A generalized conservation property for the heat semigroup on weighted manifolds

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Set up

- (M, g, μ) a weighted manifold with density $\Psi > 0$, that is $d\mu = \Psi dx_g$
- Weighted Laplacian:

$$\Delta f = \frac{1}{\Psi} \mathrm{div} \left(\Psi \nabla f \right)$$

- Sobolev space:

$$H_0^1 = \overline{C_c^\infty(M)}^{H^1}, \quad (u, v)_{H^1} = \int_M uv \, d\mu + \int_M g(\nabla u, \nabla v) \, d\mu$$

- Semigroup: $\exp(t\Delta) : L^2(M, \mu) \to L^2(M, \mu)$

$$u_t := \exp(t\Delta) f \text{ solves } (\partial_t - \Delta) u_t = 0 \text{ and } u_t \to f \text{ as } t \to 0$$

with $D(\Delta) := \{u \in H_0^1 \mid \Delta u \in L^2(M, \mu)\}$

- $\exp(t\Delta)$ extends uniquely to $T_t : L^\infty(M, \mu) \to L^\infty(M, \mu)$

Set up

We say that T_t (or M) is *conservative* ((C) for short) iff

$$T_t 1(x) \equiv 1, \quad \forall t > 0, \forall x \in M \tag{C}$$

- $\mathbb{R}^n$ is (C) (the integral of the Gaussian distribution equals to 1)
- $\Omega \subset \mathbb{R}^n$ domain is not (C) (Dirichlet boundary condition)
- $M = \mathbb{R}^n \setminus K$, where K is closed, then TFAE
  - (i) M is (C)
  - (ii) $\mathrm{cap}(K) = 0$, where

$$\mathrm{cap}(K) = \inf\{\|u\|_{H^1} \mid u \in H^1(\mathbb{R}^n),\ u|_K \geq 1\}$$

  - (iii) $H^1_0(M) = H^1(M)$

Set up

All manifolds in this slide are complete. $v(r) = \mu(B_x(r))$

Theorem 1.1 (Yau 1978)

$$\inf_{x \in M} Ric(x) > -\infty \implies (C)$$

Theorem 1.2 (Davies 1992, Takeda 1989)

$$v(r) \leq Ce^{r^2} \implies (C)$$

Remark:

$$v(r) = e^{r^{2+\delta}}, \quad \delta > 0 \not\Longrightarrow (C)$$

Theorem 1.3 (Grigoryan 1986)

$$\int^{\infty} \frac{rdr}{\log v(r)} = \infty \implies (C)$$

Set up

Important criteria for the conservation property uses "Liouville type property"

$$\begin{cases} (\partial t - \Delta)u = 0 \\ u|_{t=0} \equiv 0 \end{cases} \quad \text{with} \quad u \in L^\infty \implies u \equiv 0 \qquad \text{(K)}$$

Theorem 1.4 (Khasminskii, 1960)

$$(C) \iff (K)$$

- Khasminskii's test extends to a much wider class of operators including Dirichlet forms without killing inside.

Set up

Consider

$$L = \Delta - V, \qquad V \in C^\infty(M),\ V \geq 0$$

Recall

$$V(x) > 0 \text{ at } x \in M \implies e^{tL}1(x) < 1$$

Goal 1

To establish a conservation property in a more general sense which is characterized by Khasminskii's criteria for L.

In discrete setting, M. Keller and D. Lenz studied this problem in "Dirichlet forms and stochastic completeness of graphs and subgraphs. J. Reine Angew. Math., 2012.

Set up (Rather unusual)

- M a weighted manifold with weighted Laplacian $\Delta = \frac{1}{\Psi}(\mathrm{div}(\Psi\nabla))$
- With a strictly positive $\rho \in C^\infty(M)$,

$$L_{\rho,V} := \frac{1}{\rho}(\Delta - V)$$

$\implies$ $L_{\rho,V}$ restricted to $C_c^\infty(M)$ is symmetric in $L^2(M, \rho\mu)$.

- Sobolev spaces:

$$H^1(M,\rho) := \{f \in L^2(M,\rho\mu) \mid \nabla f \in L^2(TM,\mu)\}$$

with

$$(u,v)_{H^1} := \int_M g(\nabla u, \nabla v)\, d\mu + \int_M uv\, \rho d\mu$$

$$H_0^1(M,\rho) := \overline{C_c^\infty(M)}^{H^1(M,\rho)}$$

Set up

- Dirhclet form $(\mathcal{E}_{\rho,V}, D(\mathcal{E}_{\rho,V}))$ in $L^2(M,\rho)$:

$$D(\mathcal{E}_{\rho,V}) := H_0^1(M, V+\rho) \subset L^2(M, V+\rho) \subset L^2(M,\rho)$$

$$\mathcal{E}_{\rho,V}(u,v) := \int_M g(\nabla u, \nabla v)\, d\mu + \int_M uvV\, d\mu, \quad \forall u,v \in D(\mathcal{E}_{\rho,V})$$

