cauchy sequence such that*

$$\Re\lambda_j \neq 0 \quad \forall j \in \mathbb{N}, \qquad \lambda_j \underset{j\to\infty}{\longrightarrow} \lambda_0 \in \mathrm{i}\mathbb{R}, \qquad \lambda_0 \neq \pm\mathrm{i}(m + |\omega_0|).$$

If $\omega_0 = \pm m$, *additionally assume that*

$$\lambda_0 \neq 0.$$

Then

$$\lambda_0 \in \sigma_{\mathrm{p}}(JL(\omega_0)).$$

Let

$$\mathrm{L}(\omega) = \mathrm{D}_m - \omega + \mathrm{V}(\omega),$$

with $\mathrm{V}(\omega) : L^2(\mathbb{R}^n, \mathbb{C}^{2N}) \to L^2(\mathbb{R}^n, \mathbb{C}^{2N})$ a self-adjoint zero-order operator

Lemma (Bifurcation of point eigenvalues)

Let $J \in \mathrm{End}\,(\mathbb{C}^N)$ be skew-adjoint and invertible, such that $J^2 = -I_{\mathbb{C}^N}$, $[J, D_m] = 0$. Let $(\omega_j)_{j\in\mathbb{N}}$, $\omega_j \in [-m, m]$, be a sequence with $\lim_{j\to\infty} \omega_j = \pm m$, and there is $\varepsilon > 0$ such that

$$\lim_{j\to\infty} \|\langle r\rangle^{1+\varepsilon} V(\omega_j)\|_{L^\infty(\mathbb{R}^n, \mathrm{End}\,(\mathbb{C}^N))} = 0,$$

Let $\lambda_j \in \sigma_{\mathrm{p}}(JL(\omega_j))$, $j \in \mathbb{N}$, such that $\lim \lambda_j = \lambda_0$. Then

$$\lambda_0 \in \{0; \pm 2\mathrm{i}m\}\,.$$

Consider $\Re\lambda_j \neq 0$. For $\delta > 0$, let

$$U_\delta := \mathbb{D}_\delta(-2m\mathrm{i}) \cup \mathbb{D}_\delta(0) \cup \mathbb{D}_\delta(2m\mathrm{i}).$$

The eigenvalues of J are $\pm\mathrm{i}$ then the operator $J(D_m - \omega)$ can be represented as the direct sum of operators $\mathrm{i}(D_m - \omega)$ and $-\mathrm{i}(D_m - \omega)$. For any $s > 1/2$

$$\big(J(D_m - \omega) - z\big)^{-1} : \; L^2_s(\mathbb{R}^n, \mathbb{C}^N) \to L^2_{-s}(\mathbb{R}^n, \mathbb{C}^N), \qquad z \in \mathbb{C} \setminus (\mathrm{i}\mathbb{R} \cup U_\delta)$$

is bounded uniformly for $z \in \mathbb{C} \setminus (\mathrm{i}\mathbb{R} \cup U_\delta)$:
For appropriate values of $z \in \mathbb{C}$, the resolvent of $JL(\omega)$ is expressed as

$$(JL(\omega) - z)^{-1} = \big(J(D_m - \omega) - z\big)^{-1} \frac{1}{1 + JV\big(J(D_m - \omega) - z\big)^{-1}}.$$

Thus, the action

$$(JL(\omega) - z)^{-1} \;: L^2_s(\mathbb{R}^n, \mathbb{C}^N) \to L^2_{-s}(\mathbb{R}^n, \mathbb{C}^N)$$

is bounded uniformly in $z \in \mathbb{C} \setminus (\mathrm{i}\mathbb{R} \cup U_\delta)$ as long as ω is sufficiently close to m.

Lemma

The operator $\mathrm{L}(\omega)$ *corresponding to the linearization at a (one-frequency) solitary wave has the eigenvalue* -2ω *of geometric multiplicity (at least)* $N/2$*, with the eigenspace containing the subspace*

$$\mathrm{Span}\left\{\chi_{\omega,\eta}\ ;\ \eta \in \mathbb{C}^{N/2}\right\}.$$

The operator $\mathrm{JL}(\omega)$ *of the linearization at the solitary wave has eigenvalues* $\pm 2\omega\mathrm{i}$ *of geometric multiplicity (at least)* $N/2$.

