Edge states in near-honeycomb structures

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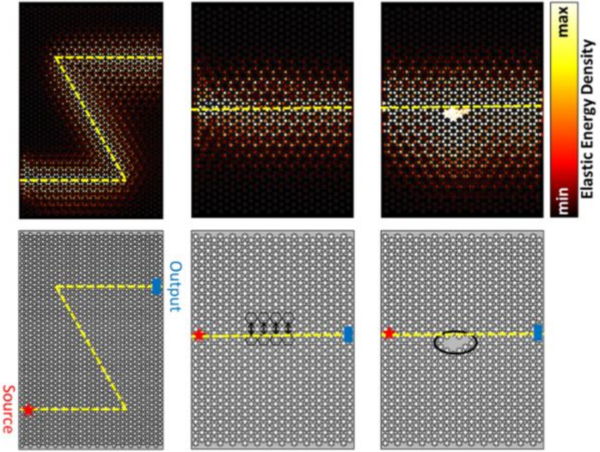
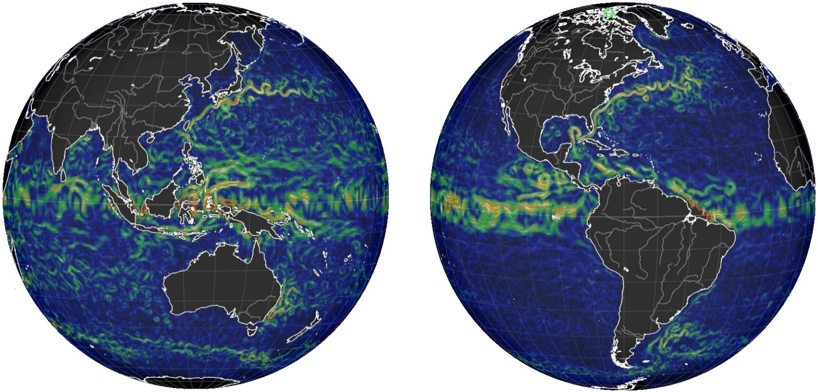
March 4th, Himeji conference on PDEs

Topological edge states in honeycomb lattices



Physical experiments due to [Yu–Ren–Lee '19].

Equatorial waves



Currents displayed on https://earth.nullschool.net as of Feb. 20th 2019.

Theoretical analysis demonstrate their **topological character** [Delplace–Martson–Venaille '17, Tauber–Delplace–Venaille '18, Faure '19] .

Plan of the talk

- **An introduction to the bulk-edge correspondence:**
  - Topological waves as a spectral problem
  - The edge index: **spectral flow**
  - The bulk index: **Chern number**

- **Edge states in magnetic graphene:**
  - **Dirac points** in honeycombs [Fefferman–Weinstein '12]
  - Conjugation-breaking and **spectral gaps** [Lee-Thorp–Weinstein–Zhu '18, D. '18]
  - A **quantitative** bulk-edge formula [D. '18, D. '19]

Floquet–Bloch theory

Let $V \in C^\infty(\mathbb{R}^2, \mathbb{R})$ and $A \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ periodic w.r.t $\mathbb{Z}^2$:

$$V(x+n) = V(x), \quad A(x+n) = A(x), \quad x \in \mathbb{R}^2, \ n \in \mathbb{Z}^2.$$

Quantum evolution in e.m. field $(\nabla_{\mathbb{R}^2} V, \nabla_{\mathbb{R}^2} \times A)$:
$P = -(\nabla_{\mathbb{R}^2} + iA)^2 + V$.

For each $\xi \in \mathbb{R}^2$ (or $\xi \in \mathbb{R}^2/(2\pi\mathbb{Z})^2 = \mathbb{T}^2$), P acts on

$$L^2_\xi \stackrel{\text{def}}{=} \left\{ u \in L^2_{\text{loc}}\left(\mathbb{R}^2, \mathbb{C}\right) : \ u(x+n) = e^{i\xi \cdot n} \cdot u(x) \right\}.$$

The L^2_ξ-spectrum of P is ξ-**dependent and discrete:**

$$\lambda_1(\xi) \leq \lambda_2(\xi) \leq \cdots \leq \lambda_j(\xi) \leq \cdots \to +\infty.$$

One **recovers the** L^2**-spectrum** of P:

$$\Sigma_{L^2}(P) = \bigcup_{j=1}^{\infty} \left\{ \lambda_j(\xi) : \xi \in \mathbb{R}^2 \right\} = \bigcup_{j=1}^{\infty} \lambda_j\left(\mathbb{R}^2\right).$$

