Semiclassical measures for Schrödinger operators with homogeneous potentials of order zero

Keita Mikami

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Egret Himeji

Introduction

Consider $P = -\triangle + V$ on $\mathbb{R}^n$.

$$
\begin{aligned}
&V(x) \text{ is potentials of order zero} \\
&\Leftrightarrow V(kx) = V(x) \quad \text{for} \quad k, |x| \geq 1 \\
&\Leftrightarrow \partial r V(x) = 0 \quad \text{for} \quad r \geq 1 \quad \text{where} \quad r = \left|x\right| \\
&\Leftrightarrow V(r, \theta) = V(\theta) \quad \text{for} \quad r \geq 1
\end{aligned}
$$

Assumption

Let $P = -\triangle + V$.

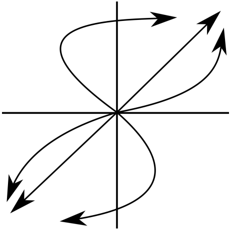
Assumption

(1) V is real valued and smooth.
(2) We can decompose V as $V = V_\infty + V_s$, where V_∞ is real valued and homogeneous of order zero i.e. V satisfies $V_\infty(x) = V_\infty(\frac{x}{|x|})$ for $|x| \geq 1$ and $V_s(x) = o(|x|^{-1})$ as $|x| \to \infty$.

Motivation

Localization in direction of Hamiltonian flow ('91 Herbst).
Let $(x(t), \xi(t))$
be a solution to the Hamiltonian equation, i.e.

$$\dot{x}(t) = \xi(t)$$
$$\dot{\xi}(t) = -\partial_x V(x)$$



Then $\frac{x(t)}{|x(t)|} \to \theta_\infty \in \mathrm{Cr}(V_\infty)$ as $t \to \infty$.

Motivation

Localization in direction of Schrödinger operators('91 Herbst, '08 Herbst-Skibsted, '04,'08 Hassell-Melrose-Vasy).
We define $\mathcal{H}_\theta$, a space of functions localizes in $\theta \in S^{n-1}$ by

$$\mathcal{H}_\theta = \{\varphi \in L^2(\mathbb{R}^n) \mid \left(\frac{x}{|x|} - \theta\right) e^{-itP}\varphi \to 0 \text{ as t} \to \infty\}.$$

Theorem

Suppose V_∞ has finite critical points. Then there exists family $\{\theta_m\}_{m=1}^M$ of critical points of V_∞ such that $\mathcal{H}_{a.c}(P) = \oplus_{m=1}^M \mathcal{H}_{\theta_m}$.

Question

Can we formulate localization in direction of Schrödinger operators in terms of microlocal(semiclassical) analysis?

Semiclassical Measures

Consider following Quasimodes problem:

$$\begin{cases} (P_h - E)u_h = R_h \\ \|u_h\|_{L^2(\mathbb{R}^n)} = 1, \end{cases}$$

where $R_h \to 0$ as $h \to 0$ and $P_h = -h^2 \triangle + V$.

Semiclassical (defect) measure

There exists a sequence of positive number h_m and a finite Radon measure μ such that $h_m \to 0$ as $m \to \infty$ and

$$\langle u_{h_m}, a^w(x, hD_x)u_{h_m}\rangle_{L^2(\mathbb{R}^n)} \to \int_{T^*\mathbb{R}^n} a d\mu \text{ as } m \to \infty,$$

Semiclassical Measures

Semiclassical measure μ satisfies

- $\text{supp}\mu \subset \{(x, \xi) \in T^*\mathbb{R}^n \mid |\xi|^2 + V(x) - E = 0\}$
- If $\|R_h\| = o(h)$, μ is invariant under Hamiltonian flow.

Idea

Understand localization in direction as a property of the support of semiclassical measure.

Difficulty and Key idea

Difficulty

$\mu = 0$ if P_h is non-trapping!