For $\eta \in \mathbb{C}^{N/2}$,

$$\chi_{\omega,\eta} = \begin{bmatrix} -\mathrm{i}\frac{x}{r}\cdot\sigma^* \, u(r,\omega)\eta \\ v(r,\omega)\eta \end{bmatrix}.$$

One has $-2\omega\chi_{\omega,\eta} = (-\mathrm{i}\alpha\cdot\nabla_x + (m-f)\beta - \omega)\chi_{\omega,\eta}$.
Notice that $\phi_\omega(x)^*\beta\chi_{\omega,\eta}(x) = 0$.

Theorem (Bifurcations from $\pm 2m\mathrm{i}$ at $\omega = m$)

Let $(\omega_j)_{j\in\mathbb{N}}$, $\omega_j \in (\omega_0, m)$, be a sequence such that $\omega_j \to m$ and assume that λ_j are eigenvalues of (NLD) *linearized at $\phi_{\omega_j} e^{-\mathrm{i}\omega_j t}$ such that $\lambda_j \to 2m\mathrm{i}$. Denote*

$$z_j = -\frac{2\omega_j + \mathrm{i}\lambda_j}{\epsilon_j^2} \in \mathbb{C}, \qquad \epsilon_j := (m^2 - \omega_j^2)^{1/2}, \qquad j \in \mathbb{N},$$

and let $Z_0 \in \mathbb{C} \cup \{\infty\}$ be an accumulation point of the sequence $(z_j)_{j\in\mathbb{N}}$. Then:

1. $Z_0 \in \{\frac{1}{2m}\} \cup \sigma_{\mathrm{d}}(\mathfrak{l}_-)$. *In particular, $Z_0 \neq \infty$.*
2. *If the edge of the essential spectrum of $\mathfrak{l}_-$ at $1/(2m)$ is a regular point of the spectrum of $\mathfrak{l}_-$ (neither a resonance nor an eigenvalue), then $Z_0 \neq 1/(2m)$.*
3. *If $Z_0 = 0$, then $\lambda_j = 2\omega_j\mathrm{i}$ for all but finitely many $j \in \mathbb{N}$.*

We apply

$$\pi_P = (1 + \beta)/2, \qquad \pi_A = (1 - \beta)/2, \qquad \pi^{\pm} = (1 \mp iJ)/2,$$

to

$$\left(\epsilon_j D_0 + \beta m - \omega_j + J\lambda_j + \epsilon_j^2 V(\omega_j)\right)\Psi_j = 0$$

and obtain

$$\begin{aligned}
\epsilon_j D_0 \pi_A^- \Psi_j + (m - \omega_j - i\lambda_j)\pi_P^- \Psi_j + \epsilon_j^2 \pi_P^- V \Psi_j = 0,\\
\epsilon_j D_0 \pi_P^- \Psi_j - (m + \omega_j + i\lambda_j)\pi_A^- \Psi_j + \epsilon_j^2 \pi_A^- V \Psi_j = 0,\\
\epsilon_j D_0 \pi_A^+ \Psi_j + (m - \omega_j + i\lambda_j)\pi_P^+ \Psi_j + \epsilon_j^2 \pi_P^+ V \Psi_j = 0,\\
\epsilon_j D_0 \pi_P^+ \Psi_j - (m + \omega_j - i\lambda_j)\pi_A^+ \Psi_j + \epsilon_j^2 \pi_A^+ V \Psi_j = 0.
\end{aligned}$$

This allow one to express $Y := \pi^+ \Psi_j$ in terms of $X := \pi^- \Psi_j$ with $\vartheta(\cdot, \epsilon, z) : L^{2,-k}(\mathbb{R}^n, \mathrm{Range}\ \pi^-) \to L^{2,-k}(\mathbb{R}^n, \mathrm{Range}\ \pi^+)$, which is analytic in z. What is z ?

Theorem (Bifurcations from the origin at $\omega = m$)

Let $(\omega_j)_{j\in\mathbb{N}}$, $\omega_j \in (\omega_0, m)$, *be a sequence such that* $\omega_j \to m$, *and assume that* λ_j *are eigenvalues of* (NLD) *linearized at* $\phi_{\omega_j} e^{-\mathrm{i}\omega_j t}$ *such that* $\lambda_j \to 0$. *Denote*

$$\Lambda_j := \frac{\lambda_j}{\epsilon_j^2} \in \mathbb{C}, \qquad \epsilon_j := (m^2 - \omega_j^2)^{1/2}, \qquad j \in \mathbb{N},$$

and let $\Lambda_0 \in \mathbb{C} \cup \{\infty\}$ *be an accumulation point of the sequence* $(\Lambda_j)_{j\in\mathbb{N}}$. *Then:*