$\Sigma_{L^2}(P)$ has a **band structure,** made up intervals $\lambda_1(\mathbb{R}^2), \ldots \lambda_j(\mathbb{R}^2), \ldots$

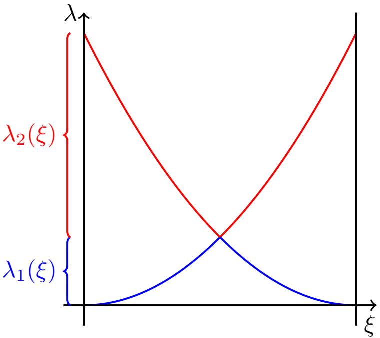
Example: $A = 0$, $V = 0$, dimension 1

L^2_ξ-spectrum of $P = -\Delta_{\mathbb{R}}$: eigenvalue problem

$$\begin{cases} \big(-\Delta_{\mathbb{R}} - E\big)u = 0 \\ u(x+1) = e^{i\xi} \cdot u(x) \end{cases}.$$

Solutions $u(x) = e^{i(\xi+2m\pi)x}$, $E = (\xi + 2\pi m)^2$.

Dispersion curves:

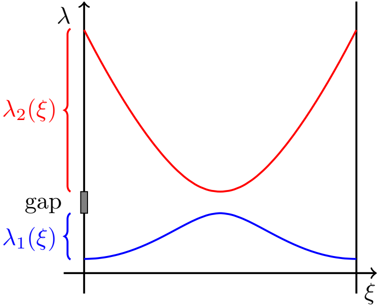


Example: $A \neq 0$, $V \neq 0$, dimension 1

L^2_ξ-spectrum of $P = -(\partial_x + iA)^2 + V$: eigenvalue problem

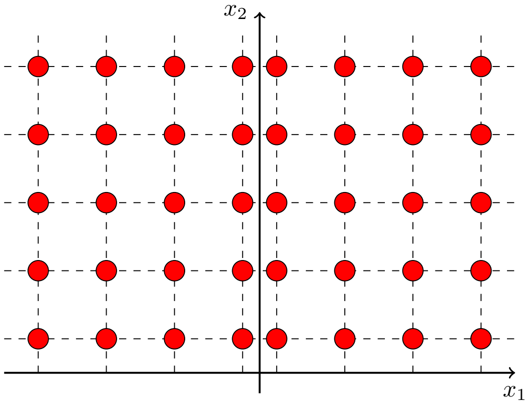
$$\begin{cases} (-(\partial_x + iA)^2 + V - E)u = 0 \\ u(x+1) = e^{i\xi} \cdot u(x) \end{cases}.$$

Generically the **first gap is open:**



Much more **complicated** in higher dimensions: no more ODEs!

Robust waves



Each red circle represent the same e.m. field.
We want to explain the following fact:

In **favorable** conditions, robust (topological) waves propagate **along** $\mathbb{R}e_2$ **but not across** $\mathbb{R}e_2$**.**

Line defect created by a magnetic field

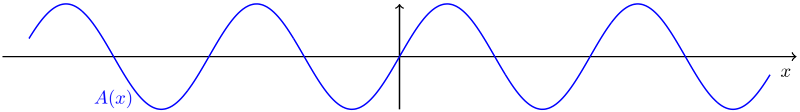
Schrödinger operator $\mathscr{P} = -(\nabla_{\mathbb{R}^2} + i\mathscr{A})^2 + V$, where:

- $V \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is $\mathbb{Z}^2$-periodic;
- $\mathscr{A}$ is periodic in x_2 and **asymptotically periodic in x_1:**

$$\mathscr{A}(x_1, x_2 + 1) = \mathscr{A}(x_1, x_2); \qquad \mathscr{A}(x_1, x_2) = \begin{cases} A(x_1, x_2), & x_1 \gg 1 \\ -A(x_1, x_2), & x_1 \ll -1 \end{cases}$$

with $A \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ periodic w.r.t. $\mathbb{Z}^2$.

