Mean Value Operators on Noncompact Symmetric Spaces

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Mean Value Operators on $\mathbb{R}^n$

$$M^r f(x) := \frac{1}{\mathrm{Vol}(S_r)} \int_{|y|=r} f(x+y)\, dS_r(y)$$

$$= \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{\omega \in S^{n-1}} f(x+r\omega)\, dS_\omega$$

$f \in C^\infty(\mathbb{R}^n)$

$r > 0$, $S_r = \{ y \in \mathbb{R}^n \mid |y| = r \}$

$S^{n-1} = \{ \omega \in \mathbb{R}^n \mid |\omega| = 1 \}$

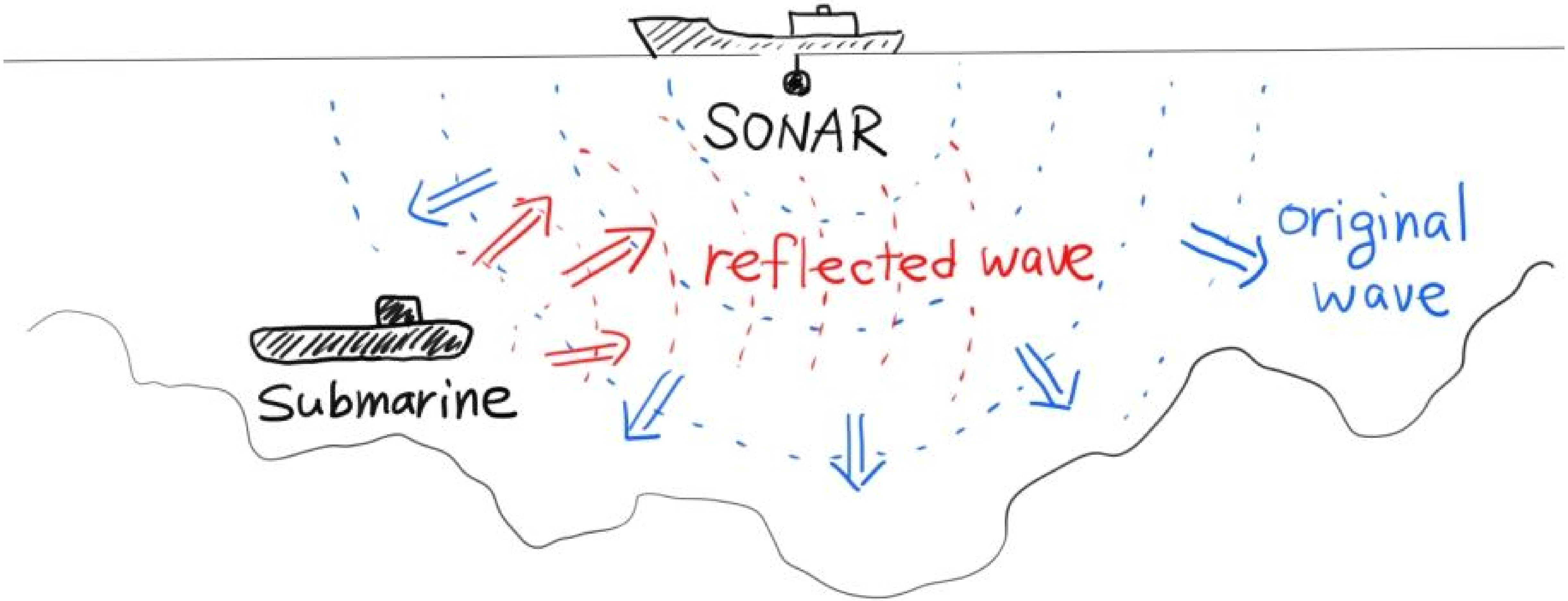
Take the average of f on S_r.

$$M^r : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$$

Mean Value Operator and Sonar Transform

SONAR receives reflected waves

$\rightsquigarrow \{M^r f\} \rightsquigarrow$ Reconstruction of f



Wave equation on $\mathbb{R}^n$ (n: odd)

$$\partial_t^2 u - \Delta u = 0, \quad u(0,x) = f(x), \quad u_t(0,x) = g(x)$$

The solution is given by $x \in \mathbb{R}^n,\ t \in \mathbb{R}$

$$u(t,x) = \frac{1}{(n-2)!}\,\partial_t\left(\frac{1}{t}\partial_t\right)^{\frac{n-3}{2}}\left\{t^{n-2}(M^t f)(x)\right\}$$

$$+ \frac{1}{(n-2)!}\left(\frac{1}{t}\partial_t\right)^{\frac{n-3}{2}}\left\{t^{n-2}(M^t g)(x)\right\}$$

$$M^r\phi(x) = \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{\omega \in S^{n-1}} \phi(x + r\omega)\, dS_\omega$$

The solution is given in terms of mean value op's.

