Quantum search on simplicial complexes

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joint work with
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Outline

1. Aim of this talk and ‘What’s is quantum walk?’
2. Quick review of Simplicial Complex
   - n-simplex and its face
   - Simplicial Complex
   - Direction of simplex and induced direction of face
3. Quantum walk on Simplicial Complex
   - Quantum walk on graph Γ, revisited
   - Definition of QW on SC
4. QW on SC and bipartite walk
5. Application: Quantum search on SC
6. Conclusion and references

Outline

Aim of this talk 1/2

1. Quantum walk is a quantum analogue of random walk.
2. QW is defined on discrete graph Γ.
3. QW has impressive features:
   1. linear spreading,
   2. localization,
   3. (quantum tunneling [Matue, Matsuoka, O, Segawa, 2018].)
4. These seem to depend on the geometrical structure of Γ.

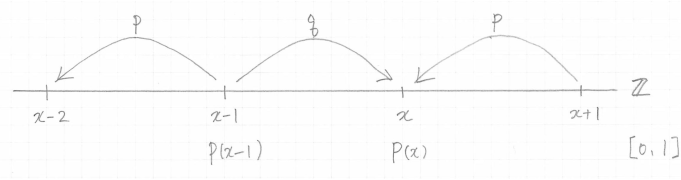
Our AIM is to propose QW on simplicial complex.

1. We discuss relation between QW on SC and bipartite walk.
2. We give an application, Glover search on simplicial complex.

What's quantum walk? 2/2

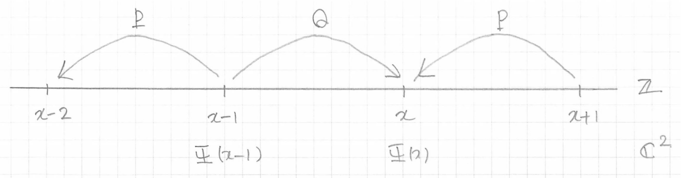
Random walk: $P(x, n+1) = qP(x-1, n) + pP(x+1, n) \in [0, 1]$

$$p \geq 0, \; q \geq 0, \; p + q = 1$$



Quantum walk: $\Psi(x, n+1) = Q\Psi(x-1, n) + P\Psi(x+1, n) \in \mathbb{C}^2$

$$U = P + Q \in U(2)$$



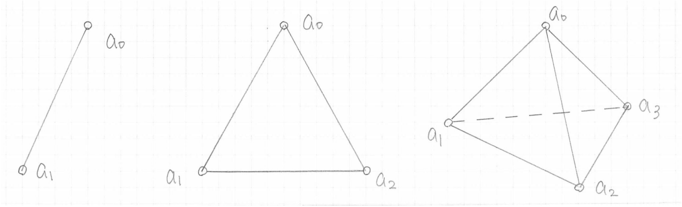
QW has probability amplitude Ψ; the distribution $P(x) = \|\Psi(x)\|^2_{\mathbb{C}^2}$.

Outline

n-simplex 1/4

$$|\sigma| := \left\{ \sum_{k=1}^{n} \tau_k \overrightarrow{a_0 a_k}\ ;\ \tau_k \geq 0, \quad \sum_{k=1}^{n} \tau_k \leq 1 \right\}, \qquad a_i \in \mathbb{R}^D.$$

$|a_0 a_1|$ $\qquad$ $|\tau| := |a_0 a_1 a_2|$ $\qquad$ $|\sigma| := |a_0 a_1 a_2 a_3|$



1-simplex $\qquad$ 2-simplex $\qquad$ 3-simplex

Note: Consider $|\sigma|$ is an unordered sequence. e.g. $|a_0 a_1| = |a_1 a_0|$.

Definition:

We say $|\tau|$ is a face of $|\sigma|$ iff $|\tau| \subset |\sigma|$, and write $|\tau| \prec |\sigma|$.

Simplicial Complex 2/4

Definition: $|K|$ is a simplicial complex iff

1. $|K|$ is a set of simplices,
2. if $|\tau| \prec |\sigma| \in |K|$, then $|\tau| \in |K|$,
3. if $|\tau| := |\sigma| \cap |\sigma'| \neq \emptyset$, then $|\tau| \prec |\sigma|$ and $|\tau| \prec |\sigma'|$.