1D-analog



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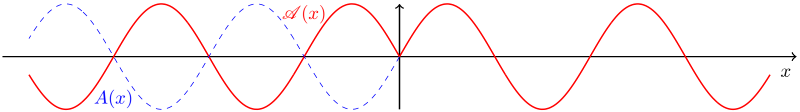

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Waves propagating along the defect $x_1 = 0$:

$$\begin{cases} i\partial_t \psi = \mathscr{P}\psi \\ \psi(t, x) = e^{i(\zeta x_2 - Et)} \cdot u(x) \end{cases}, \quad u \in L^2\left(\mathbb{R}^2/\mathbb{Z}e_2\right).$$

Associated **spectral problem:** $\mathscr{P}u = Eu$ on the space

$$L^2[\zeta] \overset{\text{def}}{=} \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2),\ u(x + e_2) = e^{i\zeta} \cdot u(x),\ \int_{\mathbb{R}^2/\mathbb{Z}e_2} |u|^2 < \infty \right\}.$$

$L^2[\zeta]$-spectral theory for $\mathscr{P} = -(\nabla_{\mathbb{R}^2} + i\mathscr{A})^2 + V$

$$\begin{cases} \mathscr{P}u = Eu \\ u(x+e_2) = e^{i\zeta} \cdot u(x) \end{cases}, \qquad \int_{\mathbb{R}^2/\mathbb{Z}e_2} |u|^2 < \infty.$$

Set $P_\pm = -(\nabla_{\mathbb{R}^2} \pm iA)^2 + V$, where $\pm A = \mathscr{A}$ near $\pm\infty$. Then

$$\Sigma_{L^2[\zeta]}(\mathscr{P}) \; = \; \Sigma_{L^2[\zeta]}(P_+) \; \cup \; \Sigma_{L^2[\zeta]}(P_-) \; \cup \; \Sigma_{L^2[\zeta],\mathrm{d}}(\mathscr{P}).$$

Floquet–Bloch theory along $\zeta e_2 + \mathbb{R}e_1$:

$$\Sigma_{L^2[\zeta]}(P_\pm) = \bigcup_{j=1}^{\infty} \lambda_{\pm,j}\big(\zeta e_2 + \mathbb{R}e_1\big).$$

$L^2[\zeta]$-spectrum of $\mathscr{P}$: **band structure + eigenvalues.**

We assume that the **first essential $L^2[\zeta]$-gap is open:**

$$\forall \epsilon, \epsilon' \in \{\pm\}, \quad \lambda_{\epsilon,1}(\zeta e_2 + \mathbb{R}e_1) \cap \lambda_{\epsilon',2}(\zeta e_2 + \mathbb{R}e_1) = \emptyset.$$

Spectral flow for $\mathscr{P} = -(\nabla_{\mathbb{R}^2} + i\mathscr{A})^2 + V$

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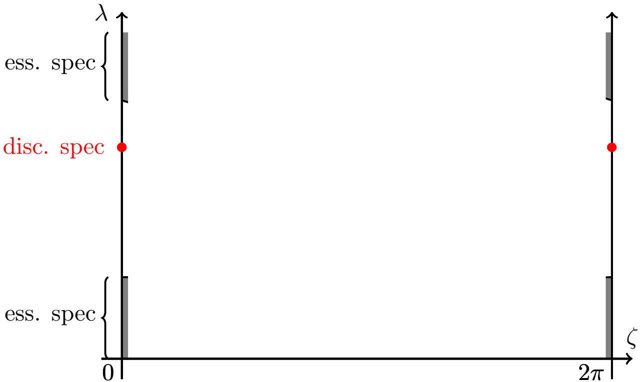

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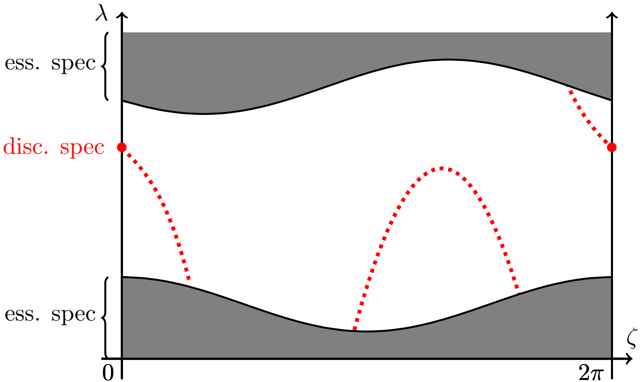

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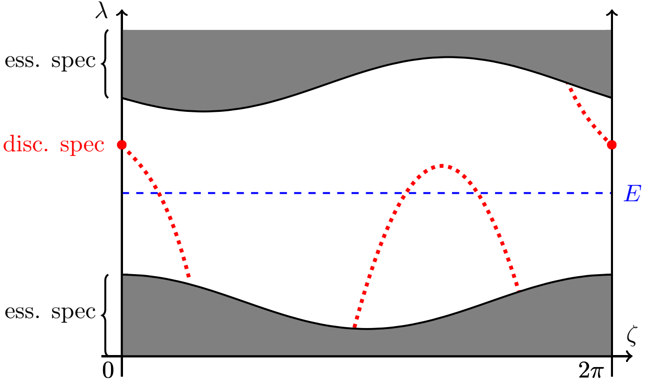


Spectral flow of $\mathscr{P}$ at E: signed number of eigenvalue crossings E.