1. $\Lambda_0 \in \sigma(\mathrm{jl}) \cup \sigma(\mathrm{il}_-) \cup \sigma(-\mathrm{il}_-)$; *in particular,* $\Lambda_0 \neq \infty$. *If moreover* $N = 2$, *then* $\Lambda_0 \in \sigma(\mathrm{jl})$.
2. *If* $\Re\lambda_j \neq 0$ *for all* $j \in \mathbb{N}$, *then* $\Lambda_0 \in \sigma_{\mathrm{p}}(\mathrm{jl}) \cap \mathbb{R}$.
3. *If* $\Re\lambda_j \neq 0$ *for all* $j \in \mathbb{N}$, *then* $\Lambda_0 = 0$ *is only possible when* $k = 2/n$ *and* $\partial_\omega Q(\phi_\omega) > 0$ *for* $\omega \in (\omega_*, m)$, *with some* $\omega_* < m$. *Moreover, in this case* $\lambda_j \in \mathbb{R}$ *for all but finitely many* $j \in \mathbb{N}$.

Explicitly, we have

$$\phi_\omega(x) = \begin{bmatrix} v(r,\omega)\xi \\ \mathrm{i}u(r,\omega)\,\frac{x}{r}\cdot\sigma\,\xi \end{bmatrix}, \qquad r = |x|, \quad \xi \in \mathbb{C}^{N/2}, \quad |\xi| = 1.$$

We denote

$$\Xi = \begin{bmatrix} \Re\xi \\ 0 \\ \Im\xi \\ 0 \end{bmatrix} \in \mathbb{C}^{2N}, \qquad |\Xi| = 1.$$

We introduce the orthogonal projection onto Ξ:

$$\Pi = \Xi\langle\Xi, \cdot\rangle_{\mathbb{C}^{2N}} \in \mathrm{End}\,(\mathbb{C}^{2N}).$$

We note that, since $\beta\Xi = \Xi$,

$$\Pi \circ \pi_P = \pi_P \circ \Pi = \Pi, \qquad \Pi \circ \pi_A = \pi_A \circ \Pi = 0.$$

By means of the Shur complement, in the non relativistic limit, the operator

$$\mathrm{K} = \frac{1}{2m} - \frac{\Delta}{2m} - u_k^{2k}(1 + 2k\Pi)$$

appears.

Theorem (Spectral stability of solitary waves of the nonlinear Dirac equation)

Assume k, K *is such that either* $0 < k < 2/n$, $K > k$ *(charge-subcritical case), or* $k = 2/n$ *and* $K > 4/n$ *(charge-critical case).*
Further, assume that $\sigma_{\mathrm{d}}(\mathfrak{l}_-) = \{0\}$, *and that the threshold* $z = 1/(2m)$ *of the operator* $\mathfrak{l}_-$ *is a regular point of its spectrum.*
Then there is $\omega_* \in (0, m)$ *such that for each* $\omega \in (\omega_*, m)$ *the corresponding solitary wave is spectrally stable.*

Let us assume that there is a sequence $\omega_j \to m$ and a family of eigenvalues λ_j of the linearization at solitary waves $\phi_{\omega_j} e^{-\mathrm{i}\omega_j t}$ such that $\Re\lambda_j \neq 0$.
The only accumulation points of the sequence $\left(\lambda_j\right)_{j\in\mathbb{N}}$ are $\lambda = \pm 2m\mathrm{i}$ and $\lambda = 0$.
As long as $\sigma_{\mathrm{d}}(\mathrm{l}_-) = \{0\}$ and the threshold of l_- is a regular point of the spectrum, $\lambda = \pm 2m\mathrm{i}$ can not be an accumulation point of nonzero-real-part eigenvalues;
If $\Re\lambda_j \neq 0$ and Λ_0 is an accumulation point of the sequence

$$\Lambda_j := \lambda_j/(m^2 - \omega_j^2),$$

then $\Lambda_0 \in \sigma_{\mathrm{p}}(\mathrm{jl}) \cap \mathbb{R}$, where jl is the linearization of the NLS in dimension n with the nonlinear term $-|\psi|^{2k}\psi$.
For $k \leq 2/n$, the spectrum of the linearization of the corresponding NLS at a solitary wave is purely imaginary: $\sigma_{\mathrm{p}}(\mathrm{jl}) \subset \mathrm{i}\mathbb{R}$.
We conclude that one could only have $\Lambda_0 = 0$; this would require that $k = 2/n$ and $\partial_\omega Q(\phi_\omega) > 0$ for $\omega \lesssim m$. On the other hand, as long as $k = 2/n$ and $K > 4/n$, this yields $\partial_\omega Q(\phi_\omega) < 0$ for $\omega \lesssim m$, hence $\Lambda_0 = 0$ would not be possible. We conclude that there is no family of eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ with $\Re\lambda_j \neq 0$.