Direction of simplex

Let

$$|K_k| := \{|\sigma| \in |K|;\ |\sigma| \text{ is k-simplex}\}, \quad \text{(unordered sequences)}$$

and

$$K_k := \{\pi(a);\ \pi \in S_{k+1},\ |a| \in |K_k|\}. \quad \text{(ordered sequences)}$$

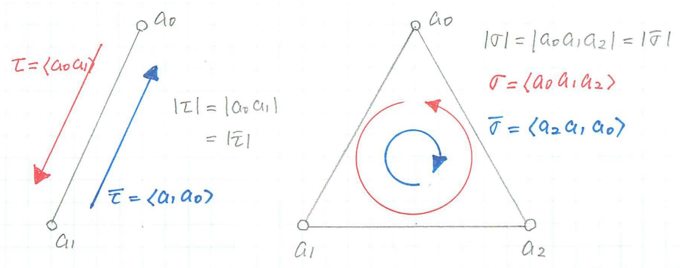
Here, $\pi(a) := (a_{\pi(0)}, a_{\pi(1)}, \ldots, a_{\pi(k)})$
for $|a| = |a_0 \ldots a_k| \in |K_k|$ and $\pi \in S_{k+1}$.

Direction of simplex 3/4

Define $\sigma \sim \sigma'$ on K_k iff $\exists \pi \in A_{k+1}$ s.t. $\pi(\sigma) = \pi(\sigma')$ and

$$\langle K_k \rangle := K_k/_\sim \; = \{\langle a_0 a_1 \dots a_k \rangle;\; a \in K_k\}.$$

Note that $\#\langle K_k \rangle = 2$ $(k \geq 2)$.

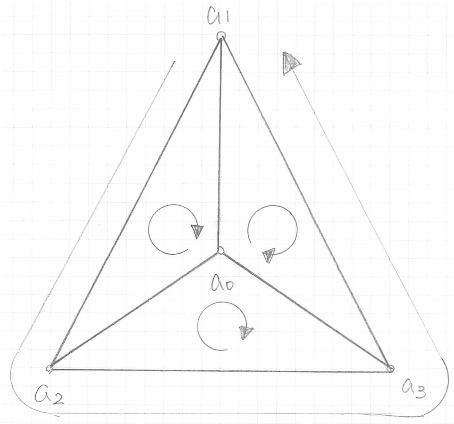


Remark: We can consider any simplex $|\sigma|$ has two *ends* σ and $\bar{\sigma}$.

Induced Direction of Face 4/4

Definition: Let $\sigma = \langle a_0 a_1 \dots a_k \rangle \in \langle K_k \rangle$, $|\tau| \in |K_{k-1}|$, and $|\tau| \prec |\sigma|$.
We say τ has the induced direction from σ

$$\text{iff} \quad \exists \pi \in A_{k+1} \quad \text{s.t.} \quad \tau = \langle a_{\pi(1)} a_{\pi(2)} \dots a_{\pi(k)} \rangle.$$



e.g.

Let $\sigma = \langle a_0 a_1 a_2 a_3 \rangle$. Then,

$$\begin{aligned}
\tau &= \langle a_1 a_2 a_3 \rangle & (\pi &= \mathrm{id}), \\
\tau &= \langle a_0 a_3 a_2 \rangle & (\pi &= (01)(23)), \\
\tau &= \langle a_3 a_0 a_1 \rangle & (\pi &= (02)(13)), \\
\tau &= \langle a_2 a_1 a_0 \rangle & (\pi &= (03)(12)).
\end{aligned}$$

Outline

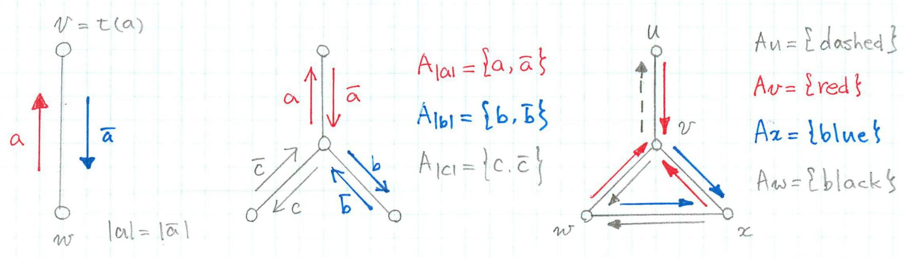
Quantum walk on graph Γ, revisited 1/6

Preparation

- Let $\Gamma = (V, E)$ be a discrete graph.
- $A := \{a, \bar{a};\ |a| \in E\}$; a, $\bar{a}$ are the induced *arcs* from an edge $|a|$.