- $\mathrm{Sf}(\mathscr{P}, E)$ counts **topological** waves: the effective conductivity.
- $\mathrm{Sf}(\mathscr{P}, E)$ is stable against compact/**gap-preserving** perturbations.
- $\mathrm{Sf}(\mathscr{P}, E)$ depends only on $P_{\pm} = -(\nabla_{\mathbb{R}^2} \pm iA)^2 + V$.

Bulk index

Sf($\mathscr{P}, E$) is invariant under compact perturbations
$\Rightarrow$ it depends **only on** $P_\pm$.

Write the spectrum of P_+ on L^2_ξ, $\xi \in \mathbb{T}^2$, as: $\lambda_{+,1}(\xi) \leq \lambda_{+,2}(\xi) \leq \dots$
The first $L^2[\zeta]$-gap of $\mathscr{P}$ is open $\Rightarrow \lambda_{+,1}(\xi) < \lambda_{2,+}(\xi)$.

Define **line bundle** $\mathscr{E}_+ \to \mathbb{T}^2$ with fibers

$$\ker_{L^2_\xi}\left(P_+ - \lambda_{+,1}(\xi)\right) \subset L^2_\xi.$$

Topology characterized by **first Chern number** $c_1(\mathscr{E}_+) \in \mathbb{Z}$.

If instead the j-th gap is open, get a rank-j bundle.

Bulk-edge correspondence

$$\mathrm{Sf}(\mathscr{P}, E) = c_1(\mathscr{E}_+) - c_1(\mathscr{E}_-)$$

Index theorem: "spectral invariant" = "topological invariant".

Mathematical proofs in:

- **Many discrete models:** [Hatsugai '93, Graf–Porta '11, Avila–Schulz-Baldes–Villegas-Blas '11, Shapiro–Tauber '18, ...]
- **Some continuous models:** [Kellendonk–Schulz-Baldes '04, Taarabt '14, Kubota '17, Bourne–Rennie '18, ...]

Problem: BEC **does not address** existence of edge states in PDEs. For that you need to:

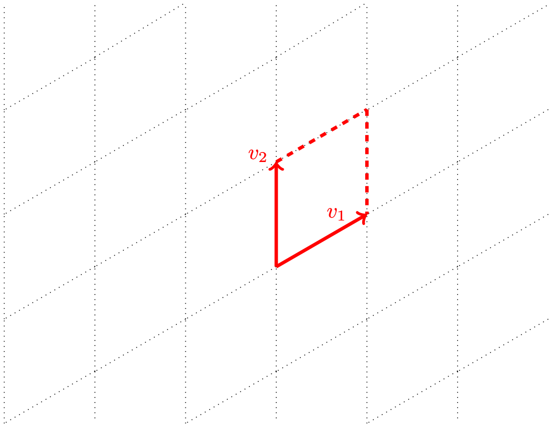
- Derive an effective equation (e.g. discrete 2×2) for the PDE.
- Compute the index of the effective model.

Some previous results: [Nakamura–Belissard '90] (vanishing Chern number), [Haldane–Raghu '08, Bal '18, Faure '19] (Dirac operators), [D. '18] (dislocation systems: explicit formula for $2\mathbb{Z}+1$-index).

Continuous graphene [Fefferman–Weinstein '12]

Let $P_0 = -\Delta_{\mathbb{R}^2} + V$, where V **is honeycomb:**

- $V \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is even: $V(x) = V(-x)$;
- V is $\mathbb{Z}v_1 \oplus \mathbb{Z}v_2$-periodic.