In particular, in the case $n=3$, the sol of

$$\partial_t^2 u - \Delta u = 0, \quad u(0,x) = f(x), \quad u_t(0,x) = g(x)$$
$$x \in \mathbb{R}^3, \ t \in \mathbb{R}$$

is given by

$$u(t,x) = \partial_t \{ t M^t f(x) \}$$
$$+ t M^t g(x)$$

In addition, if $f \equiv 0$, the mapping

$g \longmapsto \frac{1}{t} u(t,\cdot)$ is a mean value operator for each fixed $t \, (\neq 0)$.

Problem Take any $h \in C^\infty(\mathbb{R}^3)$ and fix it.

Can one find $g \in C^\infty(\mathbb{R}^3)$ such that

$u(T, \cdot) = h$?

$$\Downarrow$$

Surjectivity of the mapping $g \mapsto u(T, \cdot)$

$$\Downarrow$$

Surjectivity of the mean value operator

$$M^T : C^\infty(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$$

In this talk, we will deal with the following two problems.

We fix $r > 0$.

(1) Surjectivity of

$$M^r : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$$

Rem We will also mention the injectivity of M^r.

(2) Surjectivity of

$$M^r : C^\infty(X) \longrightarrow C^\infty(X)$$

X : noncompact symmetric space

John's formula

(Fritz John "Plane Waves and Spherical Means" page 123)

Let $g \in \mathcal{S}(\mathbb{R}^3)$. If we put

$$f(x) = \sum_{k=0}^{\infty} \frac{-1}{2\pi(2k+1)} \int_{|y|=2k+1} \Delta_x g(x+y)\, dS_y$$

Then we have

$$M^1 f = g.$$

Remark Even though $g \in \mathcal{S}(\mathbb{R}^3)$, the reconstruction formula is complicated!

So it is really a difficult problem to find $f \in C^\infty(\mathbb{R}^n)$ such that $M^r f = g$ for a general $g \in C^\infty(\mathbb{R}^n)$.

Surjectivity of $M^r : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$

Idea of the proof

- Write M^r as a convolution operator.
- Apply a theorem of Ehrenpreis.

$$M^r f(x) = \frac{1}{\mathrm{Vol}(S_r)} \int_{|y|=r} f(x-y)\, dS_r(y)$$

$$= \int_{\mathbb{R}^n} f(x-y)\mu(y)\,dy = f*\mu(x),$$

where

$$\mu(y) = \frac{1}{\mathrm{Vol}(S_r)} \delta_{S_r}(y)$$

delta function whose support is S_r.

A theorem of Ehrenpreis

$\mu \in \mathcal{E}'(\mathbb{R}^n)$: the space of compactly supported distributions

$\hat{\mu}(\zeta)$: Fourier-Laplace transform of μ ← an entire fn on $\mathbb{C}^n$

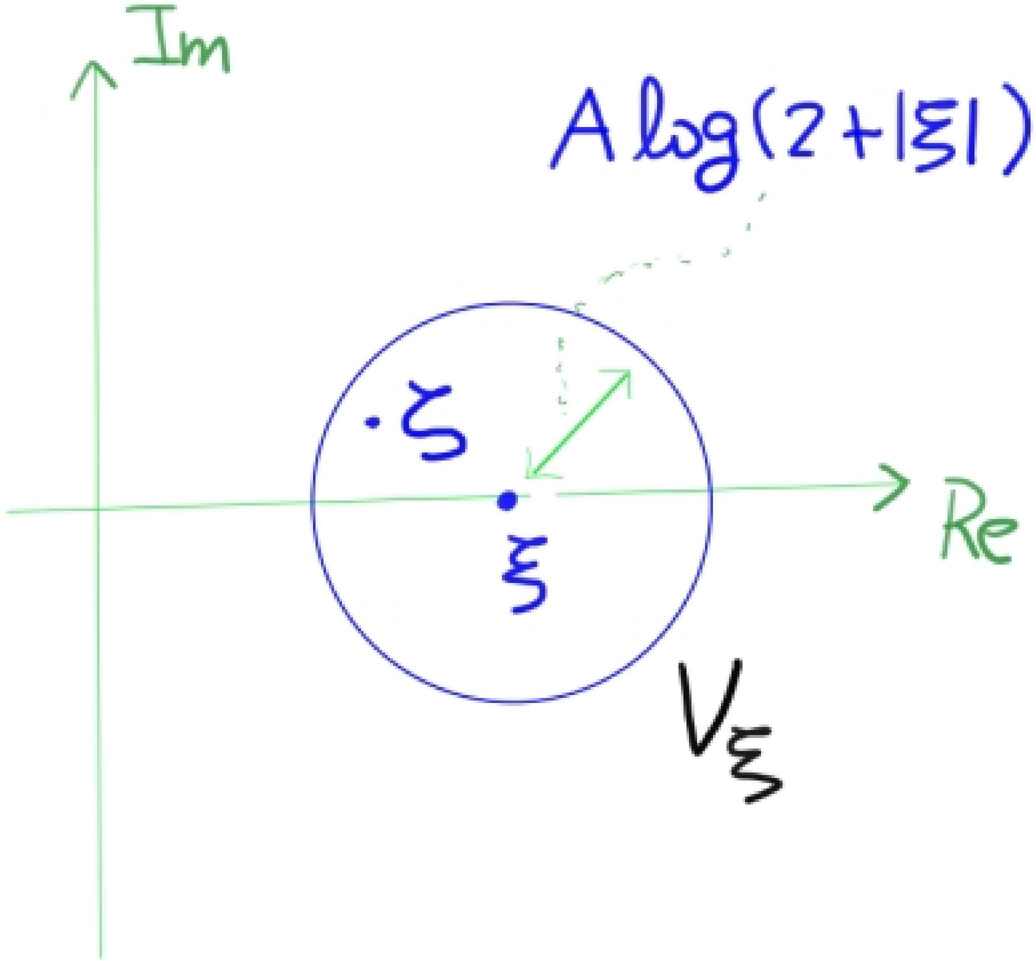
<u>Def</u> A holomorphic function F on $\mathbb{C}^n$ is slowly decreasing.