Key idea

Instead of taking "energy" to infinity, we take "position" to infinity,

$=$ defining a new quantization.

Definition of new quantization

Let $j(r) = 1$ if $r \geq 1, = 0$ if $r \leq \dfrac{1}{2}$.

For $a \in C_0^\infty(\mathbb{R} \times T^*S^{n-1}), \tilde{a}(x, \xi) = j(r)a\left(\rho, \theta, \dfrac{\eta}{r}\right) \in S(1)$ i.e.

$\forall \alpha, \beta \in \mathbb{N}^n, \sup_{(x,\xi)\in T^*\mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \tilde{a}(x, \xi)| < \infty.$

We write $Op_j(a) = \tilde{a}(hx, D_x)$.
Note
We consider $T^*\mathbb{R}^n$ as $T^*\mathbb{R}_{>0(r,\rho)} \times T^*S^{n-1}_{(\theta,\eta)}$ via polar coordinate and ignore r variable.

Definition of new semiclassical measures

We write $Op_j(a) = \tilde{a}(hx, D_x)$.

Theorem

For any bounded family v_h in $L^2(\mathbb{R}^n)$, there exist h_m and a finite Radon measure μ_j such that $h_m \to 0$ as $m \to \infty$ and

$$\langle v_{h_m}, Op_j(a) v_{h_m} \rangle_{L^2(\mathbb{R}^n)} \to \int_{\mathbb{R}\times T^*S^{n-1}} a \mathrm{d}\mu_j \text{ as } m \to \infty,$$

for all $a \in C_0^\infty(\mathbb{R} \times T^*S^{n-1})$.

Assumption (Revisit)

Let $P = -\triangle + V$.

Assumption

(1) V is real valued and smooth.
(2) We can decompose V as $V = V_\infty + V_s$, where V_∞ is real valued and homogeneous of order zero i.e. V satisfies $V_\infty(x) = V_\infty(\frac{x}{|x|})$ for $|x| \geq 1$ and $V_s(x) = o(|x|^{-1})$ as $|x| \to \infty$.

Main Theorem

We consider following asymptotic eigenvalue problem:

$$\begin{cases} (P-E)u_h = R_h \\ \|u_h\|_{L^2(\mathbb{R}^n)} = 1, \end{cases} \tag{1}$$

where $R_h \to 0$ as $h \to 0$.

Theorem

Let $\|R_h\|_{L^2(\mathbb{R}^n)} = o(h)$ as $h \to 0$. Assume there exists
$\chi \in C_0^\infty((1,\infty))$ such that
$u_h(x) = \chi(h|x|)u_h(x) +$ "*error*".
Then one can prove the following:
(1) $E \in \mathrm{Cv}(V_\infty)$.
(2) $\mathrm{supp}(\mu_j) \subset \{(0,\theta,0) \in \mathbb{R} \times T^*S^{n-1} \mid \theta \in \mathrm{Cr}(V_\infty) \cap V_\infty^{-1}(E)\}$.

Sketch of the proof for main theorem

Usual commutator argument yields $\int (ae^{2\rho t}) \circ \Phi_t \mathrm{d}\mu_j$ is is independent of t, where Φ_t is "Hamiltonian" flow for any $a \in C_0^\infty(\mathbb{R} \times T^* S^{n-1})$.
$\Longrightarrow E \in \mathrm{Cv}(V)$ and localization follows.

Remarks on main theorem

- Assumption of main theorem yields we treat modes with $|x| \sim h^{-1}$.
  Thus this theorem implies as far as we consider modes with $|x| \sim h^{-1}$, they localize in direction.
- Actually, we can construct modes with with $|x| \sim h^{-2}$ which localizes in direction of regular points.