- Generator $L_{\rho,V}$ in $L^2(M,\rho\mu)$:

$$\mathcal{E}_{\rho,V}(f,\psi) = (-L_{\rho,V} f, \psi)_{L^2(M,\rho\mu)}, \quad \forall f \in D(L_{\rho,V}), \psi \in C_c^\infty(M)$$

Set up

- Markovian Semigroup and Markovian Resovelent

$$T_t^{\rho,V} := \exp(tL_{\rho,V}) : L^\infty(M) \to L^\infty(M), \quad t > 0$$

$$G_\alpha^{\rho,V} := (\alpha - L_{\rho,V})^{-1} : L^\infty(M) \to L^\infty(M), \quad \alpha > 0$$

Definition 1.1 (Conservative in the generalized sense (CG))

With $\hat{V} = V/\rho$, let

$$H_t := T_t^{\rho,V} 1 + \int_0^t T_s^{\rho,V} \hat{V} \, ds$$

We say $L_{\rho,V}$ in $L^2(M,\rho)$ is conservative in the generalized sense (in short, CG) iff

$$H_t(x) = 1, \quad \forall t > 0, x \in M$$

Results

Denote $H = H_t$ and $\mathcal{C} := C^\infty((0,\infty)\times M)\cap C([0,\infty)\times M)$

Theorem 2.1

$H \in \mathcal{C}$, $0 \le H \le 1$, and

$$\begin{cases} (\partial t - L_{\rho,V})H = \hat{V}, & (0,\infty)\times M \\ H \equiv 1, & \{0\}\times M. \end{cases}$$

Furthermore, $1 - H$ is the largest function $u \in \mathcal{C}$ with $u \le 1$ and

$$\begin{cases} (\partial t - L_{\rho,V})u \le 0, & (0,\infty)\times M \\ u \equiv 0, & \{0\}\times M. \end{cases}$$

Results

Set

$$N_\alpha := \alpha G_\alpha^{\rho,V} 1 + G_\alpha^{\rho,V} \hat{V}, \quad \alpha > 0$$

Theorem 2.2

$$N_\alpha = \int_0^\infty \alpha e^{-t\alpha} H_t \, dt$$

$N_\alpha \in C^\infty(M)$, $0 \leq N_\alpha \leq 1$, and

$$(\alpha - L_{\rho,V}) N_\alpha = \alpha 1 + \hat{V}$$

Furthermore, $1 - N_\alpha$ is the largest function $f \in C^\infty(M)$ with $f \leq 1$ and

$$(\alpha - L_{\rho,V}) f \leq 0$$

Results

Theorem 2.3 (Khasminskii's test)

The following assertions are equivalent.

(i) $L_{\rho,V}$ *is (CG).*

(ii) $\forall/\exists\alpha > 0$, $(\alpha - L_{\rho,V})f = 0$, $f \in C_b^\infty(M) \implies f \equiv 0$.

(iii) $\forall/\exists\alpha > 0$, $(\alpha - L_{\rho,V})f \leq 0$, $f \in C_b^\infty(M)$, $f \geq 0 \implies f \equiv 0$.

(iv)

$$\begin{cases} (\partial t - L_{\rho,V})u = 0, & (0,\infty) \times M, \\ u(0,\cdot) \equiv 0 \end{cases}, \quad u \in \mathcal{C}_b \implies u \equiv 0$$

(v)

$$\begin{cases} (\partial t - L_{\rho,V})u \leq 0, & (0,\infty) \times M, \\ u(0,\cdot) \equiv 0 \end{cases}, \quad u \in \mathcal{C}_b,\ u \geq 0 \implies u \equiv 0$$

Results

Corollary 2.1

(i) $L_{\rho,0}$ *is (C)* $\implies$ $L_{\rho,V}$ *is (CG) with any* $V \in C_+(M)$.

(ii) $L_{\rho,V}$ *in* $L^2(M,\rho)$ *is (CG)* $\iff$ $L_{\rho+V,0}$ *in* $L^2(M,\rho+V)$ *is (C).*

Corollary 2.2

Of the following $(i) \implies (ii)$. *If* $\rho+V \in L^1(M,\mu)$ *then* $(i) \iff (ii)$.

(i) $L_{\rho,V}$ *in* $L^2(M,\rho)$ *is (CG)*

(ii) $H^1(M,\rho+V) = H^1_0(M,\rho+V)$

Results

Corollary 2.3

Let M be a complete weighted manifold M.

- *M admits V such that $L_{\rho,V}$ is (CG).*
- *In particular, if M is spherically symmetric (model manifold) with radially symmetric ρ and V and if$\mu = vol_g$, then TFAE*
  - (i) *$L_{\rho,V}$ in $L^2(M, \rho vol_g)$ is (CG).*
  - (ii)

$$\int^{\infty} \frac{v(r)}{s(r)} dr + \int^{\infty} \frac{\int_0^r s(\xi) V(\xi) d\xi}{s(r)} dr = \infty$$

where $v(r)$ is the volume of $B(0, r)$ and $s(r) = \frac{d}{dr} v(r)$.