$\Longleftrightarrow$ $\exists$ consts $A, B, C, D > 0$, s.t

$$\sup_{V_\xi \ni \zeta} |F(\zeta)| \geq B(C+|\xi|)^{-D}$$

holds for $\forall \xi \in \mathbb{R}^n$.

Here $V_\xi = \{\zeta \in \mathbb{C}^n \mid |\zeta - \xi| \leq A\log(2+|\xi|)\}$.



Theorem (Ehrenpreis 1960, Amer. J. Math.)

Convolution operator

$C^\infty(\mathbb{R}^n) \ni f \longmapsto f * \mu \in C^\infty(\mathbb{R}^n)$ is surjective

$\Longleftrightarrow$ $\hat{\mu}(\zeta)$ is slowly decreasing.

Rem In general, it is difficult to get such an estimate from below.

$$\sup_{V_\xi \ni \zeta} |\hat{\mu}(\zeta)| \geqq B(C+|\xi|)^{-D}.$$

$\leadsto$ We have to know an asymptotic behavior of $\hat{\mu}(\zeta)$.

Proof that $\hat{\mu}(\xi)$ is slowly decreasing.

$$\mu = \frac{1}{\mathrm{Vol}(S_r)}\delta_{S_r}$$

$$\hat{\mu}(\xi) = \frac{1}{\mathrm{Vol}(S_r)}\int_{|x|=r} e^{-i\xi\cdot x}\, dS_r(x)$$

$$C_\nu = \sqrt{\frac{2}{\pi}}\, 2^\nu \Gamma(\nu+1)$$

$$= \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{\omega\in S^{n-1}} e^{ir\xi\cdot\omega}\, dS_\omega$$

$$= j_{\frac{1}{2}n-1}(r|\xi|) \qquad j_\nu(z) = \Gamma(\nu+1)\sum_{k=0}^{\infty}\frac{(-1)^k (z/2)^{2k}}{k!\,\Gamma(k+\nu+1)}$$

$j_\nu(z)$: normalized Bessel function. $\left(\frac{z}{2}\right)^\nu j_\nu(z) = \Gamma(\nu+1) J_\nu(z)$

$$j_\nu(x) \sim C_\nu |x|^{-\frac{\nu+1}{2}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad (|x|\to\infty)$$

This formula shows that $\hat{\mu}(\xi)$ is slowly decreasing.

<u>Theorem</u> [K-T, Lim, Ph.D Thesis 2012]

$\forall r > 0$, $M^r : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$

is surjective.

Mean Value Operators on Noncompact Symmetric Spaces

$M^r : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ is surjective.

Problem Does a similar result hold for mean value operators on noncompact symmetric spaces?