Remark: $A = \sqcup_{|a| \in E} A_{|a|} = \sqcup_{v \in V} A_v$ (disjoint unions),

with $A_{|a|} = \{a, \bar{a}\}$, $A_v = \{a \in A;\ t(a) = v\}$.



$A = A_{|a|} \sqcup A_{|b|} \sqcup A_{|c|}$ $\qquad$ $A = A_u \sqcup A_v \sqcup A_x \sqcup A_w$

Quantum walk on graph Γ, revisited 2/6

Quantum walker moves on E.

Definition of QW on Γ

(1) The total state space: $\mathcal{H} := \ell^2(A)$

(2) Time evolution unitary operator U on $\mathcal{H}$:

$$\psi(n+1) := U\psi(n) \quad (n \in \mathbb{Z}\text{: time}).$$

(3) The existence probability of quantum walker at $|a|$:

$$\mu_\psi(|a|) := |\psi(a)|^2 + |\psi(\bar{a})|^2 \ \text{ for } \ \mu_\psi(E) = \|\psi\|_{\mathcal{H}}^2 = 1.$$

How to give U?

Quantum walk on graph $\Gamma = (V, A)$, revisited 3/6

Unitary operator U made from shift and coin operators

Since

$$\mathcal{H} := \ell^2(A) \;\cong\; \oplus_{v \in V} \ell^2(A_v) \;\cong\; \oplus_{|a| \in E} \ell^2(A_{|a|}),$$

we can give U by

$$(2) \qquad U := SC \quad \text{with} \quad S := \oplus_{v \in V} S_v, \quad C := \oplus_{|a| \in E} C_{|a|}.$$

Here, S_v and $C_{|a|}$ are unitary on $\ell^2(A_v)$ and $\ell^2(A_{|a|})$, respectively.

Using this idea, we define QW on SC.

Definition of QW on SC $_{4/6}$

We consider quantum walker moves on $|K_{n-1}|$. We define

$$K := K^{n,n-1} := \{(\sigma,\tau) \in \langle K_n \rangle \times \langle K_{n-1} \rangle;\ \tau \prec \sigma\}.$$

and the total state space

$$\mathcal{H} := \ell^2(K). \tag{1}$$

Using two equivalence relations on K,

$$(\sigma,\tau)R_1(\sigma',\tau') \text{ iff } \tau = \tau' \quad \text{and} \quad (\sigma,\tau)R_2(\sigma',\tau') \text{ iff } \sigma = \sigma',$$

we can decompose K in two ways:

$$K = \sqcup_{F_\sigma \in K/R_2} F_\sigma = \sqcup_{E_\tau \in K/R_1} E_\tau. \quad \text{(disjoint unions)}$$

Definition of QW on SC 5/6

Since

$$\mathcal{H} := \ell^2(K) \;\cong\; \oplus_{F_\sigma \in K/R_2} \ell^2(F_\sigma) \;\cong\; \oplus_{E_\tau \in K_1} \ell^2(E_\tau),$$

we can define U by

(2) $\quad U = \hat{F}\hat{E} \quad$ with $\quad \hat{F} = \oplus_{F_\sigma \in K/R_2} \hat{F}_\sigma, \quad \hat{E} = \oplus_{E_\tau \in K/R_1} \hat{E}_\tau.$

Here, $\hat{F}_\sigma$ and $\hat{E}_\tau$ are unitary on $\ell^2(F_\sigma)$ and $\ell^2(E_\tau)$, respectively.

(3) The existence probability of quantum walker at $|\tau|$:

$$\mu_\psi(|\tau|) := \|\psi\|^2_{\ell^2(E_\tau)} + \|\psi\|^2_{\ell^2(E_{\bar{\tau}})} \text{ for } \mu_\psi(|K_{n-1}|) = \|\psi\|^2_{\mathcal{H}} = 1.$$

Here, $\|\psi\|_{\mathcal{K}} := \|P_{\mathcal{K}}\psi\|_{\mathcal{H}}$ for a subspace $\mathcal{K} \subset \mathcal{H}$ ($P_{\mathcal{K}}$ is a projection).