Key Lemmas

Lemma 1

The space $W_0^1(M, \rho + V)$ *is an order ideal in* $W^1(M, \rho + V)$*, i.e., for* $f \in W_0^1(M, \rho + V)$, $g \in W^1(M, \rho + V)$ *the inequality* $|g| \leq |f|$ *implies* $g \in W_0^1(M, \rho + V)$.

We say $f \in W^1(M, \rho + V)$ satisfies

$$f \geq 0 \qquad \mod W_0^1(M, \rho + V)$$

if there exists $g \in W_0^1(M, \rho + V)$ such that $f + g \geq 0$.

Lemma 2

A function $f \in W^1(M, \rho + V)$ *satisfies* $f \geq 0 \mod W_0^1(M, \rho + V)$ *if and only if* $f_- \in W_0^1(M, \rho + V)$.

Lemma 3 (Parabolic maximum principle)

Let $T \in (0, \infty]$ *and* $v : (0, T) \to W^1(M, \rho + V)$ *be a path with the following properties*

1. $\partial_t v$ *exists as a strong limit in* $L^2(M, \rho\mu)$,
2. $v(t)_- \to 0$ *in* $L^2(M, \rho\mu)$ *as* $t \to 0+$,
3. *for every* $t \in (0, T)$, $v(t) \geq 0, \quad \mathrm{mod}\ W_0^1(M, \rho + V)$,
4. *for every* $s \in (0, T)$, $\partial_t v(s) \geq L_{\rho,V}(v(s))$ *as distributions.*

Then $v \geq 0$ *on* $(0, T) \times M$

Lemma 4 (Elliptic maximum principle)

Let $f \in W^1(M, \rho + V)$ *with*

$$f \geq 0 \quad \mathrm{mod}\ W_0^1(M, \rho + V).$$

If for some $\alpha > 0$ *it satisfies* $(\alpha - L)f \geq 0$, *then* $f \geq 0$.

We denote by (T_t^Ω) the $L^2(\Omega, \rho\mu)$-semigroup of the Dirichlet form

$$\mathcal{E}^{\Omega}_{\rho,V}(f,g) = \int_\Omega \nabla f \cdot \nabla g \, d\mu + \int_\Omega V f g \, d\mu$$

with domain $D(E^\Omega_{\rho,V}) = W^1_0(\Omega, \rho + V)$. We extend it to $f \in L^2(M, \rho\mu)$ by

$$T_t^\Omega f := \begin{cases} T_t^\Omega f|_\Omega & \text{on } \Omega, \\ 0 & \text{on } M \setminus \Omega. \end{cases}$$

Lemma 5 (Duhamel's principle)

Let $\Omega \subset M$ be an open relatively compact subset and let $f \in C^\infty(M)$. Let $u \in \mathcal{C}$ nonnegative. If $\partial_t u - Lu \geq f$ on $(0,\infty) \times \Omega$, then

$$u_t \geq T_t^\Omega u_0 + \int_0^t T_s^\Omega f ds.$$

Lemma 6

Let $e, f \in C_c^\infty(M)$ *and let* $u : (0,\infty) \to L^2(M, \rho\mu)$ *given by*

$$u(t) = T_t^{\rho,V} e + \int_0^t T_s^{\rho,V} f ds.$$

For each $t > 0$ *we have* $u(t) \in C^\infty(M)$*. Moreover, the function* $\tilde{u} : (0,\infty) \times M \to \mathbb{R}$, $(t,x) \mapsto u(t)(x)$ *belongs to* $\mathcal{C}$ *and satisfies*

$$\begin{cases} (\partial_t - L)\tilde{u} = f, \\ \tilde{u}_0 = e. \end{cases}$$

Now we are ready to prove Theorems 2.1 and 2.2.

Theorem 2.3 (Khasminskii's test) is proved based on Theorems 2.1 and 2.2.

Items in the proof of Corollary 2.3.

- Suffices to prove that $L_{\rho+V,0}$ is (C) in $L^2(M, \rho + V)$.
- New distance $d_{\rho+V}$ is defined as

$$d_{\rho+V} = \sup\{f(x) - f(y) \mid f \in H^1_{\mathrm{loc}}(M) \cap C(M) \text{ with } |\nabla f| \leq \rho + V\}$$

- (M, d) complete $\implies$ $(M, d_{\rho+V})$ complete
- Apply the Grigoryan-Strum volume growth criteria for a regular local Dirichlet form with a complete intrinsic distance:

$$\int_1^\infty \frac{rdr}{\max\{\log \mu(r), 1\}} = \infty \implies (C)$$