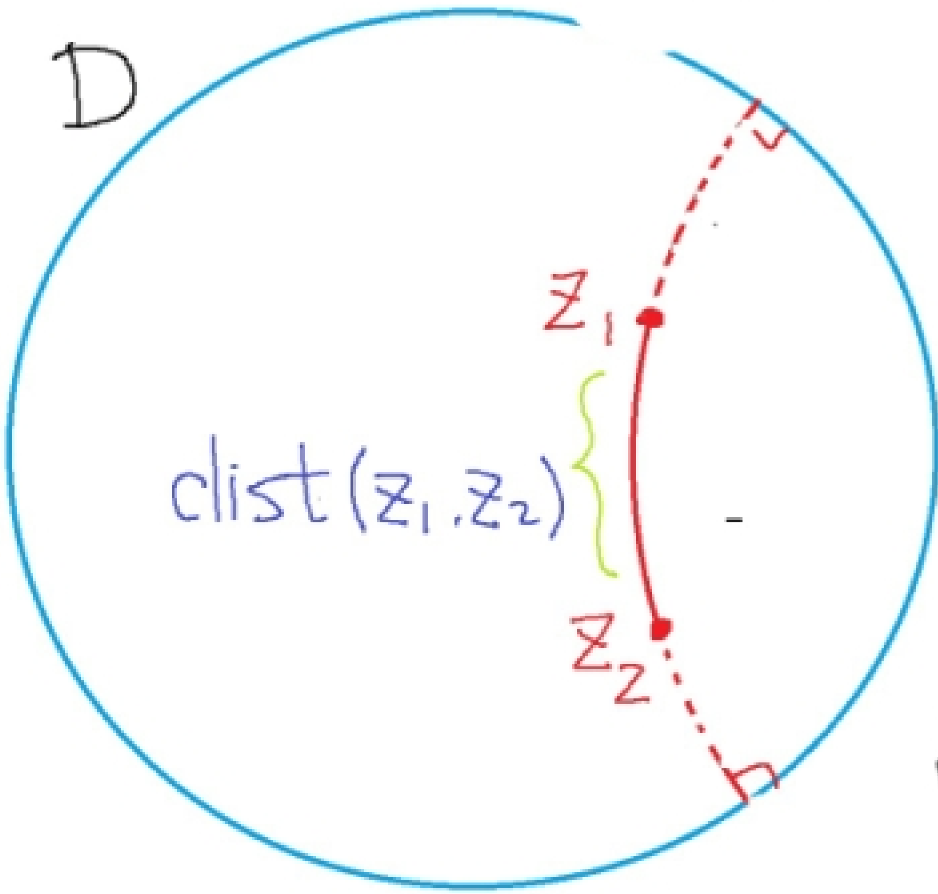
This is a joint work with
Fulton Gonzalez (Tufts University)
and Jens Christensen (Colgate University)

For simplicity, we explain our result in the case of Poincaré disk.

Poincaré disk

$$D = \{ z \in \mathbb{C} \ ; \ |z| < 1 \}$$

metric $ds^2 = \dfrac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$, $z = x + iy$



Let $r = \mathrm{dist}(z_1, z_2)$, then

$$\frac{|z_1 - z_2|}{|1 - \bar{z}_1 z_2|} = \tanh r$$

$$SU(1,1) = \left\{ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SL(2, \mathbb{C}) \right\}$$

The action of $SU(1,1)$ on D is given by

$$D \ni z \longmapsto g \cdot z = \frac{az + b}{\bar{b}z + \bar{a}} \quad \rightsquigarrow \quad D \cong SU(1,1) / S(U(1) \times U(1))$$

Laplacian Δ_D on D.