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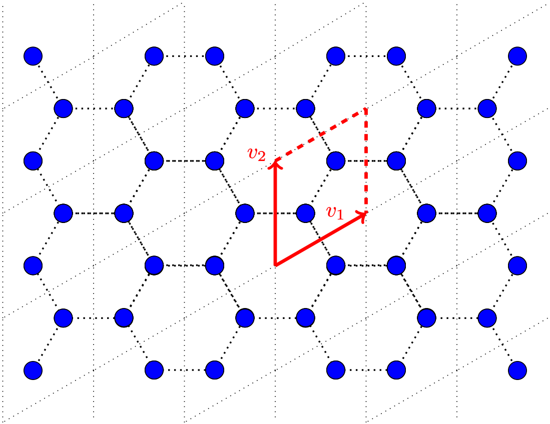

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After linear substitution, P_0 is $\mathbb{Z}^2$-periodic.

On L^2_ξ, P_0 has eigenvalues

$$\lambda_1(\xi) \leq \cdots \leq \lambda_j(\xi) \leq \dots$$

Dirac points of $P_0 = -\Delta_{\mathbb{R}^2} + V$

A **Dirac point** $(\xi_\star, E_\star)$ is a **conical singularity in band spectrum:**

- $E_\star$ is a $L^2_{\xi_\star}$-eigenvalue of multiplicity exactly 2.
- For ξ near $\xi_\star$,

$$\begin{cases} \lambda_j(\xi) \sim E_\star + b\cdot(\xi - \xi_\star) - |M(\xi-\xi_\star)| \\ \lambda_{j+1}(\xi) \sim E_\star + b\cdot(\xi - \xi_\star) + |M(\xi-\xi_\star)| \end{cases}.$$

with $b \in \mathbb{R}^2$ and $M \in M_2(\mathbb{R})$ such that $|M\cdot\xi| > |b\cdot\xi|$.

Example: If V is $2\pi/3$-rotationally invariant, then $b = 0$, $M = \nu_\star \cdot \mathrm{Id}_2$

$$\Rightarrow \quad \begin{cases} \lambda_j(\xi) \sim E_\star - \nu_\star|\xi - \xi_\star| \\ \lambda_{j+1}(\xi) \sim E_\star - \nu_\star|\xi - \xi_\star| \end{cases}.$$

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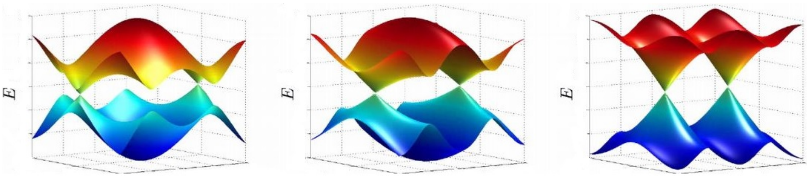

Simulations of [Hou–Chen '15] for some tight-binding lattices.

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Theorem

For a large class of honeycomb potentials V:

- *P_0 has Dirac points $(\pm\xi_\star, E_\star)$ [Fefferman–Weinstein '12, Berkolaiko–Comech '18];*
- *$E_\star$ is not a L^2_ξ eigenvalue of P_0 unless $\xi = \pm\xi_\star$* [Fefferman–Lee-Thorp–Weinstein '16, '18].

We now assume that V belongs to that class.

Dirac points of $P_0 = -\Delta_{\mathbb{R}^2} + V$

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Interest of Dirac points:

- Wavepackets localized near Dirac points follow an effective **Dirac equation** [Fefferman–Weinstein '14].
- Destroying them provide a framework for **novel topological phases.** The change of invariants can be computed via **local analysis** near Dirac points.

Breaking conjugation invariance

Dirac point can be traced down to:

- complex conjugation invariance $\mathcal{C}$;
- parity invariance.

Turning on a $\mathbb{Z}^2$-**periodic magnetic field** A breaks $\mathcal{C}$:

$$P_{\pm} = -(\nabla_{\mathbb{R}^2} \pm iA)^2 + V.$$

Conforming to the BEC setting, we look at $\mathscr{P}$ equal to $P_{\pm}$ as $x_1 \to \pm\infty$:

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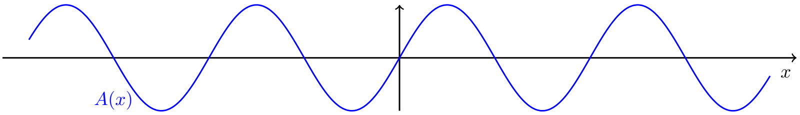
1D-analog