Lemma

Let $n \in \mathbb{N}$, $\omega \in \mathbb{R}$. *If* $v(r)$, $u(r)$ *are real-valued functions such that for some* $\omega \in [-m, m]$ *and for any* $\xi \in \mathbb{C}^{N/2}$, $|\xi| = 1$ *the function*

$$\psi(t,x) = \phi_\xi(x) e^{-\mathrm{i}\omega t},$$

with

$$\phi_\xi(x) = \begin{bmatrix} v(r)\xi \\ \mathrm{i}u(r)\sigma_r\xi \end{bmatrix}, \qquad r = |x|,$$

is a solitary wave solution to (NLD), *then for any* $\Xi, \mathrm{H} \in \mathbb{C}^{N/2}$, $|\Xi|^2 - |\mathrm{H}|^2 = 1$, *the function*

$$\begin{bmatrix} v(r)\xi \\ \mathrm{i}u(r)\sigma_r\xi \end{bmatrix} e^{-\mathrm{i}\omega t} \qquad ,$$

where $\xi = \frac{\Xi}{|\Xi|}$ *and* $\eta = \frac{\mathrm{H}}{|\mathrm{H}|}$, *is a solution to* (NLD).

Lemma

Let $n \in \mathbb{N}$, $\omega \in \mathbb{R}$. If $v(r)$, $u(r)$ are real-valued functions such that for some $\omega \in [-m, m]$ and for any $\xi \in \mathbb{C}^{N/2}$, $|\xi| = 1$ the function

$$\psi(t, x) = \phi_\xi(x) e^{-\mathrm{i}\omega t},$$

with

$$\phi_\xi(x) = \begin{bmatrix} v(r)\xi \\ \mathrm{i}u(r)\sigma_r\xi \end{bmatrix}, \qquad r = |x|,$$

is a solitary wave solution to (NLD)*, then for any Ξ, $\mathrm{H} \in \mathbb{C}^{N/2}$, $|\Xi|^2 - |\mathrm{H}|^2 = 1$, the function*

$$\begin{bmatrix} -\mathrm{i}u(r)\sigma_r^*\eta \\ v(r)\eta \end{bmatrix} e^{\mathrm{i}\omega t},$$

where $\xi = \frac{\Xi}{|\Xi|}$ and $\eta = \frac{\mathrm{H}}{|\mathrm{H}|}$, is a solution to (NLD).

Lemma

Let $n \in \mathbb{N}$, $\omega \in \mathbb{R}$. If $v(r)$, $u(r)$ are real-valued functions such that for some $\omega \in [-m, m]$ and for any $\xi \in \mathbb{C}^{N/2}$, $|\xi| = 1$ the function

$$\psi(t, x) = \phi_\xi(x) e^{-\mathrm{i}\omega t},$$

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is a solitary wave solution to (NLD)*, then for any Ξ, $\mathrm{H} \in \mathbb{C}^{N/2}$, $|\Xi|^2 - |\mathrm{H}|^2 = 1$, the function*

$$\theta_{\Xi,\mathrm{H}}(t, x) = |\Xi| \begin{bmatrix} v(r)\xi \\ \mathrm{i}u(r)\sigma_r\xi \end{bmatrix} e^{-\mathrm{i}\omega t} + |\mathrm{H}| \begin{bmatrix} -\mathrm{i}u(r)\sigma_r^*\eta \\ v(r)\eta \end{bmatrix} e^{\mathrm{i}\omega t},$$

where $\xi = \frac{\Xi}{|\Xi|}$ and $\eta = \frac{\mathrm{H}}{|\mathrm{H}|}$, is a solution to (NLD).

Spectral stability of bifrequency solitary waves

Theorem

Let $n \leq 4$, $N = 2$ *or* $N = 4$. *The bi-frequency solitary wave*

$$|\Xi| \begin{bmatrix} v(r)\xi \\ \mathrm{i}u(r)\sigma_r\xi \end{bmatrix} e^{-\mathrm{i}\omega t} + |\mathrm{H}| \begin{bmatrix} -\mathrm{i}u(r)\sigma_r^*\eta \\ v(r)\eta \end{bmatrix} e^{\mathrm{i}\omega t},$$

is spectrally stable as long as the corresponding one-frequency solitary wave solution $\phi_\omega(x)e^{-i\omega t}$ *is spectrally stable.*