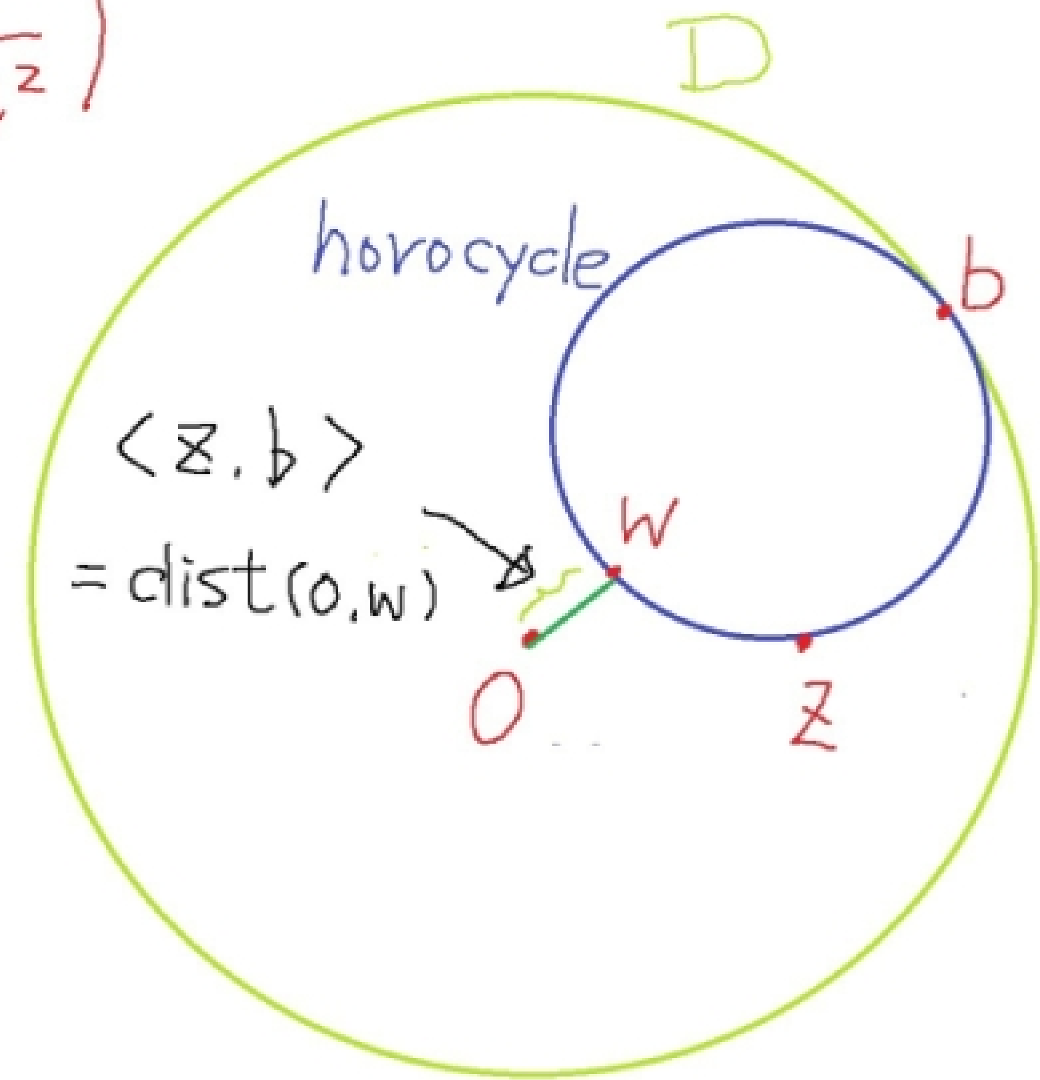
$$\Delta_D = (1-x^2-y^2)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

- gereralized eigenfunction on D

$$e^{2\langle z,b\rangle} := \frac{1-|z|^2}{|z-b|^2}$$

$$\Delta_D\left(e^{(i\lambda+1)\langle z,b\rangle}\right)$$

$$= -(\lambda^2+1)\,e^{(i\lambda+1)\langle z,b\rangle}$$



Fourier transform on D

$f \in C_0^\infty(D)$, $\lambda \in \mathbb{R}$, $b \in \partial D \cong S^1$

$$\mathcal{F}f(\lambda, b) := \int_D f(z)\, e^{(-i\lambda+1)\langle z, b\rangle} \frac{dx\,dy}{(1-|z|^2)^2}$$

$\mathcal{F}$: functions on D $\to$ functions on $\mathbb{R}\times S^1$

- Fourier inversion formula

$$f(z) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{S^1} \mathcal{F}f(\lambda, b)\, e^{(i\lambda+1)\langle z, b\rangle}$$

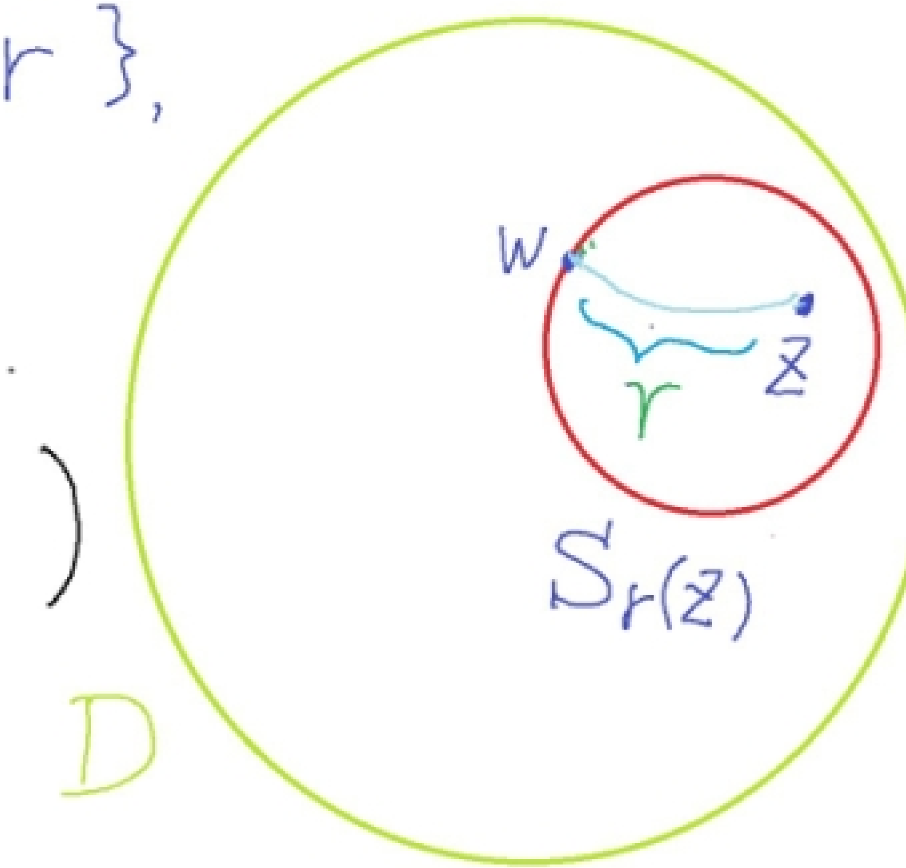
$$\times\ \lambda \tanh\frac{\pi\lambda}{2}\, db\, d\lambda$$