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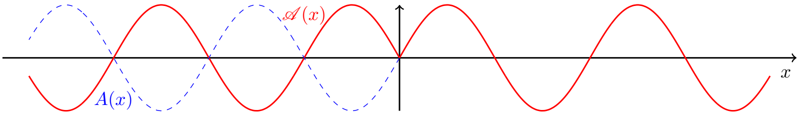
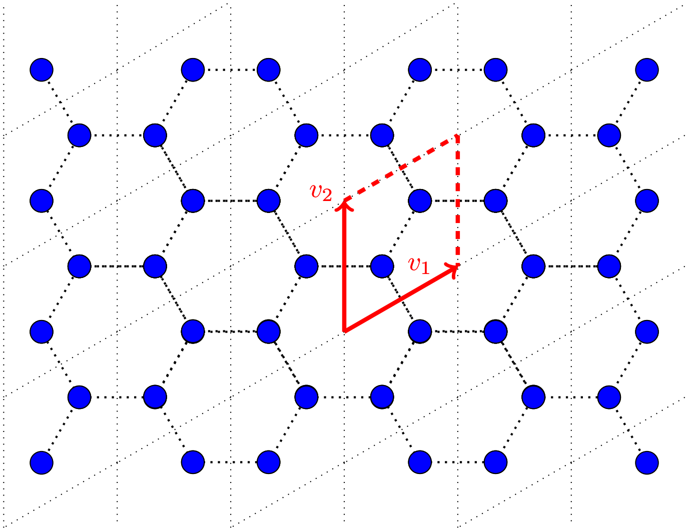
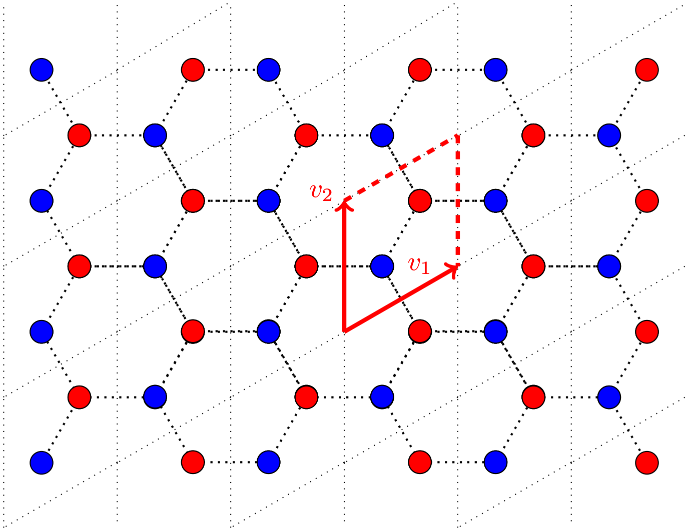


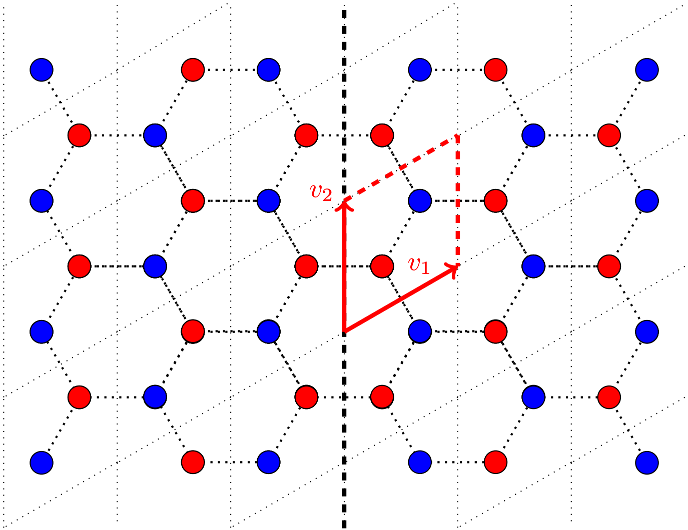
Honeycomb picture for P_0



Honeycomb picture for P_+



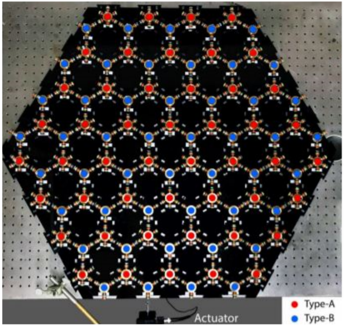
Honeycomb picture for $\mathscr{P}$

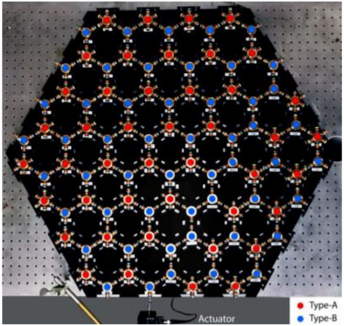


This arises in nature!