Examples

Theorem

(A) Let $E \in [\min(V_\infty), \max(V_\infty)]$, $\theta_0 \in V_\infty^{-1}(E) \subset S^{n-1}$ and $k \in \mathbb{N}^n \cup \{0\}$ be such that $\partial_\theta^{\tilde{k}} V_\infty(\theta_0) = 0$ for any $0 < |\tilde{k}| \leq |k|$. For any $C > 0$, there exists u_h a solution of (1) such that

1. $\|R_h\|_{L^2(\mathbb{R}^n)} = o(h)$ if $k > 1$ and $\|R_h\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$ if $k = 0, 1$ as $h \to 0$,
2. $j(hr)u_h(r,\theta) = u_h(r,\theta)$ further, $j(h^2 r)u_h(r,\theta) = u_h(r,\theta)$ if $k = 0$,
3. $\operatorname{supp}(u_h) \subset \{(r,\theta) \in \mathbb{R}^n \mid r > 1, \operatorname{dist}(\theta,\theta_0) < Cr^{-\ell(k)}\}$ for sufficiently small $h > 0$,

where $\ell(k)$ is such that $\ell(k) = k + 1$ if $k > 0$ and $\ell(0) = \frac{2}{3}$. Condition 3 yields $\operatorname{supp}\mu_j = \{(0, \theta_0, 0)\}$.

Examples

Theorem

(B) Let $\max(V_\infty) < E$, $\theta_0 \in S^{n-1}$. For any $C, \varepsilon > 0$, there exists u_h a solution of (1) such that

1. $\|R_h\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$ as $h \to 0$,
2. $j(hr)u_h(r,\theta) = u_h(r,\theta)$,
3. $\operatorname{supp}(u_h) \subset \{(r,\theta) \in \mathbb{R}^n \mid r > 1, \operatorname{dist}(\theta,\theta_0) < Cr^{-\ell(k)}\}$ for sufficiently small $h > 0$,

where $\ell(k)$ is the same with (1).
Condition 3 and yields $\operatorname{supp}\mu_j = \{(\rho,\theta_0,0) \mid \rho^2 + V(\theta_0) = E\}$.

Sketch of the construction

Construct $f_h \in C^\infty(\mathbb{R})$ such that

- $j(hr)f_h(r) = f_h(r)$,
- $\|(\partial_r^2 + \frac{n-1}{r}\partial_r)f_h\|_{L^2(\mathbb{R}:r^{n-1}\mathrm{d}r)} = o(h)$ as $h \to 0$.

Construct $g_h \in C^\infty(\mathbb{R})$ such that

- There exists $C > 0$ such that $|V(\theta) - E| \leq Ch^{\ell(k)}$ on $\mathrm{supp}(g_h)$.

$u_h(r,\theta) = C_h f_h(r) g_h(\theta)$ with normalizing constant C_h is what we want.

Observability

Let $\Omega \subset \mathbb{R}^n$, we say observability holds on Ω if for some $T > 0$ there exists $C_{\Omega,T} > 0$ such that

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C_{\Omega,T} \int_0^T \int_\Omega |e^{-itP} u(x)|^2 \mathrm{d}x\mathrm{d}t$$

for any $u \in L^2(\mathbb{R}^n)$.

Observability

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a domain such that

$$\Omega \cap \{x \in \mathbb{R}^n \mid |x| > R\} \subset \mathbb{R}^n \setminus \{(r, \theta) \in \mathbb{R}^n \mid r > R, \mathrm{dist}(\theta, \theta_0) < Cr^{-\ell(k)}\}$$

for some $R, C > 0$ and $\theta_0 \in S^{n-1}$ with $\partial_\theta^{\tilde{k}} V(\theta_0) = 0$ for any $\tilde{k} \leq k$.
Then the observability on Ω fails for any $T > 0$, i.e., there exists $u_m \in L^2(\mathbb{R}^n)$ such that $\|u_m\|_{L^2(\mathbb{R}^n)} = 1$ and $\int_0^T \int_\Omega |e^{-itP} u_m(x)|^2 \mathrm{d}x\mathrm{d}t \to 0$ as $m \to \infty$.