Mean value operator on D

$$M^r f(z) = \frac{1}{\mathrm{Vol}(S_r(z))} \int_{w \in S_r(z)} f(w)\, dS_r$$

$S_r(z) = \{ w \in D \,;\, \mathrm{dist}(w, z) = r \}$,

dS_r : canonical measure on $S_r(z)$.



__Theorem__ (Christensen-Gonzalez-K)

$M^r : C^\infty(D) \to C^\infty(D)$

is surjective.

Idea

- Write M^r as a convolution operator on D.

$$M^r f = f * \mu , \quad \mu = \frac{1}{\mathrm{Vol}(S_r)} \delta_{S_r} \in \mathcal{E}'(D)$$

- Show that $\mathcal{F}\mu(\zeta)$ is slowly decreasing.
- Apply a theorem of Ehrenpreis to this case.

Convolution operator on D

Let $G = SU(1,1)$

A function f on D can be viewed as a function on G by

$$f(z) \leftrightsquigarrow f(g\cdot 0) \qquad g\cdot 0 = z, \quad g \in SU(1,1)$$

Take two functions f_1, f_2 on D.

$$f_1 * f_2(g\cdot 0) :\underset{\text{def}}{=} \int_G f_1(h\cdot 0) f_2(h^{-1}g\cdot 0)\, dh$$

Outline

Let $S_r = S_r(0) = \{z \in D \,;\, \mathrm{dist}(z,0) = r\}$

δ_{S_r} : delta fn whose support is S_r.

$$\mu := \frac{1}{\mathrm{Vol}(S_r)} \delta_{S_r} \in \mathcal{E}'(D)$$

$\rightsquigarrow$ $M^r f(z) = f * \mu(z)$ $\quad f \in C_0^\infty(D)$

μ is rotationary invariant.

$\rightsquigarrow$ $\mathcal{F}\mu$ is independent of $b \in S^1$.

Moreover, we have

$$\mathcal{F}\mu(\lambda) = {}_2F_1\left(\begin{matrix} \frac{1}{2} + \frac{\lambda}{2} i, \ \frac{1}{2} - \frac{1}{2}\lambda i \\ 1 \end{matrix} ; -\sinh^2 r \right)$$

There is a serious problem.

$$\mathcal{F}\mu(\lambda) = {}_2F_1\left(\begin{matrix}\frac{1}{2}+\frac{1}{2}\lambda i, \ \frac{1}{2}-\frac{1}{2}\lambda i \\ 1\end{matrix}; -\sinh^2 r\right)$$

The asymptotic behavior of ${}_2F_1\left(\begin{matrix}a, b \\ c\end{matrix}; x\right)$ as a function of a. b. c is not well known.

$\Downarrow$ We solved this problem as follows!

Proposition [Christensen - Gonzalez - K]

$\mathcal{F}\mu(\lambda)$ has the following Bessel function expansion.