Is this QW on SC valid? 6/6

We ought to check the following;

1. linear spreading,
2. localization.

In next section, we show a relation between

our QW on SC and bipartite walk.

Outline

QW on SC and Bipartite walk 1/4

Theorem (Matsue,O,Segawa,2017)

Our QW on SC is unitary equivalent to (a kind of) bipartite walk.

More precisely,

$$\forall \text{ QW on SC with time evolution } U,$$
$$\exists \text{ a bipartite walk with time evolution } V$$
$$\text{s.t. } U \cong V.$$

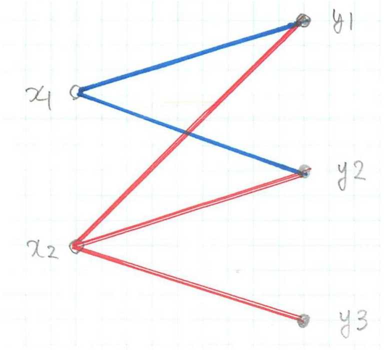
Consequently, these two QW have same distribution.

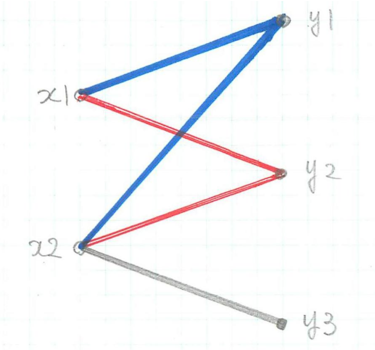
Definition of Bipartite walk 2/4

Definition: $\Gamma = (X \sqcup Y, E)$ is a bipartite graph iff

1. the vertex set $X \sqcup Y$ is the disjoint union of X and Y.
2. $E \subset X \times Y$; any edge in E connects between X and Y.

e.g. $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$.





Note: E has two natural decompositions; $E = \sqcup_{x \in X} E_x = \sqcup_{y \in Y} E_y$.

Definition of Bipartite walk on $\Gamma = (X \sqcup Y, E)$ 3/4

Bipartite walk on Γ is defined by

(1) The total state space: $\mathcal{H} := \ell^2(E) \cong \oplus_{x \in X} \ell^2(E_x) \cong \oplus_{y \in Y} \ell^2(E_y)$,

(2) Time evolution: $V := \left(\oplus_{x \in X} \hat{E}_x\right) \left(\oplus_{y \in Y} \hat{F}_y\right)$

with unitary operators $\hat{E}_x$ and $\hat{F}_y$ on $\ell^2(E_x)$ and $\ell^2(E_y)$.

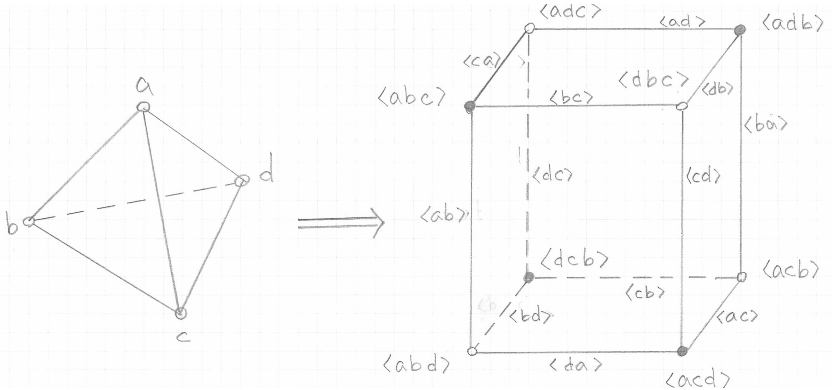
Key of Proof of Theorem

SC K induces a bipartite graph $\Gamma(K) := (K/R_1 \sqcup K/R_2, E)$ with

$$E := \{(E_\tau, F_\sigma) \in K/R_1 \times K/R_2;\ E_\tau \cap F_\sigma \neq \emptyset\}.$$

Then, we can prove
U of QW on K is unitary equivalent to V of a bipartite walk on $\Gamma(K)$.

Example of the induced bipartite graph from K 4/4



K $\qquad\qquad$ $\Gamma(K)$

QW on SC is an important subclass of bipartite walk!

(It is our hope.)