Magnetic realization

Experiments of [Qian–Apigo–Prodan–Barlas–Prodan '18]





Magnetic realization

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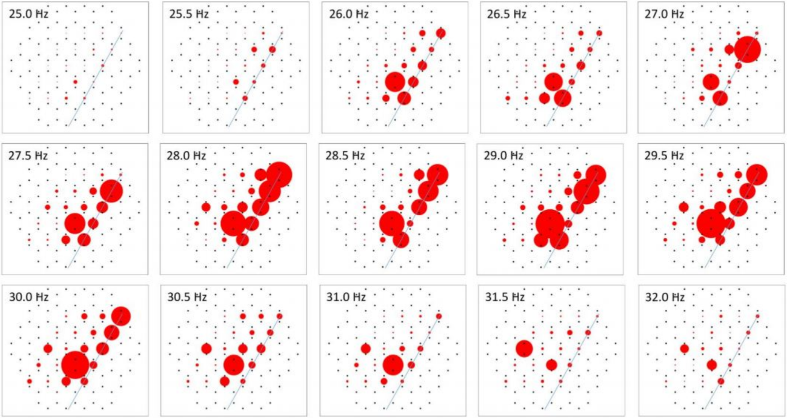

Bulk-edge correspondence for $\mathscr{P}$

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Write $\ker_{L^2_{\xi_\star}}(P_0 - E_\star) = \mathbb{C}\phi_1 \oplus \mathbb{C}\phi_2$ with $\phi_2 = \overline{\phi_1(-\cdot)}$. Assume that:

- $\theta_\star \overset{\text{def}}{=} \langle \phi_1, (A_{\text{odd}} \cdot i\nabla_{\mathbb{R}^2} + i\nabla_{\mathbb{R}^2} \cdot A_{\text{odd}})\phi_1 \rangle_{L^2_{\xi_\star}} \neq 0$;
- For all $t \in (0,1]$, the j-th L^2-gap of $-(\nabla_{\mathbb{R}^2} + itA)^2 + V$ is open.

Theorem [D. '18, '19]

Let E in the j-th gap of $\mathscr{P}$. Then

$$\mathrm{Sf}(\mathscr{P}, E) = 2 \cdot \mathrm{sgn}(\theta_\star) = c_1(\mathscr{E}_+) - c_1(\mathscr{E}_-).$$

Comments: LHS/RHS

- The assumptions depend **only** on the at-large behavior of $\mathscr{P}$, not on the transition from P_+ to P_-.
- They hold generically for small magnetic fields.

Bulk-edge correspondence for $\mathscr{P}$

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- For all $t \in (0,1]$, the j-th L^2-gap of $-(\nabla_{\mathbb{R}^2} + itA)^2 + V$ is open.

Theorem [D. '18, '19]

Let E in the j-th gap of $\mathscr{P}$. Then

$$\mathrm{Sf}(\mathscr{P}, E) = 2 \cdot \mathrm{sgn}(\theta_\star) = c_1(\mathscr{E}_+) - c_1(\mathscr{E}_-).$$

Comments: LHS

- Demonstrates that 2 **edge states must exist.**
- **Very stable:** it persists against compact and even **gap-preserving** perturbations.

Bulk-edge correspondence for $\mathscr{P}$

$$\mathscr{P} = -(\nabla_{\mathbb{R}^2} \pm i\mathscr{A})^2 + V, \qquad \mathscr{A}(x_1,x_2) = \begin{cases} A(x_1,x_2), & x_1 \gg 1 \\ -A(x_1,x_2), & x_1 \ll -1 \end{cases}.$$

Write $\ker_{L^2_{\xi_\star}}(P_0 - E_\star) = \mathbb{C}\phi_1 \oplus \mathbb{C}\phi_2$ with $\phi_2 = \overline{\phi_1(-\cdot)}$. Assume that:

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Comments: LHS

- **Previously:** in a perturbative regime (small A, adiabatic transition from $-A$ to A), two edge states had been constructed [Fefferman –Lee-Thorp–Weinstein '16, Lee-Thorp–Weinstein–Zhu '18].
- Missing ingredient for topological protection: **no other edge states.**