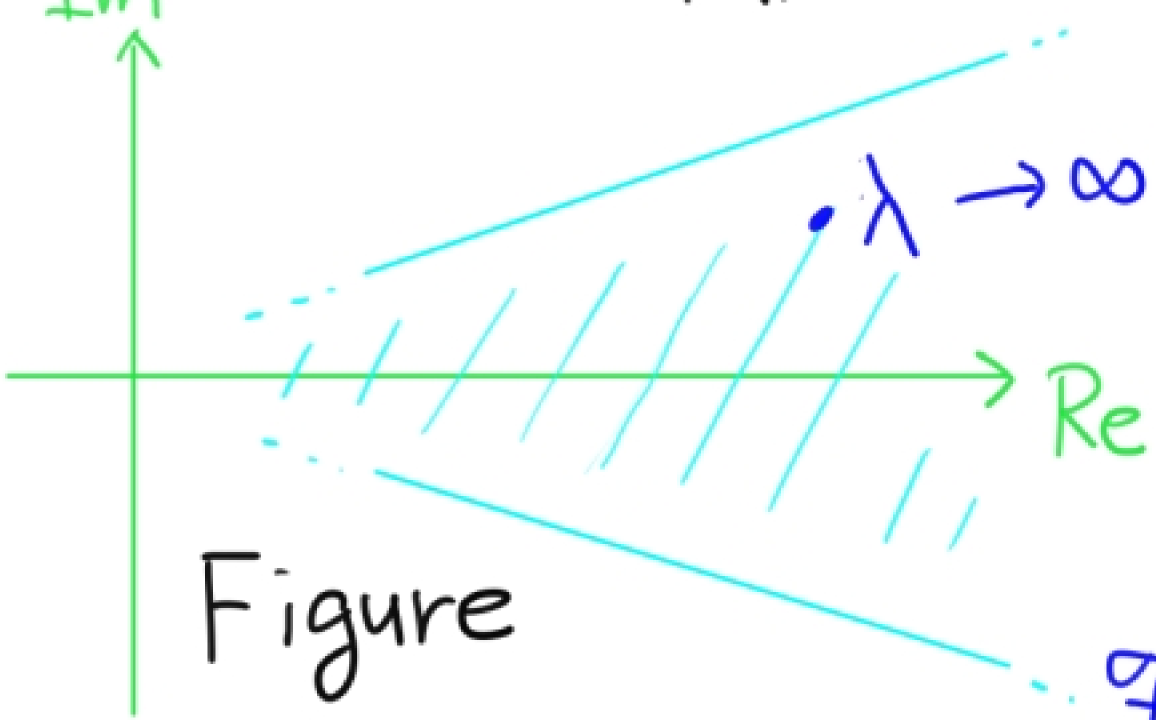
$$\mathcal{F}\mu(\lambda) \sim \sum_{m=0}^{\infty} d_m \left(\frac{r}{\lambda}\right)^m J_m(r\lambda) \quad \text{as } |\lambda| \longrightarrow +\infty$$

$$d_0 = \frac{\pi}{2} \times \left(\sinh\sqrt{r}/\sqrt{r}\right)^{-\frac{1}{2}}, \quad J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(m+k)!} \left(\frac{x}{2}\right)^{2k+m}$$

In particular,

$$\mathcal{F}\mu(\lambda) \sim d_0 J_0(r\lambda), \quad d_0 = \frac{\pi}{2}\left(\frac{\sinh\sqrt{r}}{\sqrt{r}}\right)^{-\frac{1}{2}}$$

as $|\lambda| \to \infty$



Figure

The above expansion is valid in the angular domain of the figure.

$\Downarrow$

$\mathcal{F}\mu(\lambda)$ is slowly decreasing.

$\Downarrow$

$M^r : C^\infty(D) \to C^\infty(D)$ is surjective.

Problem Is the mean value operator $M^r : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ injective?

Answer No.

We construct $f \in \operatorname{Ker} M^r$ as follows.

Take $\xi \in \mathbb{R}^n$ such that $j_{\frac{1}{2}n-1}(r|\xi|) = 0$.

Here $$j_\nu(z) = \Gamma(\nu+1) \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\,\Gamma(k+\nu+1)}$$ normalized Bessel function

Fact $j_\nu(x) \sim C_\nu |x|^{-\frac{\nu+1}{2}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \quad (|x| \to \infty)$

In particular, j_ν has infinitely many zeros.

Let $f(x) = e^{i\xi\cdot x}$. Then we have

$$M^r f(x) = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{\omega\in S^{n-1}} e^{i\xi\cdot(x+r\omega)}\, dS_\omega$$

$$= e^{i\xi\cdot x} \times \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{\omega\in S^{n-1}} e^{ir\xi\cdot\omega}\, dS_\omega$$

$$= e^{i\xi\cdot x} \times j_{\frac{1}{2}n-1}(r|\xi|) = 0.$$

We take $\xi \in \mathbb{R}^n$ such that $r|\xi|$ is one of the zeros of $j_{\frac{1}{2}n-1}$.

In particular, $\dim \operatorname{Ker} M^r = \infty$.

We have constructed $f \in \mathrm{Ker}\, M^r$ by using zeros of j_ν.

<u>Question</u> Is any $f \in \mathrm{Ker}\, M^r$ obtained in such a way?

<u>Answer</u> Yes!

Kernel of M^r

Let $\{\lambda_i\}_{i=1}^{\infty}$ $(0<\lambda_1<\lambda_2<\cdots)$ be the zeros of $j_{\frac{1}{2}n-1}(r\lambda)=0$.