Outline

Quantum search algorithm 1/5

Search Problem: Find a target in a given set.

- Problem size $N :=$ the size of the given set.
- Time complexity $:=$ the time order with respect to N.

Well-known results about time complexity:

- Classical search requires $O(N)$.
- Quantum search requires $O(\sqrt{N})$ using Glover algorithm.

We can implement Glover algorithm as *Glover walk* on a graph.

Our question:

Can we implement Glover search as QW on SC?

Glover search as Glover walk on graph 2/5

Let $\mathbf{K}_n = (V, E)$ be the **complete graph**.

Problem: Mark some $w \in V$ and find it using QW.

Using the Glover matrix, $G_n = \frac{1}{n}J_n - I_n$, we define

$$U := SC \text{ with } S := \oplus_{a \in A} G_{n(a)} \text{ and } C := \oplus_{v \in V} C_v.$$

Here, $C_w := -I_{n(w)}$ and $C_v = G_{n(v)}$ $(v \neq w)$.

Take an initial state ψ_0 a uniform state and let

$$P_t := \sum_{a \in A, a \sim w} |(U^t \psi_0)(a)|^2$$

be the finding probability at time t. Then,

$$\text{the finding time } T := \inf\{t;\ P_t > 1/2\} = O(\sqrt{N}).$$

Glover search on simplicial complex 3/5

Define SC $\mathbf{K}_{n-2,n-3}$ from $\mathbf{K}_n$ as follows:

Here, we consider $\mathbf{K}_4$ with $V(\mathbf{K}_4) = \{a, b, c, d\}$ for simplicity. $\mathbf{K}_4$ contains four 2-simplices,

$$|\mathbf{K}_2| = \{|abc|, |acd|, |adb|, |bcd|\}$$

and six 1-faces of them,

$$|\mathbf{K}_1| = \{|ab|, |ac|, |ad|, |bc|, |bd|, |cd|\}.$$

Using these, we define

$$\mathbf{K}_{2,1} := \{(\sigma, \tau) \in \langle K_2 \rangle \times \langle K_1 \rangle;\ \tau \prec \sigma\}.$$

Glover search as QW on Simplicial Complex 4/5

Consider $\mathbf{K}_{n-2,n-3}$.

Problem: Mark some $|\tau| \in |\mathbf{K}_{n-3}|$ and find it using QW.

Note $N = \#\mathbf{K}_{\mathbf{n-2,n-3}} = 2(n+1)(n+2)$.

Define

$$U := \hat{F}\hat{E} \text{ with } \hat{F} := \oplus_\sigma G_{n(\sigma)} \text{ and } \hat{E} := \oplus_\upsilon E_\upsilon.$$

Here, $E_\tau := -I_{n(\tau)}$ and $E_\upsilon = G_{n(\upsilon)}$ $(\upsilon \neq \tau)$.

Take an initial state ψ_0 a uniform state and let

$$P_t := \sum_{\sigma, \sigma \succ \tau} |(U^t \psi_0)(\sigma)|^2$$

be the finding probability at time t.

Theorem (Matsue,O,Segawa,2017)

The finding time $T := \inf\{t;\ P_t > 1/2\} = O(\sqrt{N})$.

Numerical example of Glover search on SC 5/5

1. Consider $n = 10$ case. Then, N is about 200.
2. We used another correspondence between QW on SC K and QW on a non bipartite-graph Γ, because we wanted to use a quantum search program for graphs which we already made.

   The correspondence maps the one marked face on K to four vertices on Γ. It may make you confuse. Sorry.

3. We plot the vertices of Γ on a circle.
4. You can find four peaks on the circle at finding time T in our simulation.
5. Remark: The distribution periodically varies and the time at the first peak is the finding time T.

Outline

Conclusion and references

1. We propose a definition of quantum walk on simplicial complex.
2. This definition gives a new non-trivial class of bipartite walk.
3. Our QW can achieve quantum acceleration on search problem.

References:

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2. *Quantum Walks and Search Algorithms*, Renato Portugal, Springer 2013
3. 量子情報科学入門, 石坂智, 他 3 名共著, 共立出版, 2012 (和書)
4. *Resonant-tunneling in discrete-time quantum walk*, arXiv:1708.01052, Quantum Studies: Mathematics and Foundations (2018) by Matsue, Matsuoka, O, Segawa.

Thank you.