Bulk-edge correspondence for $\mathscr{P}$

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Comments: RHS

- [Haldane–Raghu '08] proved the RHS equality for related Dirac operator $\not{D}_\pm$ with **asymptotically constant coefficients.**
- [D '19] shows that **the reduction of $P_\pm$ to $\not{D}_\pm$ holds rigorously.**

Principle of proof: deriving effective equations

$$c_1(\mathscr{E}_+) = \mathrm{sgn}(\theta_\star), \qquad \theta_\star = \langle \phi_1, (A_{\mathrm{odd}} \cdot i\nabla_{\mathbb{R}^2} + i\nabla_{\mathbb{R}^2} \cdot A_{\mathrm{odd}})\phi_1 \rangle_{L^2_{\xi_\star}} .$$

Topological transition from P_0 to $P_\delta = -(\nabla_{\mathbb{R}^2} \pm i\delta A)^2 + V$ comes from **Dirac points.**

Goal: understand P_δ as $\delta \to 0$. Say $b = 1$, $M = \mathrm{Id}$, $\xi_\star = E_\star = 0$.

- For ξ near $\xi_\star = 0$, P_0 has eigenvalues $\pm|\xi|$ near $E_\star = 0$.
- In the right basis,

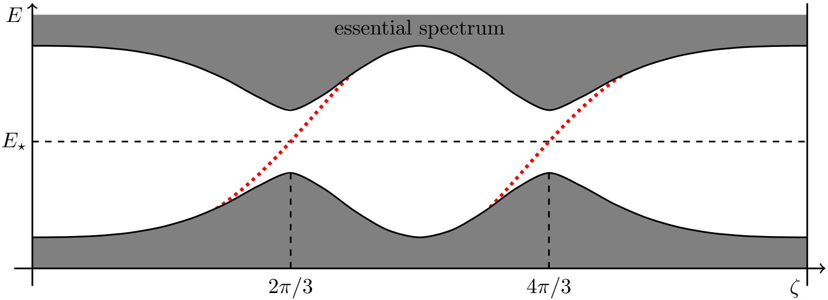
  $$P_0 : L^2_\xi \to L^2_\xi \quad \sim \begin{bmatrix} 0 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 0 \end{bmatrix}$$ – in some resolvent sense.
- **Turn on magnetic field** δA**:**

  $$P_\delta : L^2_\xi \to L^2_\xi \quad \sim \begin{bmatrix} \theta_\star\delta & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\theta_\star\delta \end{bmatrix}.$$
- **"Chern number" for** 2×2 **model:** $\frac{1}{2}\,\mathrm{sgn}(\theta_\star)$.
- **Two Dirac points** $\Rightarrow$ $c_1(\mathscr{E}_+) = \mathrm{sgn}(\theta_\star)$.

Edge states

The edge index is harder to compute. Same underlying principle: a **Dirac operator governs the effective dynamics.**

Asymptotics of **edge states** in the **perturbative** regime of [Fefferman–Lee-Thorp–Weinstein '16, Lee-Thorp–Weinstein–Zhu '18, D. '18]:



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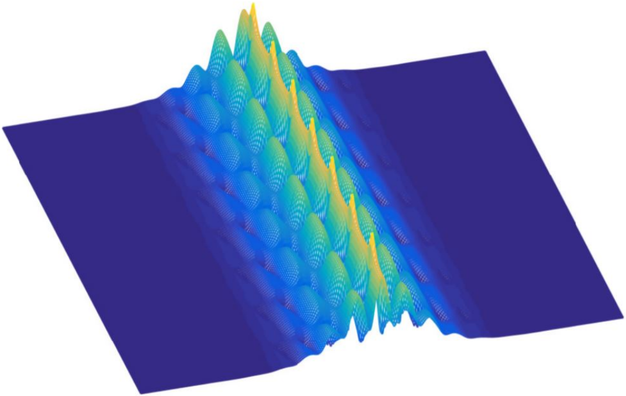
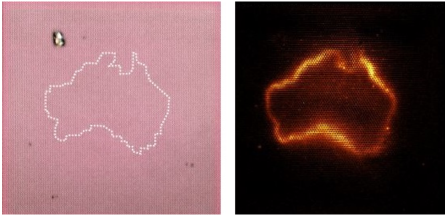


Remaining questions

- High energy – e.g. **semiclassical** – edge states?
- Edge states in the **absence of gaps?**
- Edge states with **no translation invariance?**

Photonic realization of edge states [Smirnova et al. '18]



Thank you!