Let $\mathcal{E}_\lambda = \{\varphi \in C^\infty(\mathbb{R}^n) ; \Delta_{\mathbb{R}^n}\varphi = -\lambda^2\varphi\}$

Then we have

<u>Theorem</u> $\operatorname{Ker} M^r = \overline{\bigoplus_{i=1}^{\infty} \mathcal{E}_{\lambda_i}}$

<u>Rem</u> Note that if $f \in \mathcal{E}_\lambda$ is rotationary invariant,

$f(x) = \text{const} \times j_{\frac{1}{2}n-1}(\lambda|x|)$.

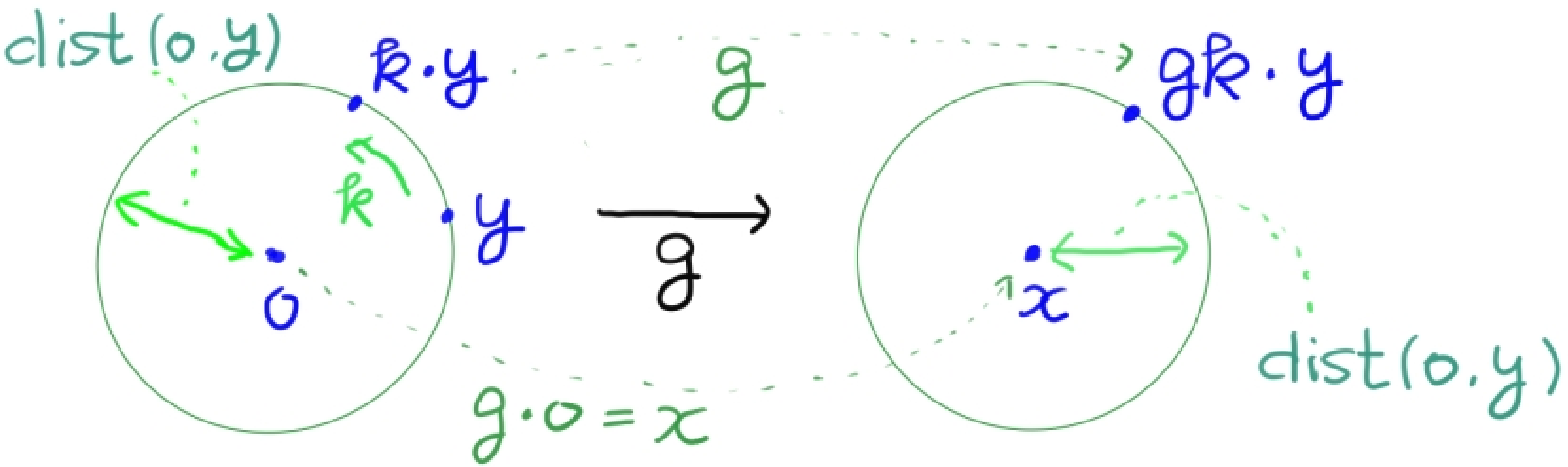
Mean value operators on general noncompact symmetric spaces.

$X = G/K$: noncompact symmetric space

Fix $y \in X$. The mean value operator M^y is defined by

$$M^y f(x) = \int_{k \in K} f(gk \cdot y)\, dk \quad (x = g \cdot o \in X)$$

dk : the normalized Haar measure on K.



$$M^y f(x) = \int_{k\in K} f(gk\cdot y)\,dk \quad (x = g\cdot o \in X)$$

M^y takes the average of f on the "sphere" in X of radius $d(o,y)$ and center $x = g\cdot o$.

Rem In the higher rank case, $y \in X$ can be taken as an element of positive Weyl chamber.

Rem If $X = D$ and if $0 < y < 1$, $y = \tanh r$.

rank one case $r > 0$

$\Updownarrow$

higher rank case $y \in$ "positive Weyl chamber"

Our old result (JFA 2017)

- dist(o, y) is sufficiently large
- y is away from Weyl wall

$\Rightarrow$ $M^y : C^\infty(X) \to C^\infty(X)$ is surjective

Our new result (in preparation)

- For "almost all" y

  $M^y : C^\infty(X) \to C^\infty(X)$ is surjective.

References

(1) Christensen, Gonzalez, and K

"Surjectivity of mean value operators on noncompact symmetric spaces"

J. Functional Analysis, Vol. 272 (2017)

rank 1 case,

higher rank case with some strong assumption

(2) K-T, Lim

"The spherical mean value operators on Euclidean and hyperbolic spaces"

Ph.D. Thesis (2012)

in the case of $\mathbb{R}^n$ and $\mathbb{H}^3$

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